q-ANALOGUES OF TWO FAMILIES OF TERMINATING ${}_2F_1(2)$-SERIES IDENTITIES

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ABSTRACT. Recently, Chu [5] derived two families of terminating ${}_2F_1(2)$-series identities. Their q-analogues will be established in this paper.

1. INTRODUCTION

Following Bailey [3], define the hypergeometric series by

$$ {}_{1+r}F_s\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix}\,\middle|\, z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k\cdots(a_r)_k}{k!\,(b_1)_k\cdots(b_s)_k} z^k, $$

where the shifted factorial is given by

$$ (x)_0 = 1 \quad \text{and} \quad (x)_n = \prod_{i=0}^{n-1}(x+i) \quad \text{for} \quad n = 1, 2, \cdots. $$

Define the symbol $X(n)$ by

$$ X(n) = \frac{1+(-1)^n}{2}. $$

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Then two known results(cf. Prudnikov et al. [13, Entries 7.3.8.2 and 7.3.8.6]) can be stated as

$$ {}_2F_1\left[\begin{matrix}-n,\ x\\ 2x\end{matrix}\,\middle|\,2\right]=\frac{(\frac{1}{2})_{\frac{n}{2}}}{(\frac{1}{2}+x)_{\frac{n}{2}}}\,X(n), \tag{1} $$

$$ {}_2F_1\left[\begin{matrix}x,\ -n\\ -2n\end{matrix}\,\middle|\,2\right]=\frac{(\frac{1+x}{2})_n}{(\frac{1}{2})_n}. \tag{2} $$

Based on (1) and (2), Chu [5] derived, by series rearrangement, two families of terminating ${}_2F_1(2)$-series identities:

$$ {}_2F_1\left[\begin{matrix}-n,\ x\\ 2x-m\end{matrix}\,\middle|\,2\right]=\frac{(1-2x-n)_m}{(1-2x)_m}\sum_{i=0}^{m}\frac{(-m)_i(-n)_i}{i!\,(1-2x-n)_i}\frac{(\frac{1}{2})_{\frac{n-i}{2}}}{(\frac{1}{2}+x)_{\frac{n-i}{2}}}\,X(n-i), \tag{3} $$

$$ {}_2F_1\left[\begin{matrix}x,\ -n\\ -m-2n\end{matrix}\,\middle|\,2\right]=\frac{(1+x+2n)_m}{(1+2n)_m}\sum_{i=0}^{m}\frac{(-m)_i(x)_i}{i!\,(1+x+2n)_i}\frac{(\frac{1+x+i}{2})_n}{(\frac{1}{2})_n}, \tag{4} $$

$$ {}_2F_1\left[\begin{matrix}-n,\ x\\ 2x+m\end{matrix}\,\middle|\,2\right]=\sum_{i=0}^{m}\Big(\frac{1}{2}\Big)^i\frac{(-m)_i(-n)_i(2x-1)_i}{i!\,(2x+m)_i(x-\frac{1}{2})_i}\frac{(\frac{1}{2})_{\frac{n-i}{2}}}{(\frac{1}{2}+x+i)_{\frac{n-i}{2}}}\,X(n-i), \tag{5} $$

$$ {}_2F_1\left[\begin{matrix}x,\ -n\\ m-2n\end{matrix}\,\middle|\,2\right]=\sum_{i=0}^{m}\Big(\frac{1}{2}\Big)^i\frac{(-m)_i(x)_i(-2n-1)_i}{i!\,(m-2n)_i(-n-\frac{1}{2})_i}\frac{(\frac{1+x+i}{2})_{n-i}}{(\frac{1}{2})_{n-i}}\ (0\le m\le n), \tag{6} $$

where m is a nonnegative integer.

We point out that (3) and (5) are both contiguous versions of (1), and (4) and (6) are both contiguous versions of (2). Considering that (3)-(6) are related to Kummer's ${}_2F_1(-1)$-series identity(cf. Bailey [3, p. 9]), Choi et al. [4] and Vidunas [21] which provided contiguous generalizations of it should be mentioned. In addition, Ibrahim et al. [7], Rakha et al [15, 16, 17] and Vidunas [22] that offered contiguous versions of known formulas should also be attended.

Following Bailey [3], define the basic hypergeometric series by

$$ {}_{1+r}\phi_s\left[\begin{matrix}a_0, & a_1, & \cdots, & a_r\\ & b_1, & \cdots, & b_s\end{matrix}\,\middle|\,q;z\right]=\sum_{k=0}^{\infty}\frac{(a_0;q)_k(a_1;q)_k\cdots(a_r;q)_k}{(q;q)_k(b_1;q)_k\cdots(b_s;q)_k}z^k, $$

where the q-shifted factorial is given by

$$ (x;q)_0=1\quad\text{and}\quad (x;q)_n=\prod_{i=0}^{n-1}(1-xq^i)\quad\text{for}\quad n=1,2,\cdots. $$

Then the terminating q-Watson formula due to Andrews [1] and the terminating q-Watson formula due to Jain [8, Eq. 3.17] can respectively be stated as follows:

$$ {}_4\phi_3\left[\begin{matrix}q^{-n},\ q^{1+n}a,\ \sqrt{c},\ -\sqrt{c}\\ q\sqrt{a},\ -q\sqrt{a},\ c\end{matrix}\,\middle|\,q;q\right]=c^{\frac{n}{2}}\frac{(q;q^2)_{\frac{n}{2}}(q^2a/c;q^2)_{\frac{n}{2}}}{(q^2a;q^2)_{\frac{n}{2}}(qc;q^2)_{\frac{n}{2}}}\,X(n), \tag{7} $$

$$ {}_4\phi_3\left[\begin{matrix}a,\ c,\ q^{-n},\ -q^{-n}\\ q^{-2n},\ \sqrt{qac},\ -\sqrt{qac}\end{matrix}\,\middle|\,q;q\right]=\frac{(qa;q^2)_n(qc;q^2)_n}{(q;q^2)_n(qac;q^2)_n}. \tag{8} $$

Setting $a = 0$ and $c = x^2$ in (7), we obtain the following q-analogue of (1):

$$ {}_3\phi_1 \left[\begin{matrix} q^{-n}, x, -x \\ x^2 \end{matrix} \;\middle|\; q; q \right] = x^n \frac{(q;q^2)_{\frac{n}{2}}}{(qx^2;q^2)_{\frac{n}{2}}} X(n). \tag{9} $$

Taking $a = 0$ and $c = x$ in (8), we get the following q-analogue of (2):

$$ {}_3\phi_1 \left[\begin{matrix} x, q^{-n}, -q^{-n} \\ q^{-2n} \end{matrix} \;\middle|\; q; q \right] = \frac{(qx;q^2)_n}{(q;q^2)_n}. \tag{10} $$

Based on (9) and (10), we shall establish, by series rearrangement, q-analogues of (3)-(6) in the next section.

2. Terminating ${}_3\phi_1(q)$-series identities

Theorem 1. *For a complex number x and two nonnegative integers m and n, there holds the following q-analogue of* (3)*:*

$$ {}_3\phi_1 \left[\begin{matrix} q^{-n}, x, -x \\ x^2 q^{-m} \end{matrix} \;\middle|\; q; q \right] = \frac{(q^{1-n}/x^2;q)_m}{(q/x^2;q)_m} (q^m x)^n \sum_{i=0}^{m} \frac{(q^{-m};q)_i (q^{-n};q)_i}{(q;q)_i (q^{1-n}/x^2;q)_i} \left(\frac{q}{x}\right)^i \times \frac{(q;q^2)_{\frac{n-i}{2}}}{(qx^2;q^2)_{\frac{n-i}{2}}} X(n-i). $$

Proof. Letting $b \to q^{k-n}$ and $c \to q^{1-n}/x^2$ for the q-Vandermonde formula(cf. Gasper and Rahman [6, p. 14]):

$$ {}_2\phi_1 \left[\begin{matrix} q^{-m}, b \\ c \end{matrix} \;\middle|\; q; q \right] = b^m \frac{(c/b;q)_m}{(c;q)_m}, $$

we gain the following equation:

$$ {}_2\phi_1 \left[\begin{matrix} q^{-m}, q^{k-n} \\ q^{1-n}/x^2 \end{matrix} \;\middle|\; q; q \right] = q^{(k-n)m} \frac{(q^{1-k}/x^2;q)_m}{(q^{1-n}/x^2;q)_m}. $$

Then we have the following composite series:

$$ {}_3\phi_1 \left[\begin{matrix} q^{-n}, x, -x \\ x^2 q^{-m} \end{matrix} \;\middle|\; q; q \right] = \sum_{k=0}^{n} \frac{(q^{-n};q)_k (x;q)_k (-x;q)_k}{(q;q)_k (x^2 q^{-m};q)_k} q^k $$

$$ = \sum_{k=0}^{n} \frac{(q^{-n};q)_k (x;q)_k (-x;q)_k}{(q;q)_k (x^2;q)_k} \frac{(q^{1-k}/x^2;q)_m}{(q/x^2;q)_m} q^{(1+m)k} $$

$$ = \sum_{k=0}^{n} \frac{(q^{-n};q)_k (x;q)_k (-x;q)_k}{(q;q)_k (x^2;q)_k} \frac{(q^{1-n}/x^2;q)_m}{(q/x^2;q)_m} q^{k+mn} {}_2\phi_1 \left[\begin{matrix} q^{-m}, q^{k-n} \\ q^{1-n}/x^2 \end{matrix} \;\middle|\; q; q \right] $$

$$ = \sum_{k=0}^{n} \frac{(q^{-n};q)_k (x;q)_k (-x;q)_k}{(q;q)_k (x^2;q)_k} \frac{(q^{1-n}/x^2;q)_m}{(q/x^2;q)_m} q^{k+mn} \sum_{i=0}^{m} \frac{(q^{-m};q)_i (q^{k-n};q)_i}{(q;q)_i (q^{1-n}/x^2;q)_i} q^i. $$

Interchanging the summation order for the last double sum, we achieve the relation:

$$
{}_3\phi_1\left[\begin{matrix} q^{-n},\ x,\ -x \\ x^2q^{-m} \end{matrix}\ \middle|\ q;q\right] = \frac{(q^{1-n}/x^2;q)_m}{(q/x^2;q)_m}\, q^{mn}\sum_{i=0}^{m}\frac{(q^{-m};q)_i(q^{-n};q)_i}{(q;q)_i(q^{1-n}/x^2;q)_i}\, q^i
\times {}_3\phi_1\left[\begin{matrix} q^{i-n},\ x,\ -x \\ x^2 \end{matrix}\ \middle|\ q;q\right]. \tag{11}
$$

Evaluating the ${}_3\phi_1$-series on the last line by (9), we attain the formula that appears in Theorem 1 to complete the proof. □

Theorem 2. *For a complex number x and two nonnegative integers m and n, there holds the following q-analogue of* (4)*:*

$$
{}_3\phi_1\left[\begin{matrix} x,\ q^{-n},\ -q^{-n} \\ q^{-m-2n} \end{matrix}\ \middle|\ q;q\right] = \frac{(q^{1+2n}x;q)_m}{(q^{1+2n};q)_m}\left(\frac{1}{x}\right)^m\sum_{i=0}^{m}\frac{(q^{-m};q)_i(x;q)_i}{(q;q)_i(q^{1+2n}x;q)_i}
\times \frac{(q^{i+1}x;q^2)_n}{(q;q^2)_n}\, q^i.
$$

Proof. Performing the exchange between x and q^{-n} for (11), we obtain the relation:

$$
{}_3\phi_1\left[\begin{matrix} x,\ q^{-n},\ -q^{-n} \\ q^{-m-2n} \end{matrix}\ \middle|\ q;q\right] = \frac{(q^{1+2n}x;q)_m}{(q^{1+2n};q)_m}\left(\frac{1}{x}\right)^m\sum_{i=0}^{m}\frac{(q^{-m};q)_i(x;q)_i}{(q;q)_i(q^{1+2n}x;q)_i}\, q^i
\times {}_3\phi_1\left[\begin{matrix} q^ix,\ q^{-n},\ -q^{-n} \\ q^{-2n} \end{matrix}\ \middle|\ q;q\right].
$$

Evaluating the ${}_3\phi_1$-series on the last line by (10), we get the identity displayed in Theorem 2 to finish the proof. □

Theorem 3. *For a complex number x and two nonnegative integers m and n, there holds the following q-analogue of* (5)*:*

$$
{}_3\phi_1\left[\begin{matrix} q^{-n},\ x,\ -x \\ x^2q^{m} \end{matrix}\ \middle|\ q;q\right] = \sum_{i=0}^{m} q^{(m+n)i}x^{n+i}\frac{(q^{-m};q)_i(q^{-n};q)_i(x;q)_i(-x;q)_i}{(q;q)_i(x^2q^m;q)_i(x^2q^{i-1};q)_i}
\times \frac{(q;q^2)_{\frac{n-i}{2}}}{(q^{1+2i}x^2;q^2)_{\frac{n-i}{2}}}\, X(n-i).
$$

Proof. Letting $a \to x^2/q$, $b \to q^{-k}$ and $c \to \infty$ for the terminating ${}_6\phi_5$-series identity (cf. Gasper and Rahman [6, p. 42]):

$$
{}_6\phi_5\left[\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ q^{-m} \\ \sqrt{a},\ -\sqrt{a},\ qa/b,\ qa/c,\ aq^{1+m} \end{matrix}\ \middle|\ q;\frac{q^{1+m}a}{bc}\right] = \frac{(qa;q)_m(qa/bc;q)_m}{(qa/b;q)_m(qa/c;q)_m},
$$

we gain the following equation:

$$
\sum_{i=0}^{m}\begin{bmatrix} k \\ i \end{bmatrix} q^{(i+m-1)i}x^{2i}\frac{(x^2q^{k+i};q)_{m-i}}{(x^2q^i;q)_{m-i}}\frac{1-x^2q^{2i-1}}{1-x^2q^{i-1}}\frac{(q^{-m};q)_i}{(x^2q^m;q)_i} = 1.
$$

Then we can proceed as follows:

$$
{}_3\phi_1\left[\begin{matrix} q^{-n},\ x,\ -x \\ x^2q^m \end{matrix}\ \middle|\ q;q\right] = \sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k\end{bmatrix} q^{\binom{k+1}{2}-nk}\frac{(x;q)_k(-x;q)_k}{(x^2q^m;q)_k}
$$

$$
= \sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k\end{bmatrix} q^{\binom{k+1}{2}-nk}\frac{(x;q)_k(-x;q)_k}{(x^2q^m;q)_k}
$$

$$
\times \sum_{i=0}^{m}\begin{bmatrix} k \\ i\end{bmatrix} q^{(i+m-1)i}x^{2i}\frac{(x^2q^{k+i};q)_{m-i}}{(x^2q^i;q)_{m-i}}\frac{1-x^2q^{2i-1}}{1-x^2q^{i-1}}\frac{(q^{-m};q)_i}{(x^2q^m;q)_i}.
$$

Interchanging the summation order, we can reformulate the last double sum as

$$
{}_3\phi_1\left[\begin{matrix} q^{-n},\ x,\ -x \\ x^2q^m \end{matrix}\ \middle|\ q;q\right] = \sum_{i=0}^{m}\begin{bmatrix} n \\ i\end{bmatrix} q^{(i+m-1)i}x^{2i}\frac{1-x^2q^{2i-1}}{1-x^2q^{i-1}}\frac{(q^{-m};q)_i}{(x^2q^m;q)_i}
$$

$$
\times \sum_{k=i}^{n}(-1)^k\begin{bmatrix} n-i \\ k-i\end{bmatrix} q^{\binom{k+1}{2}-nk}\frac{(x;q)_k(-x;q)_k}{(x^2q^i;q)_k}.
$$

Shifting the summation index $k \to i+j$ for the sum on the last line, we have

$$
{}_3\phi_1\left[\begin{matrix} q^{-n},\ x,\ -x \\ x^2q^m \end{matrix}\ \middle|\ q;q\right] = \sum_{i=0}^{m} q^{(i+m)i}x^{2i}\frac{(q^{-m};q)_i(q^{-n};q)_i(x;q)_i(-x;q)_i}{(q;q)_i(x^2q^m;q)_i(x^2q^{i-1};q)_i}
$$

$$
\times\, {}_3\phi_1\left[\begin{matrix} q^{i-n},\ xq^i,\ -xq^i \\ x^2q^{2i} \end{matrix}\ \middle|\ q;q\right]. \tag{12}
$$

Evaluating the ${}_3\phi_1$-series on the lase line by (9), we achieve the formula that appears in Theorem 3 to complete the proof. □

Theorem 4. *For a complex number x and two nonnegative integers m and n with $m \le n$, there holds the following q-analogue of* (6)*:*

$$
{}_3\phi_1\left[\begin{matrix} x,\ q^{-n},\ -q^{-n} \\ q^{m-2n} \end{matrix}\ \middle|\ q;q\right] = \sum_{i=0}^{m} q^{(i+m-2n)i}\frac{(q^{-m};q)_i(x;q)_i(q^{-n};q)_i(-q^{-n};q)_i}{(q;q)_i(q^{m-2n};q)_i(q^{i-1-2n};q)_i}
$$

$$
\times \frac{(q^{i+1}x;q^2)_{n-i}}{(q;q^2)_{n-i}}.
$$

Proof. Employing the exchange between x and q^{-n} for (12), we attain the relation:

$$
{}_3\phi_1\left[\begin{matrix} x,\ q^{-n},\ -q^{-n} \\ q^{m-2n} \end{matrix}\ \middle|\ q;q\right] = \sum_{i=0}^{m} q^{(i+m-2n)i}\frac{(q^{-m};q)_i(x;q)_i(q^{-n};q)_i(-q^{-n};q)_i}{(q;q)_i(q^{m-2n};q)_i(q^{i-1-2n};q)_i}
$$

$$
\times\, {}_3\phi_1\left[\begin{matrix} q^ix,\ q^{i-n},\ -q^{i-n} \\ q^{2i-2n} \end{matrix}\ \middle|\ q;q\right].
$$

Evaluating the ${}_3\phi_1$-series on the last line by (10), we deduce the identity displayed in Theorem 4 to finish the proof. □

Noting that (9) is the case $m = 0$ of Theorems 1 and 3, we can regard the latter as the contiguous variations of the former. Meanwhile, Theorems 2 and 4 are the contiguous variations of (10) in the same reason. Several other identities on q-contiguous relations can be found in Jain [8], Kim et al. [9] and Wei et al. [23].

For explaining the interest of Theorems 1-4, eight concrete formulas from them are laid out as follows.

Example 1 ($m=1$ in Theorem 1).

$$ {}_3\phi_1\left[\begin{matrix} q^{-n}, x, -x \\ x^2/q \end{matrix} \Big| q; q\right] = x^n \frac{(q;q^2)_{\frac{n}{2}}}{(x^2/q;q^2)_{\frac{n}{2}}} X(n) - \frac{x^{n+1}}{q} \frac{(q;q^2)_{\frac{n+1}{2}}}{(x^2/q;q^2)_{\frac{n+1}{2}}} X(n+1). $$

Example 2 ($m=2$ in Theorem 1).

$$ \begin{aligned} {}_3\phi_1\left[\begin{matrix} q^{-n}, x, -x \\ x^2/q^2 \end{matrix} \Big| q; q\right] &= x^n \frac{q^3+x^2(1-q^n-q^{n+1})}{q(q^2-x^2)} \frac{(q;q^2)_{\frac{n}{2}}}{(x^2/q;q^2)_{\frac{n}{2}}} X(n) \\ &\quad - \frac{x^{n+1}}{q^2} \frac{(q^2-x^2q^n)(1+q)}{q^2-x^2} \frac{(q;q^2)_{\frac{n+1}{2}}}{(x^2/q;q^2)_{\frac{n+1}{2}}} X(n+1). \end{aligned} $$

Example 3 ($m=1$ in Theorem 2).

$$ {}_3\phi_1\left[\begin{matrix} x, q^{-n}, -q^{-n} \\ q^{-1-2n} \end{matrix} \Big| q; q\right] = \frac{1}{x} \frac{(qx;q^2)_{n+1}}{(q;q^2)_{n+1}} - \frac{1}{x} \frac{(x;q^2)_{n+1}}{(q;q^2)_{n+1}}. $$

Example 4 ($m=2$ in Theorem 2).

$$ \begin{aligned} {}_3\phi_1\left[\begin{matrix} x, q^{-n}, -q^{-n} \\ q^{-2-2n} \end{matrix} \Big| q; q\right] &= \frac{1}{x^2} \frac{1+q-x(1+q^{3+2n})}{q(1-q^{2+2n})} \frac{(qx;q^2)_{n+1}}{(q;q^2)_{n+1}} \\ &\quad - \frac{1}{x^2} \frac{(1+q)(1-q^{2+2n}x)}{q(1-q^{2+2n})} \frac{(x;q^2)_{n+1}}{(q;q^2)_{n+1}}. \end{aligned} $$

Example 5 ($m=1$ in Theorem 3).

$$ {}_3\phi_1\left[\begin{matrix} q^{-n}, x, -x \\ qx^2 \end{matrix} \Big| q; q\right] = x^n \frac{(q;q^2)_{\frac{n}{2}}}{(qx^2;q^2)_{\frac{n}{2}}} X(n) + x^{n+1} \frac{(q;q^2)_{\frac{n+1}{2}}}{(qx^2;q^2)_{\frac{n+1}{2}}} X(n+1). $$

Example 6 ($m=2$ in Theorem 3).

$$ \begin{aligned} {}_3\phi_1\left[\begin{matrix} q^{-n}, x, -x \\ q^2x^2 \end{matrix} \Big| q; q\right] &= x^n \frac{1+qx^2-q^{n+1}x^2-q^{n+2}x^2}{1-q^2x^2} \frac{(q;q^2)_{\frac{n}{2}}}{(q^3x^2;q^2)_{\frac{n}{2}}} X(n) \\ &\quad + x^{n+1} \frac{(1-qx^2)(1+q)}{1-q^2x^2} \frac{(q;q^2)_{\frac{n+1}{2}}}{(qx^2;q^2)_{\frac{n+1}{2}}} X(n+1). \end{aligned} $$

Example 7 ($m=1$ in Theorem 4: $n \geq 1$).

$$ {}_3\phi_1\left[\begin{matrix} x, q^{-n}, -q^{-n} \\ q^{1-2n} \end{matrix} \Big| q; q\right] = \frac{(x;q^2)_n}{(q;q^2)_n} + \frac{(qx;q^2)_n}{(q;q^2)_n}. $$

Example 8 ($m=2$ in Theorem 4: $n \geq 2$).

$$ \begin{aligned} {}_3\phi_1\left[\begin{matrix} x, q^{-n}, -q^{-n} \\ q^{2-2n} \end{matrix} \Big| q; q\right] &= \frac{(1-q^{2n-1})(1+q)}{q(1-q^{2n-2})} \frac{(x;q^2)_n}{(q;q^2)_n} \\ &\quad + \frac{1+q-x(1+q^{2n-1})}{q(1-q^{2n-2})} \frac{(qx;q^2)_{n-1}}{(q;q^2)_{n-1}}. \end{aligned} $$

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