PATH DECOMPOSITIONS WHICH CONTAIN NO PROPER SUBSYSTEMS

Chandra Dinavahi[1] Department of Mathematics
1110 Cory street
The University of Findlay, Findlay, OH - 45840, USA

C. A. Rodger[2]
Department of Mathematics and Statistics
221 Parker Hall, Auburn Univeristy, AL - 36849, USA

ABSTRACT. A G-design is a partition of $E(K_v)$ in which each element induces a copy of G. The existence of G-designs with the additional property that they contain no proper subsystems has been previously settled when $G \in \{K_3, K_4 - e\}$. In this paper the existence of P_m-designs which contain no proper subsystems is completely settled for every value of m and v.

1. INTRODUCTION

For any two graphs G and H, a G-decomposition of H is an ordered pair $T = (V, D)$, where V is the vertex set of H and D is a partition of the edge set of H, each element of which induces a copy of G. For any graph G and any set L of edges in K_v, a G-packing with a leave L of order v is an ordered pair $T = (V, B)$, where V is the vertex set of K_v and B is a partition of the edge set of $K_v - L$, each element of which induces a copy of G. A G-packing of order v with leave L is said to be maximum if there is no G-packing of order v with leave L' such that $|L'| < |L|$. A proper subsystem of T is an ordered pair $S = (V', B')$ where $V' \subset V$, $B' \subset B$ and (V', B') is a G-decomposition of $K_{v'}$ for $|V| > |V'| = v' > 1$. A G-packing with $L = \emptyset$ is said to be a G-design.

When considering graph decompositions the most natural question is to find the set of values of v for which there exists a decomposition of K_v into edge-disjoint copies of a fixed graph G. This set of values is called as the spectrum of G-decompositions of K_v. This question has been settled for many G, for example, where G is K_v for $v \in \{3, 4\}$ [6, 8], a star [13], a path

[1]Email: dinavahi@findlay.edu
[2]Email: rodgec1@auburn.edu

[12], any graph with no more than four vertices [2], or a connected graph with no more than five edges [5], a cycle [1, 7, 11]. For an up-to-date survey see [3].

Another question considered in the literature concerns the existence of G-designs which contain no proper subsystems. That is, for which values of v is it possible to find a G-design (V, C) of order v such that there does not exist a G-design (W, D) where $W \subset V$ and $D \subset C$. Doyen [4] settled this question for Steiner triple systems (that is,when $G = K_3$). Rodger and Spicer solved this problem when $G = K_4 - e$ [10]. The reader may also be interested to note that the related problem for Steiner quadruple systems has been considered, but is still unsolved [9]. In this paper we solve the problem for the case $G = P_m$, a simple path with m edges, and $H = K_v$ for every value of m and v (see Theorem 6).

The existence of P_m-decompositions of K_v was solved by Tarsi [12], by proving the following result.

Theorem 1. *A necessary and sufficient condition for the existence of a decomposition of a complete multigraph λK_v into edge disjoint simple paths of length m is*

$$\begin{aligned} &v = 1, \text{ or} \\ &\lambda v(v-1) \equiv 0 \ (\text{mod } 2m) \text{ and } v \geq m+1. \end{aligned} \tag{1}$$

The approach used in proving the result involves both modifications of Tarsi's constructions and in some cases to come up with a completely new construction to make sure that the P_m-designs have no subsystems. We have created new techniques to check for subsystems in our constructions. These proof techniques can be easily applied to check for subsystems in many G-designs. The following section contains the basic ideas and notation which will be used throughout the rest of the paper.

2. Notation and Basic Ideas

For any G-decomposition $T = (V, C)$, it will be useful to let $E(T)$ denote the edges occurring in $\bigcup_{c \in C} c$; in particular, if $S = (W, D)$ is a subsystem of T, then $E(S) = E(K_{v'})$, where $v' = |W|$. In the following constructions, the set of vertices of K_v will be either $V = Z_v$ or $Z_{v-1} \cup \{\infty\}$. Let $v = |V|$ and $\varepsilon = |E(T)|$ denote the total number of vertices and edges respectively. Let the trail $T = \{\{x_0, x_1\}, \{x_1, x_2\}, \dots, \{x_{n-1}, x_n\}\}$ (not all vertices need be distinct) be denoted by $(x_0, x_1, \dots, x_n)$, and if T is a path P then let the cycle $P + \{x_0, x_n\}$ be denoted by $(x_0, x_1, \dots, x_n, x_0)$. For each trail $T = (x_0, x_1, \dots, x_n)$ on the vertex set Z_z or $Z_z \cup \{\infty\}$ (so $z = v$ or $v - 1$ respectively), let $T + i$ be the trail $(x_0 + i, x_1 + i, \dots, x_n + i)$, where each sum is reduced modulo z if $x_j \neq \infty$, and where $\infty + i$ is defined to be ∞. If $T_1 = (x_0, x_1, \ldots, x_n)$ and $T_2 = (y_0, y_1, \ldots, y_n)$ are two trails with $x_n = y_0$, then denote the concatenation of T_1 and T_2 by $T_1 + T_2 = (x_0, x_1, \ldots, x_n = y_0, y_1, \ldots, y_n)$. If $x \neq \infty$ and $y \neq \infty$ are two elements of Z_z for some $z \in \{v, v-1\}$, then the edge $\{x, y\}$ is said to be of difference k if $k = \min\{|y-x|, z - |y-x|\}$. If $k = v/2$ then the edge $\{x, y\}$ is said to be of half difference. The set of all differences will be denoted by $D_v = \{1, 2, \ldots, \lfloor v/2 \rfloor\}$.

One of the basic ingredients used in the constructions is the trail

$$C(v,k) = (0, k+1, 1, k+2, \ldots, k-1, v-1, k, 0), \tag{2}$$

where $k \in D_v$ and $k < (v-3)/2$. Notice that $C(v,k)$ has length $2v$ and contains all the edges of differences k and $k+1$. Let $\{C(v,k)+i\}$ be the trail $(i, k+1+i, 1+i, k+2+i, \ldots, k-1+i, v-1+i, k+i, i)$.

For any trail $T = (v_1, v_2, \ldots, v_k)$ and $k \geq m$, let T/m be the set of m-trails $\{(v_i, \ldots, v_{i+m}) \mid i = zm+1, 0 \leq z \leq \lfloor k - 1/m \rfloor - 1\}$. Notice that the edges in T/m partition all but at most the last $m-1$ of the edges in T. Our aim is to pick T carefully so that each element in T/m is a path. For any trail $T = (v_1, v_2, \ldots, v_k)$, if v_i and v_j are the first occurrences of a and b respectively in T then let $S(T, a, b)$ denote the subtrail $(v_i, v_{i+1}, \ldots, v_j)$ of T. For $x, y \in Z_z$ with $x \neq y$, let $I(x,y)$ be the path $(x, x+1, x+2, \ldots, y)$ consisting entirely of edges of difference 1 reducing the sums modulo z.

In order to prove that a given G-decomposition (V, C) does not have a subsystem (W, D), the argument here is usually based on the observation that if $\{x, y\} \subseteq W$, then there exists a path $c \in C$ containing the edge $\{x, y\}$, implying that $V(c) \subseteq W$. This observation is denoted by $\{x, y\} \to V(c)$. Often a specific vertex $\alpha \in V(c)$ is of specific interest, so we similarly write $\{x, y\} \to \alpha$ to indicate that since $\{x, y\} \subseteq W$ it follows that $\alpha \in W$. A common technique used here to show that a P_m-design has no subsystems when v is even is to focus on the pairs of vertices joined by an edge of difference $v/2$, showing that either the edge $\{u, u+v/2\} \to \{u+1, u+v/2+1\}$ or $\{u, u+v/2\} \to \{u-1, u+v/2-1\}$; in either case we say that the *next* half difference is also in the subsystem.

3. Preliminary Results

In order to prove the main result we first make the following useful observations.

Lemma 2. *If $m \geq 2v/3$ then every P_m-decomposition of K_v has no subsystems.*

Proof. Suppose $S = (W, D)$ is a subsystem of the P_m-decomposition (V, C) of K_v. Then since D contains a path $|W| \geq m+1$. Consider an edge

$\{x, y\}$, where $x \in W$ and $y \in V - W$. Since S is a subsystem, each edge in the path P that contains the edge $\{x, y\}$ has at least one end in $V - W$. Therefore $| V - W | \geq \lceil m/2 \rceil$. So $|V| = |W| + |V - W| \geq m + 1 + m/2 = 3m/2 + 1 \geq |V| + 1$, a contradiction. Hence the P_m-decomposition contains no subsystems. □

We now prove a lemma that is used regularly in later constructions. It considers the concatenation of various copies of $C(v, k)$ (see Equation 2).

Lemma 3. *Suppose that $i, j \in D_v$ with $v/2 > j > i$, and that $j - i$ is odd. Let T be the trail formed by the concatenation $C(v, i) + C(v, i + 2) + \cdots + C(v, j - 1)$. If T contains consecutive edges that form a cycle C, then the length of C is at least $2i + 1$.*

Proof. We prove the result by showing that if T contains x consecutive edges that form a cycle C then $x \geq 2i + 1$.

Looking at the structure of $C(v, i)$, any cycle consisting only of edges in $C(v, i)$ has length $2i + 2$. If C contains edges from both $C(v, l)$ and $C(v, l + 2)$ then C must contain precisely the first $2l$ edges in $C(v, l + 2)$ together with the edge $\{0, l\}$ in $C(v, l)$, so has length $2l + 1$. Since $l \geq i$, we can conclude that the length of the smallest cycle in T is $2i + 1$. □

Corollary 4. *Suppose that $i, j \in D_v$ with $v/2 > j > i$, and that $j - i$ is odd. Let T be the trail formed by the concatenation $C(v, i) + C(v, i + 2) + \cdots + C(v, j - 1)$. If $m \leq 2i$ then all trails in T/m are paths.*

Proof. From Lemma 3 it follows that each cycle formed by the consecutive vertices in T has length at least $2i + 1$. Since $m \leq 2i$, we can conclude that all trails in T/m are paths. □

Next we consider a similar concatenation.

Corollary 5. *Suppose that $i, j \in D_v$ with $v/2 > j > i$, that $j - i$ is odd, and that $x \in Z_v$. Let T be the trail formed by the concatenation $I(x, 0) + C(v, i) + C(v, i + 2) + \cdots + C(v, j - 1)$. If*

$$m \leq \begin{cases} \min\{v + x - 2i - 2, 2i\} & \text{when } x \geq i + 1, \text{and} \\ \min\{v - i - 1, 2i\} & \text{otherwise} \end{cases}$$

then all trails in T/m are paths.

Proof. Let C be any cycle formed by the consecutive vertices in T. If C consists only of edges in $I(x, 0) + C(v, i)$ then C must contain precisely the $v - x$ edges in $I(x, 0)$ together with the first $2x - 2i - 1$ edges of $C(v, i)$ if $x \geq i + 1$ and when $x \leq i$, C is a cycle of length $v - i$, namely the cycle $(i + 1, i + 2, \cdots, v, i + 1)$. If C contains edges from both $C(v, l)$ and

$C(v, l+2)$ then from Lemma 3 it follows that the length of the C must be $2i+1$.

Thus we can conclude that whenever

$$m < \begin{cases} \min\{v+x-2i-1, 2i+1\} \text{ when } x \geq i+1, \text{and} \\ \min\{v-i, 2i+1\} \text{ otherwise} \end{cases}$$

all trails in T/m are paths.

□

4. The Main Result

Now we state and prove the main theorem.

Theorem 6. *Let $m \geq 3$. There exists a P_m-decomposition (V, C) of K_v containing no subsystems if and only if*

$$v = 1, \text{ or } m \text{ divides } \binom{v}{2} \text{ and } v \geq m+1. \tag{3}$$

Proof. The necessary condition follows from two observations that if K_v contains at least one edge (so $v > 1$) then C must contain at least one path and so $|V| \geq m+1$; and since each of the $\binom{v}{2}$ edges in K_v occurs in exactly one path and each path contains exactly m edges.

In order to prove the sufficiency we now consider two cases depending on whether v is odd or even, each case considering various subcases in turn. In view of Theorem 1 and Lemma 2, if $m \geq 2v/3$ then P_m-decompositions exist and clearly have no subsystems; so we can assume that $m < 2v/3$. In particular, since $v \geq m+1 \geq 4$ it follows that $m \leq v-2$.

Case A: v is odd.

Let $C_0 = (v_0, v_1, v_2, \dots, v_v)$ be a hamiltonian cycle defined by

$$v_i = \begin{cases} \infty \text{ if } i \in \{0, v\}, \text{ and} \\ (-1)^i \lceil (i-1)/2 \rceil (mod \text{ v}) \text{ otherwise,} \end{cases}$$

where each sum is reduced modulo v(this is the well-known Walecki construction [14]). Let $C_i = C_0 + i$ for each $i \in Z_{v-1}$. Then clearly $C_i = C_{i+(v-1)/2}$ for $i \in Z_{(v-1)/2}$. Also note that $\{C_i \mid i \in Z_{(v-1)/2}\}$ is the standard hamiltonian decomposition of K_v. Form an Euler tour $(e_1, e_2, \dots, e_\varepsilon)$ by the concatenation $C_0 + C_1 + \cdots + C_{(v-3)/2}$. For each $i \in Z_{v(v-1)/2m}$, let π_i be the trail induced by $\{e_{im+1}, e_{im+2}, \dots, e_{(i+1)m}\}$. Then $(V, C) = (Z_{v-1} \cup \{\infty\}, \{\pi_i \mid i \in Z_{v(v-1)/2m}\})$ is a P_m-decomposition of K_v. Suppose $S = (W, D)$ is any subsystem in this P_m-decomposition of K_v. Now we consider various possibilities, arriving at the contradiction $W = V$ in each case.

Case 1: Suppose $\{\infty, i\} \in E(S)$ for some i.

We will show that $\{\infty, i\} \to i+1$. By repeating this argument we can conclude that $W = V$.

Since $m \geq 3$ and $C_i = (\infty, i, i+1, \ldots, i+(v-1)/2+1, i+(v-1)/2, \infty)$, clearly $\{\infty, i\} \to i+1$ except possibly if

(a) $\{\infty, i\}$ is the last edge of some π_j and $i < (v-1)/2$, or

(b) $\{\infty, i\}$ is the first edge of some π_j and $i \geq (v-1)/2$.

We now consider each exceptional case in turn.

Case 1a: Suppose $\{\infty, i\}$ is the last edge of some π_j and $i < (v-1)/2$.

Since $m \geq 3$, in this case $\pi_j = (\ldots, i+(v-1)/2, i-1+(v-1)/2, \infty, i)$, so clearly

$$\{\infty, i\} \to i+(v-1)/2. \tag{4}$$

Then for some k, $\{\infty, i+(v-1)/2\}$ is in the path $\pi_{j+k} = (\ldots, i+(v-1)/2, \infty, i+1, \ldots)$ or in the path $\pi_{j+k} = (\ldots, i+(v-1)/2, \infty)$ in C.

In the first case obeservation (4) implies that $V(\pi_{j+k}) \subseteq W$, so $i+1 \in W$ as required.

Otherwise $E(C_i) - \{\infty, i\} = \bigcup_{l=1}^{k} E(\pi_{j+l})$. So $|V| - 1 = |E(C_i) - \{\infty, i\}|$ is divisible by m. So this case only arises when $|V| \equiv 1 \pmod{m}$. So since $\pi_{j+k} = (\ldots, i+1+(v-1)/2, i+(v-1)/2, \infty)$, we have $\{\infty, i\} \to \{\infty, i+(v-1)/2\} \to i+1+(v-1)/2$. Then, since $|V| \equiv 1 \pmod{m}$, it follows that $E(C_{i+1}) - \{\infty, i+1+(v-1)/2\} = \bigcup_{l=k+1}^{2k} E(\pi_{j+l})$ and so $\pi_{j+(2k+1)} = (i+1+(v-1)/2, \infty, i+2, \ldots)$ implying that $\{\infty, i\} \to i+2$. But $\{i, i+2\}$ is in $\pi_{j+(k+1)} = (\infty, i+1, i+2, i, \ldots)$. Hence $\{\infty, i\} \to i+1$, so $i+1 \in W$ as required.

Case 1b: Suppose $\{\infty, i\}$ is the first edge of some π_j and $i \geq (v-1)/2$.

Since $m \geq 3$, in this case $\pi_j = (i, \infty, i-(v-1)/2+1, i-(v-1)/2+2, \ldots)$, so clearly

$$\{\infty, i\} \to i-(v-1)/2+2. \tag{5}$$

Then for some k, $\{\infty, i-(v-1)/2+2\}$ is in the path $\pi_{j+k} = (\cdots, i+1, \infty, i-(v-1)/2+2, \ldots)$ or in the path $\pi_{j+k} = (\infty, i-(v-1)/2+2, i-(v-1)/2+3, \ldots)$ in C. So in either case observation (5) implies that

$$V(\pi_{j+k}) \subseteq W \tag{6}$$

In the first case observation (6) immediately implies that $i+1 \in W$ as required.

Otherwise $E(C_{i+1} = C_{i-(v-1)/2+1}) + \{\infty, i\} = \bigcup_{l=0}^{k-1} E(\pi_{j+l})$. So $|V| + 1 = |E(C_{i+1}) + \{\infty, i\}|$ is divisible by m. So this case only arises when $|V| + 1 \equiv 0 \pmod{m}$. Therefore $\pi_{j+(2k-1)} = (\ldots, i+2, \infty, i-(v-1)/2+3)$. By observation (6), $\{\infty, i-(v-1)/2+3\} \subseteq W$, which implies that

$V(\pi_{j+2k-1}) \subseteq W$. Therefore $i+2 \in W$. Finally, notice that $\pi_{j+(k-1)} = (\dots, i, i+2, i+1, \infty)$. Hence $\{i, i+2\} \to \{i+1\}$ and so $i+1 \in W$ as required.

Case 2: Suppose $\{i, i+1\} \in E(S)$ for some $i \neq \infty$.

We will show that $\{i, i+1\} \to \{\infty, j\}$ for some j. Then the result follows by Case 1. Since the edge $\{i, i+1\}$ is either immediately precedes or follows $\{\infty, i\}$ in some C_x, clearly $\{i, i+1\} \to \{\infty, i\}$ except possibly if

(a) $\{i, i+1\}$ is the first edge of some π_j and $i < (v-1)/2$, or

(b) $\{i, i+1\}$ is the last edge of some π_j and $i \geq (v-1)/2$.

Observe that in both the exceptional cases $\{i, i+1\} \to i-1$, since $\pi_j = (i, i+1, i-1, \dots)$ or $\pi_j = (\dots, i-1, i+1, i)$ respectively. So for all x,

$$\text{either } \{x, x+1\} \to \{\infty, x\}, \text{ or } \{x, x+1\} \to \{x, x-1\}. \tag{7}$$

But, since $C_0 = (\infty, 0, 1, \dots)$ implies $\{0, 1\} \to \{\infty, 0\}$, recursively applying the observation (7) implies that for all i $\{i, i+1\} \to \{\infty, j\}$ for some j (since at worst $j = 0$).

Case 3: Suppose $\{i, i+j\} \in E(S)$ for some $i \neq \infty, j > 1$.

Notice that if $\{i, i+j\}$ is in some path π_j then π_j contains at least one of the vertices $i-1, i+1, i+j-1$ or $i+j+1$. In any of these cases $\{i, i+j\} \to \{k, k+1\}$ for some $k \in \{i-1, i, i+j-1, i+j\}$. So the result follows by Case 2.

Case B: v is even.

We will solve this case by considering different subcases in turn depending on the length of the path.

Case 1: $m = v-2$.

By Lemma 2 and from the fact that $m \geq 3$, we can conclude that P_m-decompositions of K_v contain no subsystems.

Case 2: $m = v-3$.

Since $m < 2v/3$ and in this case $m = v-3$, it follows that $v < 9$. By the necessary condition that m must divide $\binom{v}{2}$, the only situation that needs to be solved is when $v = 6$ and $m = 3$. If $v = 6$, let $Z(3)$ be the zigzag path defined by $(0, 2, 5, 3)$. So $(V, C) = (Z_6, \{\{Z(3) + i \mid i \in Z_3\} \cup (0, 1, 2, 3) \cup (3, 4, 5, 0)\})$ is a P_3-decomposition of K_6.

It is easy to check that this P_3-decomposition (V, C) has no subsystems.

Case 3: $m \leq v-4$ and $v - m \equiv 3 \pmod 4$.

Without loss of generality we can assume $m \leq v-7$, because $v - m \not\equiv 3 \pmod 4$ when $m > v-7$. Observe that in this case m is odd (since v is even in Case B and $v - m \equiv 3 \pmod 4$).

Let $Z(m)$ be the zigzag path $(v_0, v_1, \dots, v_m)$ defined by

$$v_i = \begin{cases} (-1)^{i+1}\lceil (i+1)/2 \rceil \text{ for } 0 \le i \le \lfloor m/2 \rfloor, \text{and} \\ v_{m-i} + v/2 \text{ otherwise} \end{cases}$$

where each sum is reduced modulo v. Notice that the set of m-paths $Z = \{Z(m) + i \mid i \in Z_{v/2}\}$ partitions all the edges of differences in $\{2, 3, \ldots, \lceil m/2 \rceil\} \cup \{v/2\}$.

Let $T = (v_1, v_2, \ldots, v_k)$ be the trail formed by the concatenation $I(m,0)+C(v, \lceil m/2 \rceil+1)+C(v, \lceil m/2 \rceil+3)+\cdots+C(v, v/2-2)$. Apply Corollary 5 to T using $x = m$ and $i = \lceil m/2 \rceil + 1 = (m+3)/2$. Notice that in this case, if $m \ge 5$ then $x = m \ge (m+5)/2 = i+1$ and the condition of the Corollary 5 is met, and otherwise $m = 3$ in which case $x = m = i = 3$ and by conditions of Case 3, $v \ge 10$, so clearly $m \le \min\{2i, v-i-1\}$. Thus we can conclude that all trails in T/m are paths. Note that in Case 3, $v - m \equiv 3 \pmod 4$, so $\lceil m/2 \rceil + 1 \equiv v/2 - 2 \pmod 2$. So T/m is a set of m-paths which partitions all the edges of differences in $\{\lceil m/2 \rceil + 1, \lceil m/2 \rceil + 2, \ldots, v/2 - 1\}$ and the $v - m$ edges of difference 1 from the vertex m forward to the vertex 0. So $(V, C) = (Z_v, \{Z \cup T/m \cup I(0,m)\})$ is a P_m-decomposition of K_v.

Suppose $S = (W, D)$ is any subsystem in this P_m-decomposition of K_v. Let $\pi_j \in D$ be any path of length m. Now we consider various possibilities, arriving at the contradiction $W = V$ in each case.

Case 3a: Suppose that $\pi_j = Z(m) + i$ for some $i \in Z_{v/2}$.

Each path $Z(m) + i$ contains the edge $\{k, k + v/2\}$ of half difference for some k.

Suppose $m \ge 5$. Then $Z(m) + i$ contains both $k+1$ and $k + v/2 + 1$ (if $m \equiv 1 \pmod 4$) or both $k - 1$ and $k + v/2 - 1$ (if $m \equiv 3 \pmod 4$). So W must contain one pair of vertices in the next half difference, which implies that either $Z(m) + (i+1)$ or $Z(m) + (i-1) \in D$. By repeating this argument we can conclude that $V(Z) = W = V$.

Suppose $m = 3$. Then $Z(m) + i \to \{k-2, k+v/2-2\} \to Z(m) + i - 2$. So recursively it follows that $X = \{k - 2i, k + v/2 - 2i \mid i \in Z_{v/2}\} \subseteq W$. Since $v - m \equiv 3 \pmod 4$ and $m = 3$, it follows that $v/2$ is odd so k and $k + v/2$ have different parity. So $X = V$; so $W = V$.

Case 3b: Suppose that $\pi_j \in I(0,m) \cup T/m \in D$ (so either $\pi_j = I(0,m)$ or $\pi_j \in T/m$).

We will show that in either of these cases $\pi_j \to Z(m) + i \in D$ for some i, then the result follows from Case 3a.

Suppose $m \ge 5$. Every $\pi_j \in I(0,m) \cup T/m$ contains a pair of vertices $\{k, k+2\}$ for some k, and the edge $\{k, k+2\} \in Z(m) + i$ for some i. So $Z(m) + i \in D$ as required.

Suppose $m = 3$. Then one of the following occurs.

(i) π_j contains the edge $\{k, k+2\}$. So, as above, $Z(m) + i \in D$ for some i.

(i) $\pi_j = (k, k+l, k+1, k+l+1)$ is contained in a C trail. In this case the edge $\{k, k+1\}$ is in some π_n that must contain either $k-1$ or $k+2$. So S contains an edge of difference 2 (either $\{k, k+2\}$ or $\{k-1, k+1\}$). So S contains an edge in $Z(m)+i$ for some i.

(ii) $\pi_j = (k, 0, k+3, 1)$ straddles two C trails. In this case the edge $\{0, 1\} \in I(0, 3)$. So S contains the edge $\{0, 2\} \in Z(m)+1$.

Hence the result follows by Case 3a.

Case 4: $m \le v-4$ and $v-m \equiv 1 \pmod 4$.

Without loss of generality we can assume $m \le v-5$, because $v-m \not\equiv 1 \pmod 4$ when $m > v-5$. Observe that in this case m is odd (since v is even in Case B and $v-m \equiv 1 \pmod 4$).

Let $Z_1(m)$ be the zigzag path $(v_0, v_1, \ldots, v_m)$ defined by

$$v_i = \begin{cases} (-1)^{i+1}\lceil i/2 \rceil \text{ for } 0 \le i \le \lfloor m/2 \rfloor, \textit{and} \\ v_{m-i} + v/2 \text{ otherwise}, \end{cases}$$

where each sum is reduced modulo v. Notice that the set of m-paths $Z_1 = \{Z_1(m) + i \mid i \in Z_{v/2}\}$ partitions all the edges of differences in $\{1, 2, \ldots, \lfloor m/2 \rfloor\} \cup \{v/2\}$.

Let $T = (v_1, v_2, \ldots, v_k)$ be the trail formed by the concatenation $C(v, \lfloor m/2 \rfloor + 1) + C(v, \lfloor m/2 \rfloor + 3) + \cdots + C(v, v/2-2)$; note that in Case 4 $v-m \equiv 1 \pmod 4$, so $\lfloor m/2 \rfloor + 1 \equiv v/2 - 2 \pmod 2$. Using $i = \lfloor m/2 \rfloor + 1 = (m+1)/2$, clearly $m \le 2i$, so Corollary 4 can be applied to T to conclude that all trails in T/m are paths. So T/m is a set of m-paths which partitions all the edges of differences in $\{\lfloor m/2 \rfloor + 1, \lfloor m/2 \rfloor + 2, \ldots, v/2-1\}$. So $(V, C) = (Z_v, \{Z_1 \cup T/m\})$ is a P_m-decomposition of K_v.

Suppose $S = (W, D)$ is any subsystem in this P_m-decomposition. Let $\pi_j \in D$ be any path of length m. Then either $\pi_j = Z_1(m) + i$ for some $i \in Z_{v/2}$ or $\pi_j \in T/m$. We will show that $W = V$ in both these cases.

Suppose that $\pi_j = Z_1(m) + i$ for some $i \in Z_{v/2}$. Each path $Z_1(m) + i$ contains the edge $\{k, k+v/2\}$ of half difference for some k. Since $m \ge 3$, $Z_1(m) + i$ contains both $k+1$ and $k+v/2+1$ (if $m \equiv 1 \pmod 4$) or both $k-1$ and $k+v/2-1$ (if $m \equiv 3 \pmod 4$). So W must contain one pair of vertices in the next half difference, which implies that either $Z_1(m)+(i+1)$ or $Z_1(m) + (i-1) \in D$. By repeating this argument we can conclude that $V(Z_1) = W = V$.

If $\pi_j \in T/m$, then since $m \ge 3$, $\pi_j = (k, k+l, k+1, k+1+l, \ldots)$ for some k and for some l, which implies that the edge $\{k, k+1\}$ is in some $Z_1(m) + i \in D$. So the result follows by the previous argument.

Case 5: $m \le v-4$ and $v-m \equiv 2 \pmod 4$.

We will solve this case by considering two subcases in turn. Without loss of generality we can assume $m \le v-6$, because $v-m \not\equiv 2 \pmod 4$ when $m > v-6$. Observe that in this case m is even (since v is even in

Case B and $v - m \equiv 2 \pmod 4$)).
Case 5a: $m \leq v/2$.

Let $Z_2(m)$ be the zigzag path $(v_0, v_1, \dots, v_m)$ define by

$$v_i = \begin{cases} (-1)^{i+1}\lceil (i+1)/2 \rceil \text{ for } 0 \leq i \leq m/2-1, \\ v_{m-(i+1)} + v/2 \text{ for } m/2 \leq i \leq m-1, and \\ v_{m-1}+1 \text{ for } i = m, \end{cases}$$

where each sum is reduced modulo v. Notice that the set of m-paths $Z_2 = \{Z_2(m) - i \mid i \in Z_{v/2}\}$ partitions all the edges of differences in $\{2, 3, \dots, m/2\} \cup \{v/2\}$ and the $v/2$ edges of difference 1 in $I(0, v/2)$.

Let $T = (v_1, v_2, \dots, v_k)$ be the trail formed by the concatenation $I(v/2, 0) + C(v, m/2+1) + C(v, m/2+3) + \cdots + C(v, v/2-2)$. Apply Corollary 5 to T using $x = v/2$ and $i = m/2+1$; notice that in Case 5a, $m \leq v/2$, so $x = v/2 \geq m/2+2 = i+1$ and $m \leq \min\{2i, v+x-2i-2\}$ since $m \geq 4$. So clearly the condition of the Corollary 5 is met. Thus we can conclude that all trails in T/m are $m-$paths. Note that in Case 5, $v - m \equiv 2 \pmod 4$, so $m/2 + 1 \equiv v/2 - 2 \pmod 2$. So by the above explanation T/m is a set of m-paths which partitions all the edges of differences in $\{m/2+1, m/2+2, \dots, v/2-1\}$ and the $v/2$ edges of difference 1 in $I(v/2, 0)$. So $(V, C) = (Z_v, \{Z_2 \cup T/m\})$ is a P_m-decomposition of K_v.

Suppose $S = (W, D)$ is any subsystem in this P_m-decomposition. Let $\pi_j \in D$ be any path of length m. Then either $\pi_j = Z_2(m) - i$ for some $i \in Z_{v/2}$ or $\pi_j \in T/m$. We will show that $W = V$ in both these cases.
Case 5a(i): Suppose that $\pi_j = Z_2(m) - i$ for some $i \in Z_{v/2}$.

Each path $Z_2(m) - i$ contains the edge $\{k, k+v/2\}$ of half difference for some k.

Suppose $m \geq 6$. Then $Z_2(m) - i$ contains both $k+1$ and $k+v/2+1$ (if $m \equiv 2 \pmod 4$) or both $k-1$ and $k+v/2-1$ (if $m \equiv 0 \pmod 4$)). So W must contain one pair of vertices in the next half difference, which implies that either $Z_2(m) - (i-1)$ or $Z_2(m) - (i+1) \in D$. By repeating this argument we can conclude that $V(Z_2) = W = V$.

Suppose $m = 4$. Then $Z_2(m) - i \to \{k-2, k+v/2-2\} \to Z_2(m) - i - 2$. So recursively it follows that $X = \{k-2i, k+v/2-2i \mid i \in Z_{v/2}\} \subseteq W$. Since $v - m \equiv 2 \pmod 4$ and $m = 4$ it follows that $v/2$ is odd so k and $k + v/2$ have different parity. So $X = V$; so $W = V$.
Case 5a(ii): Suppose that $\pi_j \in T/m \in D$.

Since $m \geq 4$, every $\pi_j \in T/m$ contains a pair of vertices $\{k, k+2\}$ for some k; since $\{k, k+2\} \in E(Z_2(m) - i)$, it follows that $Z_2(m) - i \in D$ for some i. So $\pi_j \to Z(m) - i$ for some i, so the result follows by Case 5a(i).
Case 5b: $v/2 < m < 2v/3$.

First observe that in this case $m \geq 6$, since when $m = 4$ there is no even v which satisfies $v/2 < 4 < 2v/3$. Recall that $S(T, a, b)$ was defined to be a subtrail of T from a to b.

Let $D_1 = I(m,0) + S(C(v, m/2+1), 0, m - v/2)$; it is easy to check D_1 is a path of length m. Denote by T_l, the final segment from $m - v/2$ to 0 remaining of $C(v, m/2+1)$; then note that $|E(T_l)| = 3v - 2m > m$. Let $T = (v_1, v_2, \ldots, v_k)$ be the trail formed by the concatenation $I(0, m - v/2) + T_l + C(v, m/2+3) + \cdots + C(v, v/2-2)$. Note that in Case 5 $v - m \equiv 2 \pmod 4$, so $m/2 + 3 \equiv v/2 - 2 (\mathrm{mod}\ 2)$, so T has all the edges of differences $m/2+1, \ldots, v/2-1$, and $v/2$ edges of difference 1 from the vertex m forward (through 0) to $m - v/2$. We now show that trails in T/m are paths by showing that if T contains consecutive vertices that form a cycle C then it has length more than m; so let C be a cycle formed by the consecutive vertices in T. Since $|E(T_l)| > m$, we need only consider 2 cases.

(i) Suppose C consists only of edges in $I(0, m - v/2) + T_l$. If C is in T_l then since T_l is a subgraph of $C(v, m/2+1)$ we can use Lemma 3 to conclude that the length of C is greater than m. If C contains edges from the path $I(0, m - v/2)$ then note that the first vertex to be repeated in T is either $(m - v/2) + m/2 + 2$ or 0. The number of edges between first two appearances of $3m/2 - v/2 + 2$ in T is $m+3 > m$; and the number of edges between first two appearances of 0 in T is $(2v-m-3)-(2m-v)+(m-v/2) = 5v/2-2m-3 > m$ since $m < 2v/3$ and $v > 8$. So the length of C is greater than m.

(ii) If C is in $T_l + C(v, m/2+3) + \cdots + C(v, v/2-2)$ then C is in $C(v, m/2+1) + \cdots + C(v, v/2-2)$. So we can use Lemma 3 to conclude that the length of C is greater than m.

Therefore, by the above observations, it follows that, $D_1 \cup T/m$ is a set of m-paths which partitions all the edges of differences in $\{m/2+1, m/2+2, \ldots, v/2-1\}$ and the $v/2$ edges of difference 1 from the vertex m forward (through 0) to $m - v/2$.

Let $Z_3(m-1)$ be the zigzag path $(v_0, v_1, \ldots, v_{m-1})$ of length $m-1$ defined by

$$v_i = \begin{cases} (-1)^{i+1}\lceil (i+1)/2 \rceil \text{ for } 0 \le i \le m/2-1, \text{and} \\ v_{m-(i+1)} + v/2 \text{ for } m/2 \le i \le m-1. \end{cases}$$

Observe that the paths in $D_1 \cup T/m$ include $v/2$ edges of difference 1, one in $L(x) = \{(x, x+1), (x+v/2, x+v/2+1)\}$ for each $x \in Z_{v/2}$. Thus the set L of the remaining $v/2$ edges of difference 1 also has exactly one edge in $L(x)$ for each $x \in Z_{v/2}$; so to each path in $\{Z_3(m-1)+i \mid i \in Z_{v/2}\}$ we can add one edge from L to form the set M of $v/2$ simple m-paths. Notice that the set of m-paths M partitions all the edges of differences in $\{2, 3, \ldots, m/2\} \cup \{v/2\}$ and the remaining $v/2$ edges in $I(m - v/2, m)$ of difference 1. So $(V, C) = (Z_v, \{M \cup D_1 \cup T/m\})$ is a P_m-decomposition of K_v.

Suppose $S = (W, D)$ is any subsystem in this P_m-decomposition. Let $\pi_j \in D$ be any path of length m. Now we consider various possibilities for π_j, arriving at the contradiction $W = V$ in each case.

Suppose that $\pi_j \in M$. Each path $\pi_j \in M$ contains the edge $\{k, k+v/2\}$ of half difference for some k. Since $m \geq 6$, π_j also contains both $k+1$ and $k+v/2+1$ (if $m \equiv 2 \pmod 4$) or both $k-1$ and $k+v/2-1$ (if $m \equiv 0 \pmod 4$). So W must contain one pair of vertices in the next half difference. By repeating the argument we can conclude that $V(M) = W = V$.

Suppose that $\pi_j \in D_1 \cup T/m$. Since $m \geq 6$, every $\pi_j \in D_1 \cup T/m$ contains a pair of vertices $\{k, k+2\}$ for some k, which implies that D contains the path $\pi_i \in M$ which contains the edge $\{k, k+2\}$. Then the result follows by the previous argument.

Case 6: $m \leq v-4$ and $v - m \equiv 0 \pmod 4$.

We will solve this case by considering three subcases in turn. Observe that in this case m is even (since v is even in Case B and $v - m \equiv 0 \pmod 4$).

Case 6a: $m < v/2$.

Let $Z_4(m)$ be the tailed zigzag path $(v_0, v_1, \ldots, v_m)$ defined by

$$v_i = \begin{cases} (-1)^i \lceil (i+2)/2 \rceil \text{ for } 0 \leq i \leq m/2-1, \\ v_{m-(i+1)} + v/2 \text{ for } m/2 \leq i \leq m-1, \text{and} \\ v_{m-1} - 1 = v/2 \text{ for } i = m, \end{cases}$$

where each sum is reduced modulo v. Notice that the set of m-paths $Z_4 = \{Z_4(m) - i \mid i \in Z_{v/2}\}$ partitions all the edges of differences in $\{3, 4, \ldots, m/2+1\} \cup \{v/2\}$ and the $v/2$ edges of difference 1 in $I(1, v/2+1)$.

Let $C_2(x) = (x, x+2, x+4, \ldots, x)$ be the trail (it is a cycle) of length $v/2$. Let $T = (v_1, v_2, \ldots, v_k)$ be the trail (in fact, a cycle) formed by the concatenation $C_2(v/2+1) + \{v/2+1, v/2+2\} + C_2(v/2+2) + I(v/2+2, 1) + \{C(v, m/2+2)+1\} + \cdots + \{C(v, v/2-2)+1\}$; note that in Case 6, $v - m \equiv 0 \pmod 4$, so $m/2+2 \equiv v/2-2 \pmod 2$. So T has all the edges of difference 2, of differences $m/2+2, \ldots, v/2-1$, and the edges in $I(v/2+1, 1)$. If T contains consecutive vertices that form a cycle C then we now show that the length of C is more than m by considering the following 3 cases.

(i) Suppose C consists only of edges in $C_2(v/2+1) + \{v/2+1, v/2+2\} + C_2(v/2+2) + I(v/2+2, 1)$. Observe that the least number of edges between two appearances of any vertex in $C_2(v/2+1) + \{v/2+1, v/2+2\} + C_2(v/2+2) + I(v/2+2, 1)$ is clearly at least $v/2$. Since $m < v/2$, it follows that the length of C is greater than m.

(ii) Suppose C is in $I(v/2+2, 1) + \{C(v, m/2+2)+1\}$. Since $C(v, m/2+2)+1$ is isomorphic to $C(v, m/2+2)$, we can apply Lemma 3 to conclude that any cycle consisting only of edges in $C(v, m/2+2)+1$ has length $m+6$. If C contains edges from the path $I(v/2+2, 1)$ then since the second vertex in $C(v, m/2+2)+1$ is less than $v/2+1$ (all differences in T are at most $v/2$) which is not in $I(v/2+2, 1)$, it follows that the length of C is greater than $v/2 > m$.

(iii) Suppose C is in $\{C(v, m/2+2)+1\} + \cdots + C\{(v, v/2-2)+1\}$. Then observe that $C(v, m/2+j)+1$ is isomorphic to $C(v, m/2+j)$ for all j, so we can use Lemma 3 to conclude that the length of C is greater than m.

Therefore, by the above observations, T/m partitions into paths of length m all the edges of differences in $\{m/2+2, \ldots, v/2-1\} \cup \{2\}$ and the $v/2$ edges of difference 1 in $I(v/2+1, 1)$. So $(V, C) = (Z_v, \{Z_4 \cup T/m\})$ is a P_m-decomposition of K_v.

Suppose $S = (W, D)$ is any subsystem in this P_m-decomposition. Let $\pi_j \in D$ be any path of length m. Then either $\pi_j = Z_4(m) - i$ for some $i \in Z_{v/2}$ or $\pi_j \in T/m$. We will show that $W = V$ in both these cases.

Case 6a(i): Suppose that $\pi_j = Z_4(m) - i$ for some $i \in Z_{v/2}$.

Each path $Z_4(m) - i$ contains the edge $\{k, k+v/2\}$ of half difference for some k.

Suppose $m \geq 6$. Then $Z_4(m) - i$ contains both $k+1$ and $k+v/2+1$ (if $m \equiv 0 \pmod 4$) or both $k-1$ and $k+v/2-1$ (if $m \equiv 2 \pmod 4$). So W must contain one pair of vertices in the next half difference which implies that either $Z_4(m) - (i+1)$ or $Z_4(m) - (i-1) \in D$. By repeating this argument we can conclude that $V(Z_4) = W = V$.

Suppose $m = 4$ (so, being Case 6, $v \equiv m \equiv 0 \pmod 4$). Then $Z_4(m) - i \rightarrow \{k+3, k+v/2+3\} \rightarrow Z_4(m) - i + 3$. So recursively it follows that $X = \{k+3i, k+v/2+3i \mid i \in Z_{v/2}\} \subseteq W$. So $X = V = W$ unless $v \equiv 0 \pmod 3$.

Hence the only case that remains to be solved is when $m = 4$ and $v \equiv 0 \pmod 3$ (so actually $v \equiv 0 \pmod{12}$, since in this case $v \equiv 0 \pmod 4$ as well). So finally suppose that $v \equiv 0 \pmod{12}$ and $m = 4$. Notice that in this exceptional case,

$$\text{for any } x \in V, \{x, x+v/2\} \rightarrow \{x+3, x+3+v/2\}. \tag{8}$$

So if $\{x, x+v/2\} \subseteq W$ then we can recursively apply Equation 8 to $\{x, x+v/2\}$ to see that $\{y \in V \mid y \equiv x \pmod 3\} \subseteq W$. In particular, since $\{k, k+v/2\} \subseteq W$, it follows that $A = \{a \in V \mid a \equiv k \pmod 3\} \subseteq W$. But since each path containing an edge of half difference joining vertices in A also contains an edge of difference 1, we in fact know that $A' = \{b \in V \mid b \equiv k-1 \pmod 3, 1 \leq b \leq v/2\} \subseteq W$. Then observe that if $b \geq 4$ and $b \in A'$ then $\{b, b-3\} \rightarrow \{b-3, b-3+v/2\}$. So by applying Equation (8) recursively to $\{b-3, b-3+v/2\}$, where $b \in A'$ and $b \geq 4$ we will get that $B = \{b \in V \mid b \equiv k-1 \pmod 3\} \subseteq W$. So $\{k, k+v/2\} \rightarrow \{a \mid a \equiv$

k *or* $k-1 \pmod 3\}$. In particular $\{k-1, k+v/2-1\} \subseteq W$, so similarly $\{k-1, k+v/2-1\} \rightarrow \{a \mid a \equiv k-1$ *or* $k-2 \pmod 3\}$. Hence $W = V$ in this case.

Case 6a(ii): Suppose that $\pi_j \in T/m$.

Now we consider various possibilities for π_j. If π_j contains two vertices that are joined by an edge that occurs in $Z_4(m) - i$ for some i, then the result follows from Case 6a(i). Notice that if $m \geq 4$ then each edge of difference 3 occurs in $Z_4(m) - i$ for some i, and if $m \geq 6$ then each edge of difference 4 occurs in $Z_4(m) - i$ for some i.

Suppose $m \geq 6$. Every $\pi_j \in T/m$ contains the vertices in $\{x, x+3\}$ or $\{x, x+4\}$ for some x. So every subsystem S containing π_j contains an edge of difference 3 or 4; so S contains an edge in $Z_4(m) - i$ for some i.

Suppose $m = 4$. Then one of the following occurs.

(a) $\pi_j = (\ldots, k, k+2, k+4, \ldots)$. In this case the edge $\{k, k+4\}$ is in a path that must contain either $k-1$ or $k+5$. So S contains an edge of difference 3 (either $\{k-1, k+2\}$ or $\{k+2, k+5\}$), so S contains $Z_4(m) - i$ for some i.

(b) π_j contains 3 consecutive edges in $I(v/2+2, 1)$. In this case π_j contains a pair of vertices distance 3 apart, so S contains an edge of difference 3. So S contains $Z_4(m) - i$ for some i.

(c) π_j contains edges in $I(v/2+2, 1) + \{C(v, m/2+2) + 1\}$. In view of the Case(b) we can assume that π_j contains at most 2 edges from $I(v/2+2, 1)$. So S contains an edge of difference 2 or 3. If S contains an edge of difference 2 then S contains a path that was just considered in Case(a). If S contains an edge of difference 3, then S contains $Z_4(m) - i$ for some i.

(d) $\pi_j = (k, k+l, k+1, k+l+1, k+2)$. In this case the edge $\{k, k+2\}$ is in a path that was just considered in Case(a). So S contains $Z_4(m) - i$ for some i.

Case 6b: $v/2 \leq m < 3v/4$, and $m \leq v - 8$.

First define the sub zigzag path $(v'_0, v'_1, \ldots, v'_{m/2-1})$ of length $m/2 - 1$ by

$$v'_i = (-1)^{i+1}\lceil (i+1)/2 \rceil \text{ for } 0 \leq i \leq m/2 - 1.$$

Then let $Z_5(m)$ be the zigzag path $(v_0, v_1, \ldots, v_m)$ defined by

$$v_i = \begin{cases} v'_{(m/2-1)-i} \text{ for } 0 \leq i \leq m/2-1, \\ v_{m/2-1} + 1 \text{ for } i = m/2, \mathit{and} \\ v_{m-i} + v/2 \text{ for } m/2+1 \leq i \leq m, \end{cases}$$

where each sum is reduced modulo v. Notice that the set of m-paths $Z_5 = \{Z_5(m) - i \mid i \in Z_{v/2}\}$ partitions all the edges of differences in

$\{2, 3, \ldots, m/2\}$, $v/2$ edges of difference $v/2 - 1$, and the $v/2$ edges of difference 1 in $I(v/2, 0)$.

Let A be the trail defined by $(0, v/2, 1, v/2+1, 2, \ldots, v-1, v/2)$ of length v which covers the edges of difference $v/2$ and remaining edges of difference $v/2 - 1$. Notice that the only vertex appearing more than once in A is $v/2$ which appears twice.

If $m > v/2$ then let $C = S(A, 0, v - m) + I(v - m, v/2)$. Then C is a trail of length $3v/2 - m > m$. Let T_l be the final segment of A after the subtrail $S(A, 0, v - m)$ has been removed. Observe that the length of T_l is $2m - v$. Clearly $B = I(0, v - m) + T_l$ is an m-path.

Note that if $m = v/2$ then let $B = I(0, v/2)$ and $C = A$.

In either case, let F be the subtrail of C containing the last m edges and let E be the subtrail of C formed by removing F. Then F is an m-path because the only vertex repeated in C is $v/2$ which appears as both the second and the last vertex. Note that

$$E = \begin{cases} S(C, 0, (3v/2 - 2m)/2) \text{ if } 3v/2 - 2m \text{ is even}, \textit{and} \\ S(C, 0, v/2 + \lfloor (3v/2 - 2m)/2 \rfloor) \text{ otherwise.} \end{cases}$$

Let $T = (v_1, v_2, \ldots, v_k)$ be the trail formed by the concatenation $C(v, m/2+1) + C(v, m/2+3) + \cdots + C(v, v/2-3) + E$. Again we show that the trails in T/m are paths by showing that each cycle C in T has length more than m; so let C be a cycle in T.

(i) Suppose C consists only of edges in $C(v, m/2+1) + C(v, m/2+3) + \cdots + C(v, v/2-3)$ then by Lemma 3 we can conclude that the length of C is greater than m.

(ii) C contains edges from both $C(v, v/2 - 3)$ and E. First observe that, since $m \geq \max\{4, v/2\}$ and since $m \leq v - 8$, it follows that $v \geq 16$. Hence $3v/4 - m < v/2 - 3 < v/2 + \lfloor (3v/2 - 2m)/2 \rfloor$. The first vertex to be repeated in $C(v, v/2-3) + E$ is $v/2 - 3$ and the number of edges between its appearances is $v - 5 (> m)$, which implies that the length of C is greater than m.

Therefore, by the above observations, we can conclude that all trails in T/m are paths. Note that in Case 5, $v - m \equiv 2 \pmod 4$, so $m/2 + 3 \equiv v/2 - 3 \pmod 2$. So $T/m \cup B \cup F$ is a set of m-paths which partitions all the edges of differences in $\{m/2+1, m/2+2, \ldots, v/2-2\} \cup \{v/2\}$, the remaining $v/2$ edges of difference $v/2 - 1$, and the $v/2$ edges of difference 1 in $I(0, v/2)$. So $(V, C) = (Z_v, \{Z_5 \cup B \cup F \cup T/m\})$ is a P_m-decomposition of K_v.

Suppose $S = (W, D)$ is any subsystem in this P_m-decomposition. Let $\pi_j \in D$ be any path of length m. We will now consider various possibilities for π_j and show that $W = V$ in each possibility. First observe that since $v \geq 16$, $m \geq v/2$ implies that $m \geq 8$.

Suppose that $\pi_j = Z_5(m) - i$ for some $i \in Z_{v/2}$.

Each $Z_5(m)-i$ contains the edge $\{k, k+(v/2-1)\}$ for some k. Therefore π_j contains both $k-1$ and $k-1+(v/2-1)$. So W must contain the edge $\{k-1, k-1+(v/2-1)\}$, which implies that $Z_5(m) - (i+1) \in D$. By repeating this argument we can conclude that $V(Z_5) = W = V$.

Suppose that $\pi_j \in B \cup F \cup T/m$.

Every $\pi_j \in B \cup F \cup T/m$ contains a pair of vertices $\{k, k+2\}$ for some k, which implies that the edge $\{k, k+2\}$ is in some $Z_5(m) - i$ for some i. Hence the result follows by the previous argument.

Case 6c: $v/2 \leq m \leq 2v/3, m > v - 8$.

Without loss of generality we can assume that $m = v - 4$, because $v - m \equiv 0 \pmod 4$ in Case 6. By Lemma 2 $m \leq 2v/3$, so in this case $v/2 \leq v - 4 \leq 2v/3$, implying $8 \leq v \leq 12$. Since v is even and m has to divide $\binom{v}{2}$, this implies that the only exceptional case that needs to be solved is when $v = 8$ and $m = 4$.

So finally suppose that $v = 8$ and $m = 4$. Then let $(V, C) = (Z_7 \cup \{\infty\}, \{Z_6 + i \mid i \in Z_7\})$ is a P_4- decomposition of K_8 where $Z_6 = (\infty, 0, 6, 1, 5)$. This decomposition contains no subsystems because whenever $\{\infty, x\}$ is in any subsystem, $\{\infty, x\} \to x+1$ for some $x \in Z_7$, which implies that whenever $Z_6 + i$ is in any subsystem $Z_6 + (i+1)$ is also in the same subsystem. By repeating this argument we can conclude that $V(\{Z_6 + i \mid i \in Z_7\}) = V$. □

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