A Note on General Neighbor-Distinguishing Total Coloring of Graphs

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Abstract

The general neighbor-distinguishing total chromatic number $\chi''_{gnd}(G)$ of a graph G is the smallest integer k such that the vertices and edges of G can be colored by k colors so that no adjacent vertices have the same set of colors. It is proved in this note that $\chi''_{gnd}(G) = \lceil \log_2 \chi(G) \rceil + 1$, where $\chi(G)$ is the vertex chromatic number of G.

Key word: General neighbor-distinguishing total coloring; Chromatic number; Graph

1 Introduction

Only simple graphs are considered in this note unless otherwise stated. For a graph G, we denote its vertex set, edge set, and maximum degree by $V(G)$, $E(G)$, and $\Delta(G)$, respectively. A *k-total-coloring* of a graph G is a mapping f from $V(G) \cup E(G)$ to the set $\{1, 2, \cdots, k\}$. Let

$$C_f(v) = \{f(xv) \mid xv \in E(G)\} \cup \{f(v)\}$$

denote the set of colors assigned to a vertex v and the edges incident to v, $C_f(v)$ is called the color set of vertex v. A k-total-coloring f of G is *general neighbor-distinguishing*, or a *k-gndt-coloring*, if $C_f(u) \neq C_f(v)$ whenever $uv \in E(G)$. The *general neighbor-distinguishing total chromatic number*,

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denoted by $\chi''_{gnd}(G)$, is the smallest integer k such that G has a k-gndt-coloring.

The general neighbor-distinguishing total chromatic number is related to two known graph invariants. The one is *general neighbor-distinguishing chromatic index*. A *k-edge-coloring* of a graph G is a mapping ϕ from $E(G)$ to the color set $\{1, 2, \cdots, k\}$. Let $C'_\phi(v) = \{\phi(xv) \mid xv \in E(G)\}$ denote the set of colors assigned to the edges incident to v. A k-edge-coloring ϕ of G is *general neighbor-distinguishing*, or a *k-gnd-coloring*, if $C'_\phi(u) \neq C'_\phi(v)$ whenever $uv \in E(G)$. The general neighbor-distinguishing chromatic index, denoted by $\text{gndi}(G)$, is the smallest integer k such that G has a k-gnd-coloring. This concept was introduced by Győri et al. [3]. They characterized the general neighbor-distinguishing chromatic index for bipartite graph and computed $\text{gndi}(K_n) = \lceil \log_2 n \rceil + 1$ for any $n \geq 3$, where $\chi(G)$ denotes the ordinary (vertex) chromatic number of G. Furthermore, they proved that $\text{gndi}(G) \leq 2\lceil \log_2 \chi(G) \rceil + 1$ for any graph G without isolated edges. If $\chi(G) \geq 3$, Horňák and Soták [5] improved the above bound by showing that $\lceil \log_2 \chi(G) \rceil + 1 \leq \text{gndi}(G) \leq \lfloor \log_2 \chi(G) \rfloor + 2$. More recently, it was shown in [4] that $\text{gndi}(G) = \lceil \log_2 \chi(G) \rceil + 1$ for any connected graph G with $\chi(G) \geq 3$.

Proposition 1 *For any graph G without isolated edges, we have* $\chi''_{gnd}(G) \leq \text{gndi}(G)$.

Proof. Let ϕ be a k-gnd-coloring of G using the colors $1, 2, \cdots, k$. Based on ϕ, we can define a k-total-coloring f as follows.

(1) For each edge $e \in E(G)$, let $f(e) = \phi(e)$.

(2) For each isolated vertex $v \in V(G)$, let $f(v) = 1$.

(3) For each vertex $v \in V(G)$ that is not isolated, let $f(v)$ be any color in $C'_\phi(v)$.

Let uv be an arbitrary edge of G. By the definition of ϕ, we conclude that $C'_\phi(u) \neq C'_\phi(v)$. Thus, $C_f(u) = C'_\phi(u) \neq C'_\phi(v) = C_f(v)$. This shows that f is a k-gndt-coloring of G. Therefore, $\chi''_{gnd}(G) \leq \text{gndi}(G)$. □

The other is *adjacent vertex distinguishing total chromatic number*. A k-total-coloring of a graph is *proper* if any two adjacent or incident elements in $V(G) \cup E(G)$ receive different colors. A proper k-total-coloring f of G is *adjacent vertex distinguishing* if $C_f(u) \neq C_f(v)$ whenever $uv \in E(G)$. The adjacent vertex distinguishing total chromatic number, denoted by $\chi''_a(G)$, is the smallest integer k such that G has an adjacent vertex distinguishing k-total-coloring.

In [8], Zhang et al. first introduced this concept and conjectured that $\chi''_a(G) \leq \Delta(G) + 3$ for any connected graph G with at least two vertices.

Note that $\chi''_a(K_{2n+1}) = \Delta(K_{2n+1}) + 3 = 2n + 3$ for any $n \geq 1$. This example shows that the upper bound $\Delta(G) + 3$ of $\chi''_a(G)$ is tight if their conjecture is true. In [8], $\chi''_a(G)$ is determined if G is a path, a cycle, a fan, a wheel, a tree, a complete graph, and a complete bipartite graph. Chen [2] and Wang [6], independently, confirmed that $\chi''_a(G) \leq 5$ for a graph G with $\Delta(G) \leq 3$. In [7], outerplanar graphs were completely characterized using the adjacent vertex distinguishing total chromatic number.

Proposition 1 and the result of [4] imply immediately that $\chi''_{gnd}(G) \leq \lceil \log_2 \chi(G) \rceil + 1$ for any graph G. In this note, we will establish a similar and neat expression $\chi''_{gnd}(G) = \lceil \log_2 \chi(G) \rceil + 1$, independent of the result in [4].

2 Main Result

For integers p, q with $q > p$, we use $[p, q]$ to denote the integer interval bounded by p and q, i.e., $[p, q] = \{p, p+1, \cdots, q-1, q\}$.

Let G be a connected graph with $\chi(G) = k \geq 3$. Clearly, a proper (vertex) k-coloring of G admits a k-partition $(V_1, V_2, \cdots, V_k)$ of $V(G)$ such that $G[V_i]$, the subgraph of G induced on V_i, is edgeless. Let $\Lambda_k(G)$ denote the set of all such k-partitions $(V_1, V_2, \cdots, V_k)$ of $V(G)$. Given $\lambda = (V_1, V_2, \cdots, V_k) \in \Lambda_k(G)$ and $i, j \in \{1, 2, \cdots, k\}$, let $E_{i,j}(\lambda)$ denote the set of edges of G joining a vertex in V_i to a vertex in V_j, and $e_{i,j}(\lambda) = |E_{i,j}(\lambda)|$. Further, we set $e(\lambda) = (e_1(\lambda), e_2(\lambda), \cdots, e_k(\lambda))$, where

$$e_i(\lambda) = \sum_{j=1,\ j\neq i}^{k} e_{i,j}(\lambda).$$

Suppose that $A = (a_1, a_2, \cdots, a_n)$ and $B = (b_1, b_2, \cdots, b_n)$ are two distinct real sequences with $n \geq 1$. We say that A is greater than B in a lexicographical order if there is an index $1 \leq i \leq n$ such that $a_i > b_i$ and $a_j = b_j$ for all $j = 1, 2, \cdots, i-1$.

Lemma 2 *Let G be a connected graph with $k = \chi(G) \geq 3$. Let $\lambda^* = (V_1^*, V_2^*, \cdots, V_k^*) \in \Lambda_k(G)$ be a lexicographically maximal sequence in $\Lambda_k(G)$ according to $e(\lambda^*) = (e(V_1^*), e(V_2^*), \cdots, e(V_k^*))$. Then for any $i \in [2, k]$, $x \in V_i^*$ and $j \in [1, i-1]$, there exists a vertex $y \in V_j^*$ such that x is joined to y in G.*

Lemma 2 is obviously right.

If G is a disconnected graph with $n \geq 2$ components $G_1, G_2, \cdots, G_n$, then it is straightforward to derive that $\chi''_{gnd}(G) = \max_{1\leq i\leq n} \{\chi''_{gnd}(G_i)\}$. Thus, in what follows, we only consider connected graph.

Theorem 3 *For a connected graph G, $\chi''_{gnd}(G) = \lceil \log_2 \chi(G) \rceil + 1$.*

Proof. Let $k = \chi(G)$. If $k = 1$, then G is K_1 and $\chi''_{gnd}(G) = 1$. If $k = 2$, then G is a bipartite graph with bipartition $V(G) = X \cup Y$. We define a mapping f as follows:

$$f(x) = \begin{cases} 1 & \text{if } t \in X, \text{ or } t \in E(G); \\ 2 & \text{if } t \in Y. \end{cases}$$

It is easy to inspect that f is a 2-gndt-coloring of G, and hence $\chi''_{gnd}(G) \leq 2$. On the other hand, it is evident that $\chi''_{gnd}(G) \geq 2$. Consequently, $\chi''_{gnd}(G) = 2$.

Assume that $k \geq 3$. We first prove that $\chi''_{gnd}(G) \leq \lceil \log_2 k \rceil + 1$. Let $\ell = \lceil \log_2 k \rceil + 1$ and $\mathcal{A} = \{A | 1 \in A \text{ and } A \subseteq [1, \ell]\}$. Then

$$|\mathcal{A}| = 2^{\ell-1} = 2^{\lceil \log_2 k \rceil} \geq 2^{\log_2 k} = k.$$

According to the lexicographical order, we can arrange all the elements of $\mathcal{A}$ as follows:

$$A_1 = \{1\}, A_2 = \{1,2\}, \cdots, A_\ell = \{1,\ell\}, A_{\ell+1} = \{1,2,3\}, \cdots, A_{2^{\ell-1}} = \{1,2,\cdots,\ell\}$$

Let $\lambda^* = (V_1^*, V_2^*, \cdots, V_k^*) \in \Lambda_k(G)$ be a lexicographically maximal sequence in $\Lambda_k(G)$ according to $e(\lambda^*) = (e(V_1^*), e(V_2^*), \cdots, e(V_k^*))$. We define a function f on $V(G) \cup E(G)$ in the following ways.

(1) For each vertex $v \in V_i^*$, let $f(v) = i$ if $1 \leq i \leq \ell$, and $f(v) = 1$ if $\ell + 1 \leq i \leq k$.

(2) For each edge $e \in E_{j,i}(\lambda^*)$ with $1 \leq j < i \leq k$, let $f(e) = j$ if $j \in A_i$, and $f(e) = 1$ if $j \notin A_i$.

We have to prove that f is an ℓ-gndt-coloring of G. Let $v \in V(G)$. So, $v \in V_i^*$ for some $1 \leq i \leq k$. We observe the construction of $C_f(v)$ by considering the following two possibilities:

Case 1. $1 \leq i \leq \ell$.

By (1), $f(v) = i$. If $i = 1$, it is easy to see that $C_f(v) = A_1 = \{1\}$. Suppose that $i > 1$. By Lemma 2, there exists $u \in V_1^*$ such that $uv \in E(G)$, hence $f(uv) = 1$ by (2). This implies that $A_i = \{1, i\} \subseteq C_f(v)$. Let e be any edge incident to v. Then either $f(e) = 1$ or $f(e) = i$ by (2), implying that $C_f(v) \subseteq A_i = \{1, i\}$. Therefore, $C_f(v) = A_i$.

Case 2. $\ell + 1 \leq i \leq k$.

We see that $f(v) = 1$ by (1). For any $j \in A_i \subseteq [1, \ell]$, there exists $u \in V_j^*$ such that $uv \in E(G)$ by Lemma 2, so $f(uv) = j$ by (2). It follows that $j \in C_f(v)$ and henceforth $A_i \subseteq C_f(v)$. Conversely, let e be any edge incident to v. Then either $f(e) = 1$ or $f(e) \in A_i$ by (2), i.e., $C_f(v) \subseteq A_i$. Therefore, $C_f(v) = A_i$.

Since $A_i \neq A_j$ for any $i \neq j$ and $C_f(v) = A_i$ for $v \in V_i^*$ with $1 \leq i \leq k$, any two adjacent vertices u and v have distinct color sets. So, f is an ℓ-gndt-coloring of G. It turns out that $\chi''_{gnd}(G) \leq \ell$.

Second, we need to prove that $\chi''_{gnd}(G) \geq \lceil \log_2 k \rceil + 1$. Suppose that $\chi''_{gnd}(G) = t$. Let f be an t-gndt-coloring of G using colors $1, 2, \ldots, t$, and let $T = \{1, 2, \cdots, t\}$. For any edge $e = uv \in E(G)$, we have that $C_f(u) \neq C_f(v)$ by definition, and $C_f(u) \cap C_f(v) \neq \emptyset$ as $f(e) \in C_f(u) \cap C_f(v)$. Let $\mathcal{A} = \{A \subseteq T | 1 \in A\}$. Then $|\mathcal{A}| = 2^{t-1}$. We write that $\mathcal{A} = \{A_1, A_2, \cdots, A_{2^{t-1}}\}$. Since f is an t-gndt-coloring of G, for any vertex $v \in V(G)$, we have $C_f(v) \subseteq T$. Thus, $C_f(v) \in \mathcal{A}$ or $T - C_f(v) \in \mathcal{A}$, by the definition of $\mathcal{A}$.

Based on f, we define a 2^{t-1}-vertex coloring π of G using the colors $c_1, c_2, \cdots, c_{2^{t-1}}$. For a vertex $v \in V(G)$, we set $\pi(v) = c_i$, where c_i can be chosen to satisfy, by the definition of f, that $1 \leq i \leq 2^{t-1}$, and $C_f(v) = A_i$ or $T - C_f(v) = A_i$. In order to show that π is a proper vertex coloring of G, we assume to the contrary that there exist two adjacent vertices v and u such that $\pi(v) = \pi(u) = c_j$. By the definition of c_j, there exists A_j, $1 \leq j \leq 2^{t-1}$, such that $C_f(v) = A_j$ or $T - C_f(v) = A_j$, and $C_f(u) = A_j$ or $T - C_f(u) = A_j$. By the definition of f, $C_f(u) \neq C_f(v)$. So, we derive that $C_f(u) = T - C_f(v)$. This means that $C_f(u) \cap C_f(v) = \emptyset$, which is impossible because $f(vu) \in C_f(u) \cap C_f(v)$. Thus, π is a proper vertex coloring of G, and hence $\chi(G) \leq 2^{t-1}$. It follows easily that $t \geq \lceil \log_2 \chi(G) \rceil + 1 = \lceil \log_2 k \rceil + 1$. Consequently, $\chi''_{gnd}(G) = \lceil \log_2 k \rceil + 1$. □

Using Theorem 3 and the result of [3], we obtain:

Corollary 4 *For any connected graph G without isolated edges, we have $\chi''_{gnd}(G) = \text{gndi}(G)$.*

The Four-Color Theorem [1] says that every planar graph is 4-colorable. This fact together with Theorem 3 establish the following.

Corollary 5 *If G is a planar graph, then $\chi''_{gnd}(G) \leq 3$.*

Since $\chi(K_n) = n$, the following result follows immediately from Theorem 3.

Corollary 6 $\chi''_{gnd}(K_n) = \lceil \log_2 n \rceil + 1$.

Corollary 7 *Let G be a connected graph with at least two vertices. Then $\chi''_{gnd}(G) = 2$ if and only if G is bipartite.*

The well-known Brooks' Theorem asserts that $\chi(G) \leq \Delta(G)$ if G is neither a complete graph nor an odd cycle. Using this fact and Theorem 3, we obtained the following corollary.

Corollary 8 *If connected graph G is neither complete nor an odd cycle, then* $\chi''_{gnd}(G) \leq \lceil \log_2 \Delta(G) \rceil + 1$.

Since χ is a monotone graph parameter, χ''_{gnd} is also monotone by Theorem 3.

Corollary 9 *If H is a subgraph of G, then* $\chi''_{gnd}(H) \leq \chi''_{gnd}(G)$.

References

[1] K. Appel and W. Haken, Every planar graph map is four colorable, Bull. Amer. Math. Soc. **82** (1976), 711-712.

[2] X. Chen, On the adjacent vertex distinguishing total coloring numbers of graphs with $\Delta = 3$, Discrete Math. **308** (2008), 4003-4007.

[3] E. Győri, M. Horňák, C. Palmer and M. Woźniak, General neighbor-distinguishing index of a graph, Discrete Math. **308** (2008), 827-831.

[4] E. Győri and C. Palmer, A new type of edge-derived vertex coloring, Discrete Math., **309** (2009), 6344-6352.

[5] M. Horňák and R. Soták, General neighbor-distinguishing index via chromatic number, Discrete Math., **310** (2010), 1733-1736.

[6] H. Wang, On the adjacent vertex distinguishing total chromatic number of the graphs with $\Delta(G) = 3$, J. Comb. Optim. **14** (2007), 87-109.

[7] Y. Wang and W. Wang, Adjacent vertex distinguishing total colorings of outerplanar graphs, J. Comb. Optim. **19** (2010), 123-133.

[8] Z. Zhang, X. Chen, J. Li, B. Yao, X. Lu and J. Wang, On adjacent-vertex-distinguishing total coloring of graphs, Sci. China Ser. A **48** (2005), 289-299.