Some new optimal generalized Sidon sequences[†]

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Abstract. A sequence A is a $B_h^*[g]$ sequence if the coefficients of $(\sum_{a\in A} z^a)^h$ are bounded by g. The standard Sidon sequence is a $B_2^*[2]$ sequence. Finite Sidon sequences are called Golomb rulers, which are found to have many applications such as error correcting codes, radio frequency selection and radio antennae placement. Let $R_h(g,n)$ be the largest cardinality of a $B_h^*[g]$ sequence contained in $\{1,2,\cdots,n\}$, and $F(h,g,k) = \min\{n : R_h(g;n) \geq k\}$. In this paper, computational techniques are applied to construct optimal generalized Sidon sequences, and 49 new exact values of $F(2,g,k)$ are found.

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1 Introduction

Let A be a subset of an abelian group,

$$A^*(k) = \#\{(a_1, a_2) \in A \times A | a_1 + a_2 = k\}$$

and

$$A^o(k) = \#\{(a_1, a_2) \in A \times A | a_1 - a_2 = k\}$$

In 1932, the Hungarian mathematician Simon Sidon [1] considered sets of integers with both A^* and A^o bounded. It is not hard to show that $A^*(k) \leq 2$ for all k if and only if $A^o(k) \leq 1$ for all $k \neq 0$. Sidon asked Erdös how large a subset of $\{1, 2, \cdots, n\}$ can be with the property that $A^*(k) \leq 2$ (for all k). Since that time such sets have been known as Sidon sets. A sequence A is a $B_h^*[g]$ sequence if the coefficients of $(\sum_{a \in A} z^a)^h$ are bounded by g. We denote by $R_h(g, n)$ the largest cardinality of a $B_h^*[g]$ sequence contained in $\{1, 2, \cdots, n\}$. The standard Sidon sequence is a $B_2^*[2]$ sequence. Cilleruelo et. al. [6] gave sharp asymptotic estimates for the cardinality of a set of residue classes with the property that the representation function $R_h(g, n)$ is bounded by a prescribed number.

Surprisingly, the standard Sidon sequence has many applications. Finite Sidon sequences are also called Golomb rulers. It has been widely studied in radio frequency selection, radio antennae placement and error correcting codes [2–4]. There are other related problems, such as the Golomb rectangle, the disjoint Golomb ruler and the sparse ruler, related to Sidon sequences, and there is vast literature to research these problems. In [7], Robinson gave the extension of some results to an arbitrary number of dimensions of Golomb rulers. In [8], more results on Golomb rectangles were given. In [10], Shearer presented some larger unpublished results for Golomb rectangles. In [9], disjoint Golomb rulers were investigated.

Let $F(g, k) = \min\{n : R_2(g; n) \geq k\}$. K. O'Bryant [11] presented a table of some values of $F(g, k)$. In this paper, we apply some computational techniques to construct optimal generalized Sidon sequences and find some new results.

Let $\{0, 1\}^*$ be the set of binary strings. That is to say, if $\alpha \in \{0, 1\}^*$, α is a bit array where each bit is either 0 or 1. We will consider elements $\{0, 1\}^*$ and a finite positive integer set interchangeably, and use the following notation.

Definition 1 *Let* $\sigma \in \{0, 1\}^n$.

1. *σ is identified with the set $\{i | \sigma(i) = 1\}$, $|\sigma|$ and $\#(\sigma)$ denote the length of σ and the number of ones in σ, respectively.*

2. Let $\sigma = \alpha\tau$ where $\alpha, \tau \in \{0,1\}^*$. The α is a prefix of σ and τ is a suffix of σ. In particular, $\alpha 0$ and $\alpha 1$ are α concatenated with 0 and 1, respectively.

2 Search for $B_2^*[g]$ sequences

For given positive integers g and k, we will find $F(g,k)$. That is to say, let $S = \{1, 2, \cdots, n\}$, we will find the largest cardinality of a $B_2^*[g]$ sequence as a subset of S. Let $c_{g,i}$ be the largest cardinality of a generalized Sidon sequence $B_2^*[g]$ as a subset of $\{1, 2, \cdots, i\}$. Then we have:

$$F(g,k) = \max\{i | c_{g,i} = k\}. \tag{1}$$

2.1 Lemmas for computation

In order to speed up the search, we establish the following lemmas.

Lemma 1 *For integers $g, k_1, k_2 \geq 1$, $c_{g,k_1} + c_{g,k_2} \geq c_{g,k_1+k_2}$.*

In fact, we can do a little better.

Lemma 2 *For integers $g, k_1, k_2 \geq 1$, $c_{g,k_1} + c_{g,k_2} - 1 \geq c_{g,k_1+k_2-1}$.*

Proof. Suppose $m = c_{g,k_1+k_2-1}$ and $A = \{a_1, \cdots, a_{k_1+k_2-1}\} \subseteq \{1, \cdots, m\}$ is a $B_h^*[g]$ sequence, where $a_1 < \cdots < a_{k_1+k_2-1}$. Let $A_1 = \{a_1, \cdots, a_{k_1}\}$, and $A_2 = \{a_{k_1}, \cdots, a_{k_1+k_2-1}\}$. It is not difficult to see that A_1 is a $B_h^*[g]$ sequence and A_2 is a $B_h^*[g]$ sequence. So their cardinalities are no larger than c_{g,k_1} and c_{g,k_2}, respectively. Thus $c_{g,k_1} + c_{g,k_2} - 1 \geq c_{g,k_1+k_2-1}$. □

We can see the following lemma is immediate.

Lemma 3 *Let $c(g, A)$ be the largest cardinality of a generalized Sidon sequence $B_2^*[g]$ as a subset of A. If q is an integer, then $c(g,A) = c(g, A+q)$, where $A + q = \{x + q | x \in A\}$.*

By reflection symmetry, we have the following two results.

Lemma 4 *If S is a $B_2^*[g]$ sequence as a subset of $\{1, 2, \cdots, n\}$, α and τ are the prefix and suffix of S of length $\lceil n/2 \rceil$, respectively, then either $\#(\alpha) \geq c_{g,n/2}$ or $\#(\tau) \geq c_{g,n/2}$.*

Lemma 5 *For $w \geq 1$, $c_{g,w+1} = c_{g,w}$ or $c_{g,w+1} = c_{g,w} + 1$. Furthermore, $c_{g,w+1} = c_{g,w} + 1$ if and only if the length of the longest sequence contains $w + 1$.*

2.2 Dynamic programming

The well-known dynamic programming technique is to use the accumulation of information about smaller subproblems to solve larger problems. For example, Östergård [12] used this technique to solve the maximum clique problem. We will establish many solutions to smaller subproblems and make use of the information to prune the search tree.

We will establish the upper bounds $u_{g,w}$ of $c_{g,w}$ for $w < n$. First, we compute the values of $c_{g,w}$ for some $1 \leq x \leq n$. For $w > x$, we can set $u_{g,w} = \min_{1 \leq t \leq x/2} \{u_{g,t} + u_{g,w-t}\}$ based on Lemma 2. That is to say, we establish a table u as follows:

$$u_{g,w} = \begin{cases} c_{g,w}, & w \leq x \\ \min_{1 \leq t \leq x/2} \{u_{g,t+1} + u_{g,w-t} - 1\}, & w > x. \end{cases} \tag{2}$$

By formula (1), we need to compute $c_{g,n}$. Therefore, we consider the numbers $1, 2, \cdots, n$ to be added to a generalized Sidon sequence one by one. This can be done with a backtrack algorithm. Let the current set be S and the current number be w. By Lemmas 1 and 2, we can use the following criteria to prune the search tree:

$$\#(S) + u_{g,n-w} < t_n, \tag{3}$$

where $\#(S)$ is cardinality of S and t_n is the desired cardinality of the generalized Sidon sequence.

2.3 The algorithm

We use $\sigma \in \{0,1\}^n$ to represent the sequence, and it is initialized to be empty. The variable *size* is the number of σ, *max* is the current maximum cardinality of sequence found. The function *SearchSidon* is the backtrack algorithm to search optimal generalized Sidon sequences. Lemma 3 is applied in Lines 12-14 and Lemma 4 in Lines 4-8. Algorithm 2 is the main function which shows how to call the function *SearchSidon*.

By the definition of $B_2^*[g]$ sequences, we ask the following question to determine $F(g,k)$ for positive integers g and k:

Problem 1 *Given positive integers g and k, If there exists a $B_2^*[g]$ sequence of length k as a subset of $\{1, 2, \cdots, n\}$?*

In order to answer Problem 1, we input the parameters x, n and k as global variables. Then we perform Algorithm 2. If Algorithm 2 returns TRUE, there is a generalized Sidon sequence of length k as a subset of $\{1, 2, \cdots, n\}$.

Algorithm 1 SearchSidon(σ, np, ip, $size$, max)

Require:

A generalized Sidon sequence σ as a subset of length $np - 1$;

```
if ip = np + 1 then
   return FALSE;
end if
if size > max then
   max ← size;
   save σ;
   return TRUE;
end if
if size + u_{g,np-ip} ≤ max then
   return FALSE;
end if
if size = (max + 1)/2 and ip > (np + 1)/2 then
   return FALSE;
end if
if σ1 is not a generalized Sidon sequence then
   if SearchSidon(σ0, np, ip + 1, size, max)=TRUE then
      return TRUE;
   else
      return FALSE;
   end if
end if
if SearchSidon(σ1, np, ip + 1, size + 1, max)=TRUE then
   return TRUE;
end if
if SearchSidon(σ0, np, ip + 1, size, max)=TRUE then
   return TRUE;
else
   return FALSE;
end if
```

2.4 Results

The algorithm is implemented in C++ language and is verified by the known results in [5, 11, 13], in which a table of $F(g, k)$ was presented. By using our algorithm, the table is extended. Table 1 presents these extended exact values, where $1-x$ denotes the numbers $1, 2, \cdots, x$. For the old results on exact values in Table 1 and bounds in Table 2, "-" denotes unknown result.

Algorithm 2 Main

for $w \leftarrow 1$; $w \leq x$; $w \leftarrow w + 1$ **do**
  $\sigma \leftarrow 1$; $max \leftarrow 1$;
  SearchSidon($\sigma, w, 1, 1, max$);
  $c_{g,w} \leftarrow max$; $u_{g,w} \leftarrow max$
end for
for $w \leftarrow x + 1$; $w \leq n$; $w \leftarrow w + 1$ **do**
  $u_{g,w} \leftarrow \min_{1 \leq t \leq x/2} \{u_{g,t+1} + u_{g,w-t} - 1\}$;
end for
if SearchSidon($\sigma, n, 1, 1, k - 1$) **then**
  return TRUE;
else
  return FALSE;
end if

Table 1: Exact values of $F(g, k)$

g	k	old	new	construction
3	12	≤ 72	72	$\{1,3,5,9,17,31,36,51,54,61,71,72\}$
3	13	≤ 101	87	$\{1,2,12,18,22,35,43,58,61,73,80,85,87\}$
3	14	≤ 128	106	$\{1,3,5,12,21,37,40,52,62,79,92,100,105,106\}$
3	15	–	127	$\{1,2,3,23,27,37,44,51,81,96,108,111,114,119,127\}$
3	16	–	151	$\{1,3,5,16,27,37,55,58,75,83,102,116,139,145,146,151\}$
4	15	≤ 52	51	$\{1,2,7,8,11,17,19,20,23,37,42,44,47,49,51\}$
4	16	–	58	$\{1,2,4,5,17,20,22,25,29,35,44,46,51,52,57,58\}$
4	17	–	67	$\{1,2,6,13,14,17,25,27,32,38,44,58,60,61,64,66,67\}$
4	18	–	76	$\{1,3,5,7,11,21,23,30,33,34,51,54,59,62,67,68,75,76\}$
4	19	–	85	$\{1,3,4,6,8,10,17,22,23,36,47,51,57,59,67,74,75,84,85\}$
5	15	≤ 47	47	$\{1,2,3,4,5,8,9,15,17,20,27,36,37,45,47\}$
5	16	–	53	$\{1,2,3,7,9,14,17,18,22,25,27,41,44,47,50,53\}$
5	17	–	61	$\{1,4,9,12,15,17,21,22,29,31,35,36,57,58,59,60,61\}$
5	18	–	69	$\{1,4,9,12,15,17,21,25,26,33,35,39,40,65-69\}$
5	19	–	79	$\{1,2,3,10,14,18,23,25,30,40,43,44,62,65,68,71,74,76,79\}$
6	17	≤ 42	41	$\{1,2,5,7,8,10,11,12,16,18,25,27,28,37,39,40,41\}$
6	18	–	46	$\{1-4,9,12,13,15,16,18,21,25,36,38,40,42,44,46\}$
6	19	–	52	$\{1-6,8,15,16,17,22,23,26,35,39,43,45,51,52\}$
6	20	–	57	$\{1-5,7,8,10,12,18,21,23,32,33,37,44,45,49,56,57\}$
6	21	–	64	$\{1-5,14,17,19,20,22,24,26,29,37,50,51,55,56,59,63,64\}$
6	22	–	71	$\{1-6,9,10,12,23,25,29,38,41,43,50,53,55,59,65,68,71\}$
7	17	≤ 38	38	$\{1-7,9,11,13,16,21,22,28,29,35,38\}$
7	18	–	43	$\{1-7,11,13,15,22,23,28,29,35,36,42,43\}$

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Table 1 – continued from previous page

g	k	old	new	construction
7	19	–	48	$\{1,4,8,10,12,15,16,18,21,22,24,28,29,43-48\}$
7	20	–	53	$\{1,3,5,8,10,12,15,16,18,21,22,24,27,28,48-53\}$
7	21	–	60	$\{1,2,7,8,12,13,14,18,20,21,24,27,28,36,38,55-60\}$
7	22	–	66	$\{1-6,8,10,12,19,21,29,32,33,39,44,45,51,54,57,65,66\}$
8	19	≤ 38	38	$\{1-8,11,13,14,16,19,23,29,30,33,37,38\}$
8	20	–	42	$\{1-7,9,10,14,15,16,20,22,25,32,34,36,41,42\}$
8	21	–	46	$\{1-5,11,12,13,15,16,17,18,20,22,26,34,38,39,42,44,46\}$
8	22	–	51	$\{1-8,11,13,14,17,19,23,29,30,36,37,43,44,50,51\}$
8	23	–	56	$\{1-7,9,11,13,18,20,22,30,32,33,36,42,43,46,52,55,56\}$
9	19	≤ 36	36	$\{1-9,11,13,15,18,21,25,26,29,34,36\}$
9	20	–	39	$\{1,2,4,5,6,8,9,10,11,12,15,19,24,25,26,27,36,37,38,39\}$
9	21	–	43	$\{1,2,4,5,6,7,9,10,12,13,14,16,24,25,26,27,32,40-43\}$
9	22	–	48	$\{1-9,11,13,21,22,23,24,31,34,35,37,45,46,48\}$
9	23	–	53	$\{1-9,12,13,15,19,21,24,28,29,38,39,43,47,52,53\}$
9	24	–	57	$\{1,3,4,8,10,11,12,15,17,18,19,22,24,27,28,30,32,51-57\}$
10	21	≤ 37	37	$\{1-10,12,14,17,18,25,26,27,28,35,36,37\}$
10	22	–	40	$\{1-10,12,14,18,19,20,25,28,29,32,38,39,40\}$
10	23	–	45	$\{1-10,13,14,19,20,21,23,28,31,33,35,42,43,44\}$
10	24	–	48	$\{1-10,12,14,21,22,23,24,29,32,33,37,38,46,47,48\}$
10	25	–	52	$\{1-9,11,12,14,18,23,24,25,26,33,36,37,38,46,50,51,52\}$
10	26	–	57	$\{1,2,3,5,6,8,10-13,15,16,18,25,26,34-38,41,52-56\}$
11	22	≤ 38	38	$\{1-11,14,17,18,19,24,26,28,30,35,37,38\}$
11	23	–	42	$\{1-11,13,15,17,20,21,29,30,31,32,38,41,42\}$
11	24	–	45	$\{1,2,3,7,8,9,10,12,14,15,16,18,19,21,22,24,26,39-45\}$
11	25	–	49	$\{1-10,12,13,16,22,23,24,25,30,33,35,38,40,47,48,49\}$
11	26	–	53	$\{1,2,3,5,6,9-17,19,25,28,32,33,35,40,45,47,50,52,53\}$

Based on Algorithm 1 and 2, the lower bounds for some $F(g,k)$ are obtained, and the upper bounds are found by partial search. These results are presented in Table 2, where lb and ub denote the lower and upper bounds, respectively.

Table 2: Bounds on $F(g,k)$

g	k	lb	ub	construction
3	17	159	185	$\{1,2,3,5,8,13,24,45,70,85,99,119,137,146,155,172,185\}$
3	18	–	222	$\{1,2,3,5,8,13,21,39,61,82,96,110,137,160,183,192,207,222\}$
4	20	87	99	$\{1-4,6,8,14,16,23,24,35,44,48,60,71,74,83,88,97,99\}$

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Table 2 – continued from previous page

g	k	lb	ub	construction
4	21	–	112	{1–4, 6, 8, 13, 17, 27, 30, 49, 61, 62, 67, 70, 82, 90, 98, 100, 106, 112}
4	22	–	130	{1–4, 6, 8, 12, 16, 24, 38, 50, 57, 63, 78, 87, 90, 97, 103, 114, 121, 128, 130}
5	20	85	90	{1 – 5, 7, 9, 31, 33, 44, 46, 54, 55, 64, 65, 71, 77, 80, 87, 90}
5	21	–	106	{1 – 5, 7, 9, 12, 16, 22, 27, 35, 43, 47, 53, 70, 76, 82, 92, 98, 106}
5	22	–	119	{1–5, 7, 9, 12, 16, 21, 29, 33, 48, 63, 64, 70, 80, 86, 96, 104, 114, 119}
5	23	–	133	{1–5, 7, 9, 12, 16, 21, 28, 41, 44, 54, 62, 70, 76, 85, 98, 106, 115, 128, 133}
6	23	77	82	{1 – 6, 8, 10, 12, 16, 20, 25, 29, 32, 44, 45, 50, 55, 61, 66, 73, 77, 82}
6	24	–	91	{1–6, 8, 10, 12, 16, 20, 24, 33, 36, 43, 52, 54, 65, 67, 72, 78, 81, 90, 91}
6	25	–	98	{1–6, 8, 10, 12, 16, 20, 24, 39, 40, 46, 48, 65, 68, 71, 74, 81, 87, 92, 97, 98}
6	26	–	108	{1–6, 8, 10, 12, 16, 20, 24, 32, 42, 51, 53, 60, 65, 66, 71, 80, 85, 95, 98, 104, 108}
7	23	70	74	{1 – 7, 9, 11, 13, 19, 20, 25, 27, 35, 38, 44, 47, 51, 61, 64, 71, 74}
7	24	–	82	{1 – 7, 9, 11, 13, 16, 20, 22, 34, 35, 38, 43, 55, 59, 60, 67, 75, 81, 82}
7	25	–	91	{1–7, 9, 11, 13, 16, 20, 22, 37, 38, 42, 45, 61, 62, 67, 68, 75, 85, 90, 91}
8	24	58	61	{1 – 8, 10, 12, 15, 18, 21, 23, 25, 34, 35, 36, 48, 49, 50, 57, 60, 61}
8	25	–	67	{1 – 8, 10, 12, 14, 16, 28, 30, 32, 33, 43, 44, 47, 50, 51, 60, 63, 66, 67}
8	26	–	74	{1–8, 10, 12, 14, 16, 20, 30, 32, 33, 41, 42, 48, 49, 53, 56, 63, 66, 73, 74}
8	27	–	80	{1 – 8, 10, 12, 14, 16, 20, 25, 28, 33, 34, 44, 45, 52, 55, 59, 62, 69, 74, 79, 80}
8	28	–	88	{1 – 8, 10, 12, 14, 16, 20, 24, 29, 33, 36, 44, 50, 51, 61, 62, 68, 69, 77, 80, 85, 88}
8	29	–	96	{1 – 8, 10, 12, 14, 16, 20, 24, 28, 32, 43, 49, 52, 58, 59, 64, 69, 75, 76, 83, 86, 92, 96}
9	25	61	64	{1 – 9, 11, 13, 15, 17, 20, 25, 29, 30, 33, 40, 46, 47, 55, 56, 63, 64}
9	26	–	69	{1 – 9, 11, 13, 15, 17, 28, 29, 30, 34, 38, 45, 48, 52, 53, 60, 63, 68, 69}
9	27	–	76	{1–9, 11, 13, 15, 17, 20, 25, 29, 32, 33, 41, 42, 51, 52, 61, 62, 67, 74, 76}
9	28	–	82	{1 – 9, 11, 13, 15, 17, 20, 27, 29, 30, 35, 38, 45, 49, 51, 62, 65, 68, 73, 81, 82}
9	29	–	89	{1 – 9, 11, 13, 15, 17, 20, 24, 28, 29, 40, 41, 45, 51, 59, 60, 69, 70, 75, 80, 88, 89}
9	30	–	96	{1 – 9, 11, 13, 15, 17, 20, 24, 28, 30, 40, 44, 45, 50, 56, 62, 68, 74, 75, 80, 86, 95, 96}
10	27	59	62	{1–10, 12, 14, 16, 18, 21, 27, 28, 30, 33, 41, 43, 44, 51, 52, 60, 61, 62}
10	28	–	66	{1–10, 12, 14, 16, 18, 20, 24, 29, 35, 36, 38, 41, 45, 48, 54, 55, 59, 65, 66}
10	29	–	71	{1–10, 12, 14, 16, 18, 20, 25, 32, 33, 35, 36, 44, 49, 50, 53, 58, 61, 69, 70, 71}
10	30	–	77	{1–10, 12, 14, 16, 18, 20, 25, 31, 35, 36, 39, 48, 49, 51, 56, 57, 61, 68, 71, 76, 77}
10	31	–	84	{1–10, 12, 14, 16, 18, 20, 24, 28, 33, 39, 40, 42, 53, 55, 56, 62, 64, 67, 73, 80, 82, 84}
11	27	54	58	{1–10, 12, 15, 16, 17, 19, 22, 30, 31, 33, 35, 39, 41, 47, 50, 52, 57, 58}
11	28	–	63	{1–11, 13, 15, 17, 19, 23, 24, 33, 35, 36, 38, 43, 46, 48, 53, 59, 61, 63}
11	29	–	68	{1–11, 13, 15, 17, 19, 21, 24, 29, 36, 37, 38, 41, 47, 48, 55, 58, 65, 66, 68}
11	30	–	74	{1–11, 13, 15, 17, 19, 21, 24, 29, 35, 36, 37, 44, 47, 53, 54, 57, 64, 69, 73, 74}

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Table 2 – continued from previous page

g	k	lb	ub	construction
11	31	–	79	$\{1-11, 13, 15, 17, 19, 21, 24, 29, 33, 34, 36, 40, 51, 54, 57, 58, 64, 69, 75, 76, 79\}$
11	32	–	85	$\{1-11, 13, 15, 17, 19, 21, 24, 28, 33, 34, 36, 47, 48, 49, 59, 60, 65, 70, 71, 81, 84, 85\}$

3 Conclusions

Generalized Sidon sequences are interesting in Combinatorics, which have been extensively researched by many mathematicians and computer scientists. In this note, we use dynamic programming and pruning techniques to construct generalized Sidon sequences, and hence compute the values and bounds for $F(2, g, k)$. The program performs well and 49 new exact values are found. In addition, many upper bounds are given.

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