On Constructions which yield Fully Magic Graphs

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Dedicated to Professor A. Rosa

ABSTRACT. For any abelian group A, we denote **A*=A-{0}.** Any mapping $l: E(G) \rightarrow A^*$ is called a labeling. Given a labeling on the edge set of G we can induce a vertex set labeling $l^+: V(G) \rightarrow A$ as follows:

$$l^+(v) = \Sigma \{l(u,v): (u,v) \in E(G)\}.$$

A graph G is known as **A-magic** if there is a labeling $l: E(G) \rightarrow A^*$ such that for each vertex v, the sum of the labels of the edges incident to v are all equal to the same constant; i.e., $l^+(v) = c$ for some fixed c in A. We will call $<G,l>$ an A-magic graph with sum c.

We call a graph G **fully magic** if it is A-magic for all non-trivial abelian groups A. Low and Lee showed in [11] if G is an eulerian graph of even size, then G is fully magic. We consider several constructions that produce infinite families of fully magic graphs. We show here every graph is an induced subgraph of a fully magic graph.

Keywords: magic labeling, A-magic, fully magic, eulerian, edge expansion, generalized L-product, chain-sum, two point join.

Mathematic, Subject Classification (2000). 05C78, 05C45

1. Introduction.

For any abelian group A, we denote A* =A-{0}. Any mapping 1: E(G) →A* is called a labeling. Given a labeling on edge set of G we can induce a vertex set labeling 1^+: V(G) →A as follows:

$$1^+(v) = \Sigma\,\{1(u,v): (u,v) \in E(G)\}.$$

__Definition 1__. A graph G is called __A-magic__ if there is a labeling 1: E(G) →A* such that for each vertex v, the sum of the labels of the edges incident to v are all equal to the same constant; i.e., $1^+(v) = c$ for some fixed c in A. We will call <G,1> an A-magic graph with sum c.

The concept of an A-magic graph was introduced in [8]. For other properties of A·magic graphs, the reader can refer to [5,8,9,10, 11,15,16].

__Definition 2__. We call a graph G **fully magic** if it is A-magic for all non-trivial abelian groups A.

__Example 1__. The following bipartite graph G is fully magic. We see that if x is a non-zero element for any non-trivial abelian group A, then G admits an A-magic labeling with sum 0 (Figure 1).

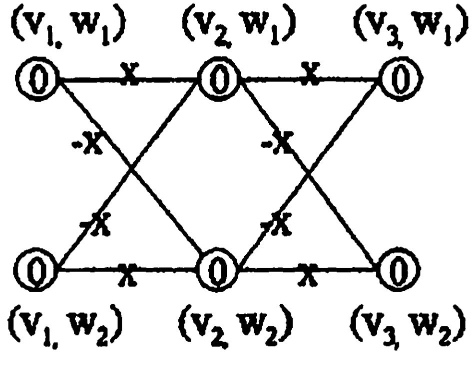


Figure 1.

When $A = Z$, the Z-magic graphs had been considered in [3, 18, 19, 20, 21]. Stanley pointed out that the theory of magic labelings can be put into the more general context of linear homogeneous diophantine equations [18,19].

When the group is Z_k, we shall refer to the Z_k -magic graph as *k-magic*. Graphs which are k-magic had been studied in [4,5,7,8,12].

The original concept of A-magic graph is due to J. Sedlacek [12,13], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices. Magic graphs of this type had been characterized by S. Jerzy and M. Trenkler in [2, 22].

In [8], we show that

Theorem 1.1. The graph G is 2-magic if and only if all vertices are odd degree or even degree.

An eulerian graph is a connected graph consisting of a single closed walk such that each edge is traversed once. It is well-known that a graph is eulerian if and only if all the degrees of its vertices are even. Low and Lee showed in [11] the following result.

Theorem 1.2. If G is an eulerian graph of even size, then G is fully magic.

The following result is obvious.

Theorem 1.3. Every eulerian graph is 2k-magic for all $k \geq 1$ and A-magic where A is any elementary 2-group.

However, eulerian graphs of odd size, need not be fully magic.

Example 2. The following eulerian graph is not 3-magic.

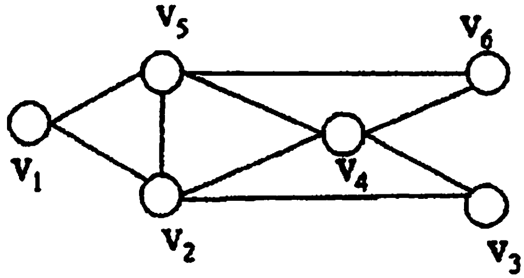


Figure 2.

Lee and Wen [11] provide eight constructions of infinite families of eulerian graphs of odd sizes which are fully magic.

The aim of this paper is to provide some constructions which will yield fully magic graphs. In section 2, the generalized product of graphs is introduced. The merit of this construction is that we can show that any graph is an induced subgraph of a fully magic graph. In section 3, the generalized L-product construction is extended to partially generalized L-product. Several infinite families of eulerian graphs of odd sizes by partially generalized L-product are presented. In section 4, chain sum and edge expansion of graphs are considered. In section 5, another construction called two-point join is also considered.

2. Fully magic graphs which are generalized L-products of graphs.

In this paper, we will use the following notations:

Gph = the class of all connected graphs.

Euler(even) = the class of all eulerian graphs of even sizes. Similarly,

Euler (odd) = the class of all eulerian graphs of odd sizes.

REG(k) = the class of all k-regular graphs.

Let Gph * be the class of pairs (H,s) where H is a connected graph and s is a distinguished vertex of H.

For any graph G and a mapping Φ: V(G)→Gph* we construct a new graph G $\times_L$ Φ as follows:

We form the disjoint union of G and $\{\Phi(v) = (H,s) : v \in V(G)\}$ and identify v with s for each v in V(G).

The resulting graph is called the **generalized L-product** of G and Φ.

If Φ is the constant map $\Phi(v) = (H,s)$ for all $v \in V(G)$, then we denote G $\times_L$ Φ simply by G $\times_L$ (H,s) and call the resulting graph the **L-product** of G and (H,s).

For any $n \geq 1$, we denote the tree with n+1 vertices of diameter two by St(n). The star has a center c and n append edges to c.

__Theorem 2.1__. For any G in Gph and Φ :V(G) → Gph*, where $\Phi(v)$ = (St(deg(v)+1,c) for each v in V(G), the generalized L-product G $\times_L$ Φ of G and Φ is fully magic.

Proof. Let x be a non-zero element of a non-trivial abelian group A. We label all the edges of G $\times_L$ Φ except edges in E(G) by x and all the edges in E(G) by -x . For each v in V(G) if degree v is n then v has an extra n+1 appended edges, then it is clear that the vertex label of v is $-nx+(n+1)x = x$. Thus the labeling is A-magic with sum x (see Figure 3).

We have the following result immediately.

__Corollary 2.2__. Every graph G is an induced subgraph of a fully magic graph.

__Example 3__. Figure 3 shows a (8,11)-graph which is non magic. We see that $deg(v_7)=1$, $deg(v_0)=deg(v_3)=deg(v_6)=2$, $deg(v_1)=3$, $deg(v_2)=deg(v_4)=deg(v_5)=4$. However, its generalized L-product G $\times_L$ Φ is fully magic.

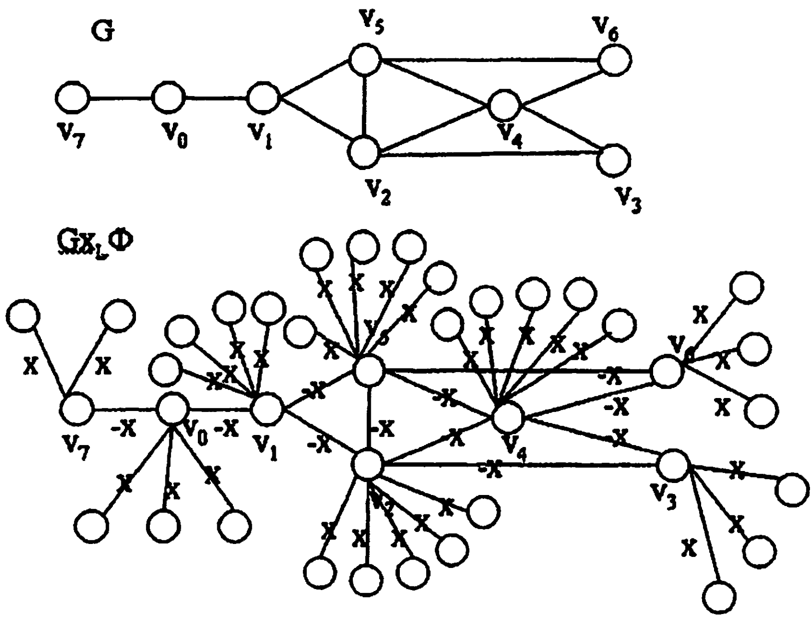


Figure 3.

For k≥0, we denote the class of all fully magic graphs G such that for each non-trivial abelian group A and any x in A*, there exists a A-magic labeling l:E(G) →A* where l(e) is x or -x, and $l^+(v) = kx$ for all v in V(G) by **Ω(k).**

Observation 1. The minimal graph in Ω (1) is P_2.

Observation 2. **Euler**(even) ⊆ Ω(0).

Observation 3. The cycle C_{2n+1} is in Ω(2) but not in Ω(1) and Ω(0).

We have the following general result.

Theorem 2.3. For any G in **Euler**(even), and Φ : V(G)→Gph*, the generalized L-product G x $_L$ Φ is in Ω(k) if Φ(v) is in Ω(k) for each v in V(G).

Proof. Let x be a non-zero element of a non-trivial abelian group A. If G is eulerian of even size then we have an eulerian trail from any vertex v back to itself. We label the edges of the trail by x, -x alternately. We see that each vertex in G has a zero sum. Now as $\Phi(v)$ is in $\Omega(k)$, it admits an A-magic labeling with sum kx. Hence the generalized L-product $G \times_L \Phi$ is in $\Omega(k)$.

Corollary 2.4. For any G in **Euler**(even), the L-product $G \times_L P_2$ is in $\Omega(1)$.

Remark. The above result is not true if we replace **Euler**(even) by **Euler**(odd). Consider C_3 which is clearly eulerian of size 3. We see that the L-product $C_3 \times_L P_2$ (Figure 4) is not 2k+1-magic for all $k \geq 1$. For if $C_3 \times_L P_2$ is A-magic for some A, then we would have x+y+y = x which implies 2y =0.

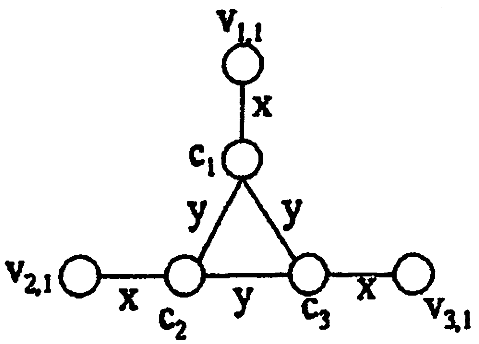


Figure 4.

In fact, Theorem 2.1 can be generalized as follows

Theorem 2.5. If G is a graph in $\Omega(0)$, then the generalized L-product $G \times_L \Phi$ is in $\Omega(k)$ if $\Phi(v)$ is in $\Omega(k)$ for each v in V(G).

3. Fully magic graphs in $\Omega(1)$.

Theorem 2.1 shows that every graph is an induced subgraph of a $\Omega(1)$ fully magic graph. In this section, we will construct several new families of fully magic graphs in $\Omega(1)$. It is obvious that REG(3) $\in \Omega(3)$. However, we have

Theorem 3.1. If H is a graph in REG(3) with a perfect matching T, then H is in $\Omega(1)$.

Proof. Let x be a non-zero element of a non-trivial abelian group A. We label all the edges of E(H) except T by x and all the edges in T by -x . For each v in V(G) , it has one edge in T, then it is clear that the vertex label of v is -x+2x =x. Thus the labeling is A-magic with sum x.

Corollary 3.2. For any $n \geq 3$, the cylinder graph $C_n \times P_2$ is in $\Omega(1)$.

Let the wheel $W(n) = K_1 + C_n$. Suppose $V(W(n) = \{c, v_1, v_2, \ldots, v_n\}$, we see that it has size 2n.

Theorem 3.3. (a)For $m \geq 2$, and any graph G in REG(2m), the L-product graph $G \times_L (W(2m\text{-}1),c)$ is in $\Omega(1)$.

(b)For $m \geq 2$ and any graph G in REG(2m+1), the L-product graph $G \times_L (W(2m),c)$ is in $\Omega(1)$.

Proof. (a)Let x be a non-zero element of a non-trivial abelian group A. We label all the edges of W(2m-1) except the spokes by x and all the spokes by -x. For each v in W(2m-1), we see the vertex label is -x+2x =x and the center c has vertex label -(2m-1)x. Now if we label all the edges of G by x, then each vertex in $G \times_L(W(2m\text{-}1),c)$ has label 2mx. Thus the labeling of $G \times_L (W(2m\text{-}1),c)$ is A-magic with sum x.

Using a similar argument, we can show that (b) is true.

Example 4. Figure 5 illustrates the labeling scheme for the L-product of K_5 with (W(3),c).

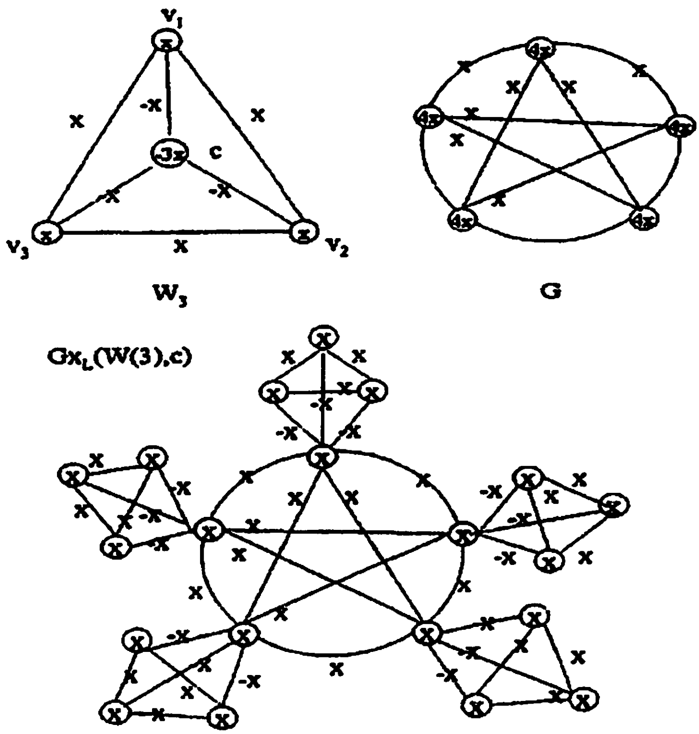
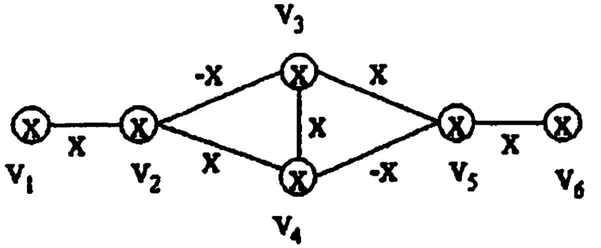


Figure 5.

Theorem 3.4. Among the 19 connected (6,7)-graphs, the graph G115 depicts in Figure 6 is the only one in Ω(1).



G115

Figure 6.

Theorem 3.5. For any $n \geq 3$, the graph C_n x_L (G 115, v) for any v in V(G115) is in Ω(1) if and only if n is even.

Proof. If n is even, then C_n is in $\Omega(0)$. Thus, $C_n\ x_L$ (G 115, v) for any v in V(G115) is in $\Omega(1)$.
If n is odd, then C_n is in 0(2) but not in $\Omega(0)$. Thus, $C_n\ x_L$ (G115, v) for any v in V(G 115) is not in $\Omega(1)$.

<u>**Theorem 3.6**</u>. The graph H depicts in Figure 7 is in $\Omega(1)$.

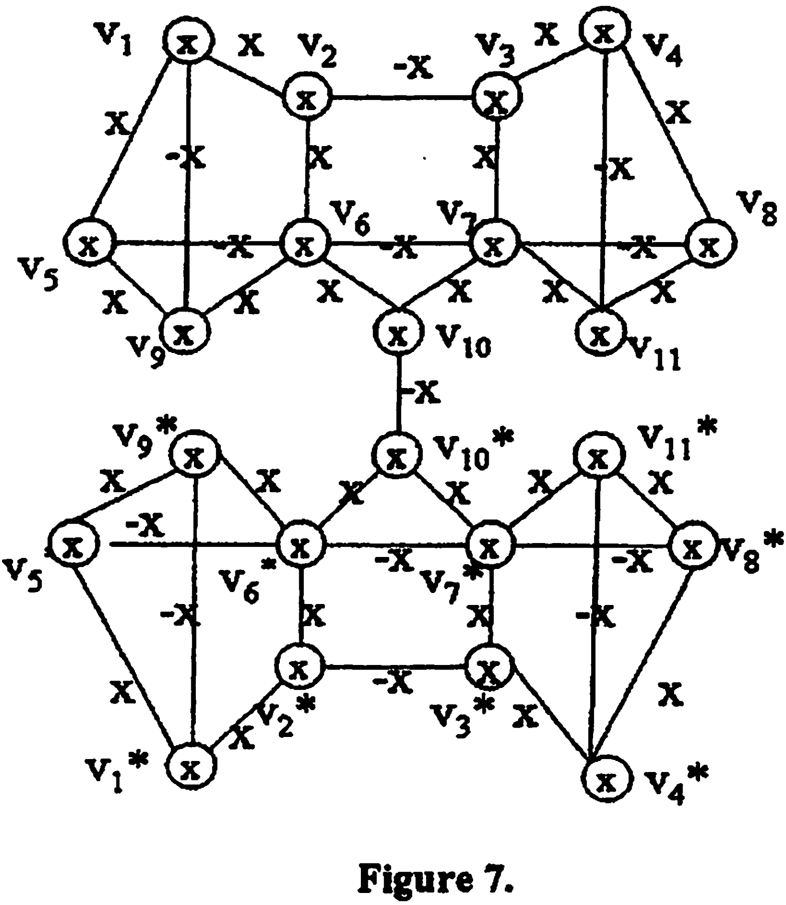


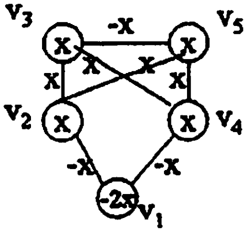
Figure 7.

Theorem 3.7. Let J be the (5,7)-graph as shown in Figure 8. For any $G \in REG(3)$, the L-product graph $G \times_L (J,v_1)$ is in $\Omega(1)$.

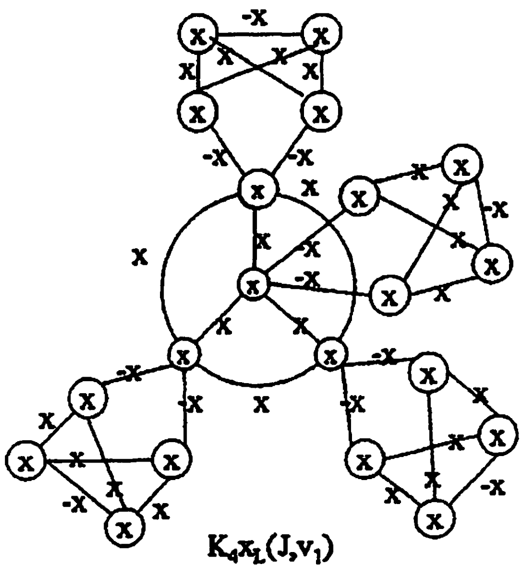
Proof. Let x be a non-zero element of a non-trivial abelian group A. We label all the edges to each J component as shown in Figure 8.

All the edges of the G are x labeled. We see that all the vertices in the L-product $G \times_L (J,v_1)$ are x. Thus $G \times_L(J,v_1)$ is in $\Omega(1)$.

Example 5. Let G be K_4. Figure 8 shows that $K_4, \times_L (J, v_1)$ is in $\Omega(1)$.



J



$K_4 \times_L (J,v_1)$

Figure 8.

Definition 3. In the definition of **generalized L-product** of G and Φ, we relax the condition $\Phi : V(G) \rightarrow Gph^*$, by $\Phi : S \rightarrow Gph^*$ where $S \subseteq V(G)$, then the construction is called the **partially generalized L-product**. We will denote this graph by $(G,S) \times_L \Phi$.

Theorem 3.8. For any $m \geq 2$ and $n \geq 3$, let $G = P_m x P_n$ be the grid graph. Let S be the set of all even vertices of G. For any $\Phi : S \rightarrow \Omega(1)^*$, the partially generalized L-product $(G,S)\ x_L\Phi$ is in $\Omega(1)$.

Proof. Let x be a non-zero element of a non-trivial abelian group A. We label all the edges of $P_m x P_n$ as follows:

All the edges of the rows of $P_m x P_n$ are labeled by x and all the edges of the columns of $P_m x P_n$ are labeled by -x. We see that all the odd vertices in $P_m x P_n$ have labeled x and all the even vertices in $P_m x P_n$ have labeled 0.

Thus the partial generalized L-product $(G,S)\ x_L\Phi$ is in $\Omega(1)$ if each even vertex is amalgamated with a $\Omega(1)$ fully magic graph.

Example 6. Figure 9 shows the partial generalized L-product $(P_2 x P_4, S)\ x_L\Phi$, is in $\Omega(1)$ where $\Phi(v) = P_2$ for all v in S.

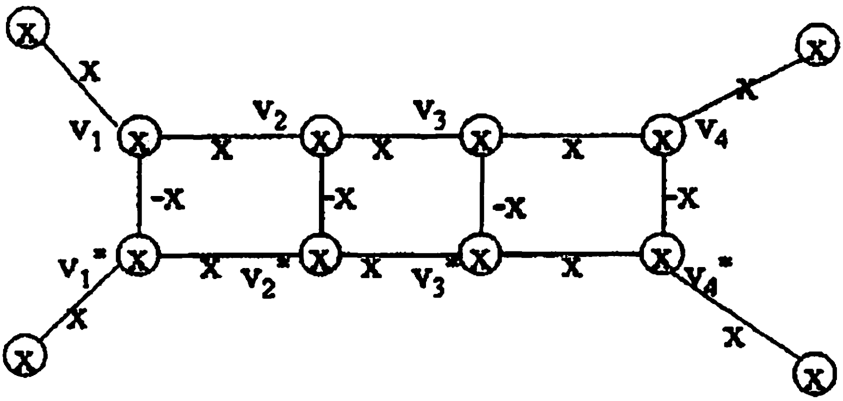


Figure 9.

Theorem 3.9. For any $m \geq 3$ and $n \geq 2$, let $G = C_m x P_n$ be the cylinder graph. Let S be the set of all even vertices of G. For any $\Phi : S \rightarrow \Omega(1)^*$, the partial generalized L-product $(G,S)\ x_L\Phi$ is in $\Omega(1)$.

Proof. Let x be a non-zero element of a non-trivial abelian group A. We label all the edges of $C_m x P_n$ as follows:

All the edges of the cycles of $C_m x P_n$ are labeled by x and all the edges of the paths of C_mxP_n are labeled by -x .

We see that all the odd vertices in C_mxP_n are labeled x and all the even vertices in C_mxP_n are labeled 0.

Thus the partially generalized L-product (G,S) $x_L\Phi$ is in $\Omega(1)$ if each even vertex is amalgamated with a $\Omega(1)$ fully magic graph.

Example 7. Figure 10 shows that the partially generalized L-product (C_3xP_3,S) x_L Φ is in $\Omega(1)$ where $\Phi(v) = P_2$ for all v in S.

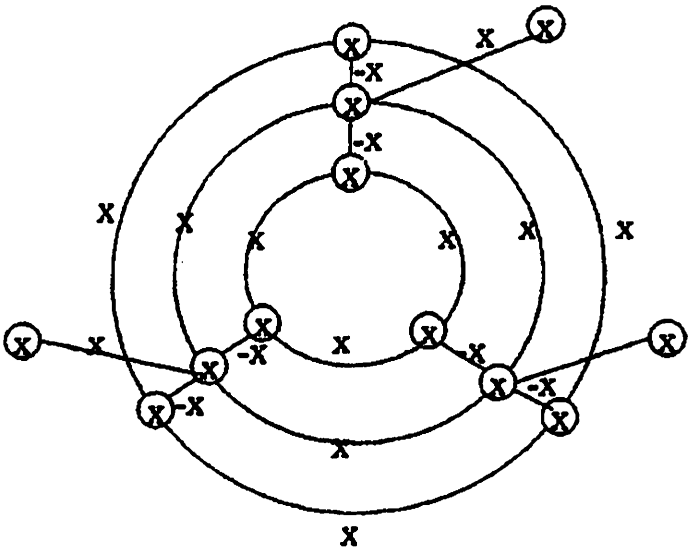


Figure 10.

Let Gph** be the class of pair (H, $\{u_1,u_2\}$) where H is a connected graph and s,t are two distinguished vertices of H.

For any two pairs (H,$\{u_1,u_2\}$) (H_1,$\{v_1,v_2\}$) in Gph**, we construct a new graph (H,$\{u_1,u_2\}$)# (H_1,$\{v_1,v_2\}$) as follows:

We form the disjoint union of H and H_1 and identify u_1 with v_1 and u_2 with v_2. The resulting graph is called the two-point join of (H,$\{u_1,u_2\}$), (H_1,$\{v_1,v_2\}$).

We can generalize the above construction in the following way:

Suppose n pairs $\{u_1,u_2\}$, $\{u_3,u_4\}$, ... , $\{u_{2n-1},u_{2n}\}$ of vertices in H are given. Let $D=\{\{u_1,u_2\}, \{u_3,u_4\}, \ldots, \{u_{2n-1},u_{2n}\}\}$ and $\partial : D \rightarrow Gph^{**}$ with

$$\partial(\{u_{2i-1},u_{2i}\}) = (H_i, \{v_{2i-1} v_{2i}\}) \text{ for } i = 1,2\ldots,n$$

We set $TJ(1) = (H,\{u_1,u_2\}) \# (H_1,\{v_1,v_2\})$

$TJ(2) = (TJ(1), \{u_3,u_4\}) \# (H_2\{v_3,v_4\})$,

$TJ(3) = (TJ(2), \{u_5,u_6\}) \# (H_3\{v_5,v_6\})$,

Finally $TJ(n) = (TJ(n-1), \{u_{2n-1},u_{2n}\}) \# (H_n, (v_{2n-1}, v_{2n}\})$.

We will say TJ(n) is the **two-point join** of H with $\{(H_1,\{v_1,v_2\})\ldots, (H_n, (v_{2n-1}, v_{2n}\})$. and will denote this graph simply by $(H,D) \# \partial(D)$.

Theorem 3.10. Let $G = C_{4t}$ where $t \geq 1$ and $V(C_{4t}) = \{u_1,u_2, u_3,u_4, \ldots u_{4t-1},u_{4t}\}$.
Let $D=\{\{u_1,u_2\}, \{u_3,u_4\},\ldots\{u_{4t-1},u_{4t}\}\}$ and $\partial: D \rightarrow Gph^{**}$ with
$\partial(\{u_{2i-1},u_{2i}\})=(St(3), \{v_1,v_2\})$ for $i=1,2,\ldots,2t$.
Then the two-points join $(H, D) \# \partial(D)$ is in $\Omega(1)$.

Example 8. Figure 11 demonstrates the magic labeling scheme for $H = (St(3), \{v_1,v_2\})$ and C_{4t} for t=1 and 2 respectively.

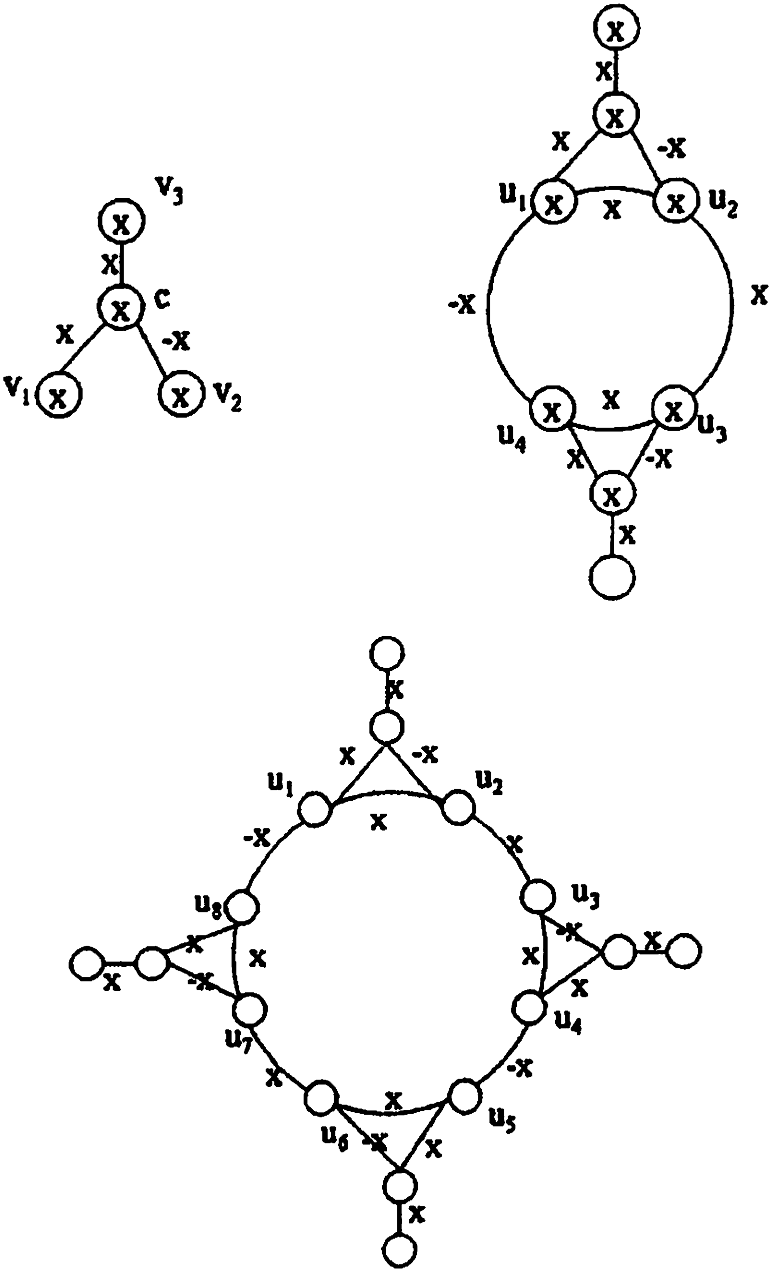


Figure 11.

4. $\Omega(2)$-and $\Omega(3)$-Full magic graphs.

Definition 3. The generalized theta graph $\Theta(l_1,l_2,\dots l_k)$ is the graph consisting of k pairwise internally disjoint paths with common end vertices s_1,s_2 and length $l_1,l_2,\dots l_k$.

It is clear that the size of the generalized theta graph $\Theta(l_1,l_2,\dots l_k)$ is $l_1+l_2+\dots+l_k$.

For each $k \geq 1$, we denote the class of all generalized theta graphs $\Theta(l_1,l_2,\dots l_k)$ by GT(k). For convenience, we consider that GT(1) consists of all paths as generalized theta graphs. We denote $\cup\{GT(k) : k \geq 1\}$ by GT.

Theorem 4.1. The generalized theta graph $\Theta(l_1, l_2,l_3)$ is in $\Omega(2)$.

Proof. Let x be a non-zero element of a non-trivial abelian group A. We label all the edges of $\Theta(1, l_2,l_3)$ except (s_1,s_2) by x and (s_1,s_2) by -x. It is clear that the labeling is A-magic with sum 2x.

For any two pairs $(H_1,\{u_1,u_2\})$, $(H_2,\{u_3,u_4\})$, in Gph**, we construct a new graph $(H,\{u_1,u_2\})$ © $(H_1,\{u_3,u_4\})$, as follows:

We form the disjoint union of H_1 and H_2 and identify u_2 with u_3). The resulting graph is called the **chain sum** of $(H_1,\{u_1,u_2\})$, $(H_2,\{u_3,u_4\})$.

We can generalize the above construction in the following way:

Suppose n pairs $(H_1,\{u_1,u_2\})$, $(H_2,\{u_3,u_4\})$,... $H_n,\{u_{2n-1},u_{2n}\})$ in Gph** are given. We set TJ(l) = $(H,\{u_1,u_2\})$ © $(H_1,\{u_3,u_4\})$,

TJ(2)= (TJ(1), $\{u_1,u_4\}$) © $(H_3,\{u_5,u_6\})$,

TJ(3) = (TJ(2), $\{u_1,u_6\}$) © $(H_4,\{u_7,u_8\})$.

Finally $TJ(n) = (TJ(n-1),\{u_1,u_{2n-2}\})$ © $(H_n,\{u_{2n-1},u_{2n}\})$.
We will say TJ(n) is the **chain sum** of $(H_1,\{u_1,u_2\})$, $(H_2,\{u_3,u_4\}),\ldots H_n,\{u_1,u_2\})$

__Theorem 4.2__. For any odd integer $n \geq 3$, the chain sum of n generalized theta graphs $(\Theta(1,1_{2,1}),\{s_1 s_2\}),(\Theta(1,1_{2,2}),\{s_3,s_4\}), \ldots(\Theta(1,1_{2,n}),\{s_{2n-1},s_{2n}\})$, is in $\Omega(2)$.

Proof. Let x be a non-zero element of a non-trivial abelian group A.

For i =1,2, ... ,n, we label all the edges of the odd component of the chain sum by x and all the edges of the even component except (s_{2i-1},s_{2i}) by x and (s_{2i-1},s_{2i}) by -x. It is clear that the labeling is A-magic with sum 2x (see Figure 10).

__Example 9__. Consider the chain sum of $(\Theta(1,2),\{s_1,s_2\}),(\Theta(1,2),\{s_3,s_4\})$, $(\Theta(1,3),\{s_5,s_6\})$, $(\Theta(1,2),\{s_7,s_8\})$, $(\Theta(1,4),\{s_9,s_{10}\})$, in Figure 12. It is easy to see that the graph is in $\Omega(2)$.

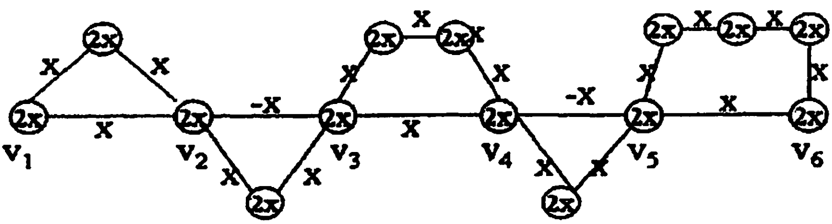


Figure 12.

Given a graph G, and $T \subseteq E(G)$, we define a mapping $\delta : T \rightarrow GT$.

Now we construct a new graph from (G,T, δ) as follows:

If $\delta(e) = \Theta(1_1,1_2, \ldots ,1_k) \in GT(k)$, where $e = (u,v) \in T$, we will identity s_1,s_2, of $\Theta(1_1,1_2, \ldots ,1_k)$ to u,v respectively. We will denote the final graph by $\Gamma(G,T, \delta)$.

We will call $\Gamma(G,T, \delta)$ as the **edge expansion** of (G,T, δ).

Theorem 4.3. Suppose the wheel W(2m+2) has 2m+3 vertices and T = { (v_1,v_2), (v_3,v_4),... (v_{2m+1},v_{2m+2}). For any mapping δ : $T \to GT(l)$, the edge expansion of $((W(2m+2),T, \delta)$,c) has the following property:

For any G in **REG(2m)**, the L-product G x_L ((W(2m+2),T, δ),c) is in $\Omega(2)$.

Proof. Let x be a non-zero element of a non-trivial abelian group A. If we label the edges in the edge expansion of (W(2m+2),T, δ) except those in T by x, and all the edges in T by -x, we see that all the vertices, except c have label 2x. The center has label (2m+2)x. Now if all the edges in G are labeled by -x, then the vertex label for each vertex in G is -2mx. Thus the L-product G x_L ((W(2m+2),T, δ),c) is in $\Omega(2)$.

Example 10. Figure 13 illustrates the labeling for (W(4),T, δ) where T= {(v_1,v_2), (v_3,v_4)} and δ : $T \to GT(1)$ is defined by $\delta((v_1,v_2)) = P_3$ and $\delta((v_3,v_4)) = P_4$.

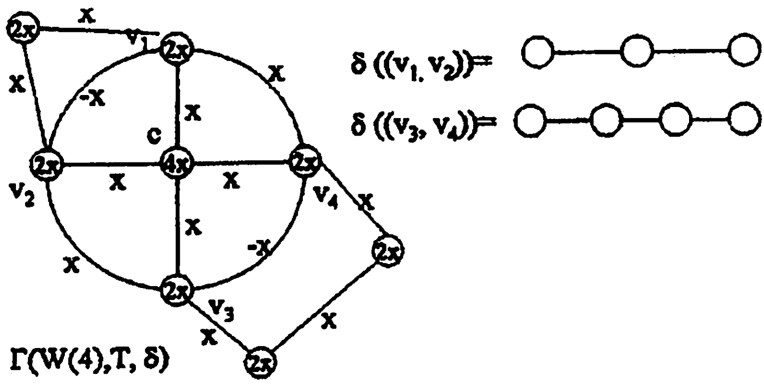


Figure 13.

In general, not all 4-regular graphs have a perfect matching. Carbonera and Shepherd [1] gave a 4-regular graph of order 30 without a perfect matching and showed that 4-regular graphs derived from quadrilateral meshes always have a perfect matching.

Theorem 4.4. If G in REG(4) and p(G) is odd, such that there exists a vertex u in G with G- {u} has a perfect matching then the L-product $C_n \times_L (G, u)$ is in $\Omega(2)$ for any $n \geq 3$ ..

Proof. Suppose G in REG(4) and $V(G) = \{ v_1, v_{2\ldots} v_{2t+1})$. For any u in V(G) and for any x in A*, where A is a non-trivial abelian group, we define a labeling on each G component as follows: the four edges in G which are incident on u are labeled by x. For the subgraph G\{u} which containing the other 2t vertices, we labeled the edges in a perfect matching of G\{u} by -x and all the other edges by x. We see that each vertex of G has vertex label 2x, except the vertex u which has label 4x. Now if we define all the edges in the cycle by

-x, then it is obvious that the L-product $C_n \times_L (G, u)$ has magic labeling with sum 2x.

Example 11. Figure 14 illustrates the labeling for $C_3 \times_L (K_5, u)$.

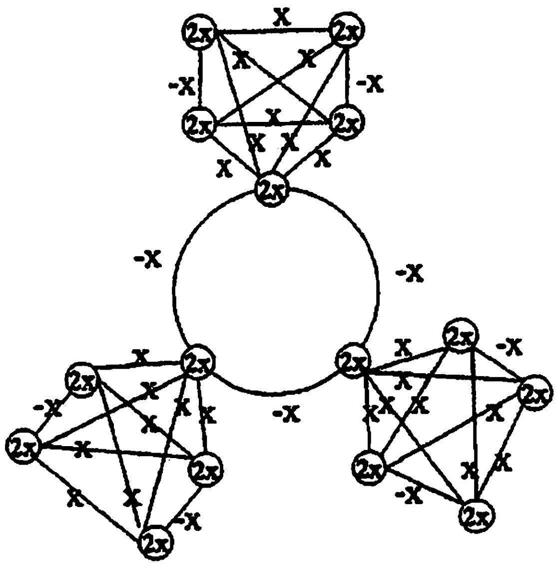


Figure 14.

Let J be K_4-e, where e =(a,b). Given a graph G∈REG(3),and

T ⊆ E(G), where all the edges in T are independent, i.e.no two edges are incident on one vertex,we define a mapping δ : T →Gph.** where δ((u,v)) = (K_4-e, {a,b}).

Now we construct a new graph from (G,T, δ) as follows:

If e = (u,v) ∈ T, we will identity u with a and v with b respectively. We will denote the final graph by Γ(G,T, δ). Let Γ(G,T, δ) be the **edge expansion** of (G,T, δ).

Theorem 4.5. For any graph G∈REG(3), and any T ⊆ E(G) with independent edges, the edge expansion of (G,T, δ).is in Ω(3).

Proof. Let x be a non-zero element of a non-trivial abelian group A. We label the edges of Γ(G,T, δ) with J component by −x on (u,v) and all the other edges by x. We see that Γ(G,T, δ) has magic sum 3x.

Example 12.. The following (6.10)-graph in Figure 15 is in Ω(3) which is Γ(K_4,T, δ) where T = {(v_2,v_3)}.

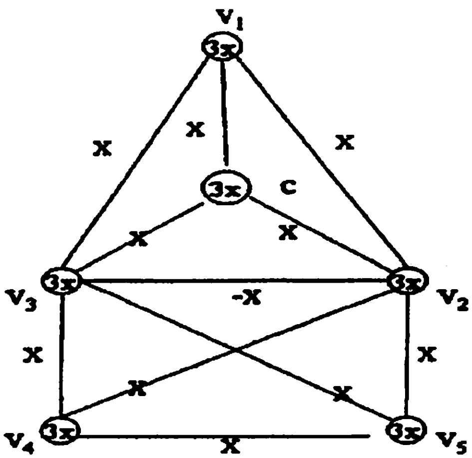


Figure 15.

5. **Ω(k)-Full magic graph.**

Finally, we will use the two point-join construction to create a large family of fully magic graphs in Ω(k) for any $k \geq 3$.

For any $k \geq 3$, and any H ∈REG(k). If e=(u,v) ∈ E(H), we consider the pairs (H \ {e}, {u,v}), (P_3, {s_1,s_2}) and forms their two-point join. We denote the resulting graph simply by H*. If V(P_3) = { s_1,s,s_2} and E(P_3) ={(s_1,s) {s,s_2}) then we consider the pair (H*,s).

__Theorem 5.1.__ For any $k \geq 3$, and any G ∈ **Ω(k-2),** the L-product G x_L (H*,s) is in **Ω(k).**

Proof. Let x be a non-zero element of a non-trivial abelian group A. If we label the edges in each H* component of G x_L (H*,s) by x, we see that all the vertices in H* component except s have label kx and the vertex label of s is 2x. Now as any G ∈**Ω(k-2),** it has an A-magic labeling with edges labeled by -x or x , and the vertex labels in G are (k-2)x. Thus the L-product G x_L (H*,s) is in **Ω(k).**

__Example 12__. Figure 16 shows that H ∈REG(3) and e =(u_2,u_3). The graph G115 of Figure 6 is in Ω(1). and the L-product G115 x_L (H*,s) is in **Ω(3).**

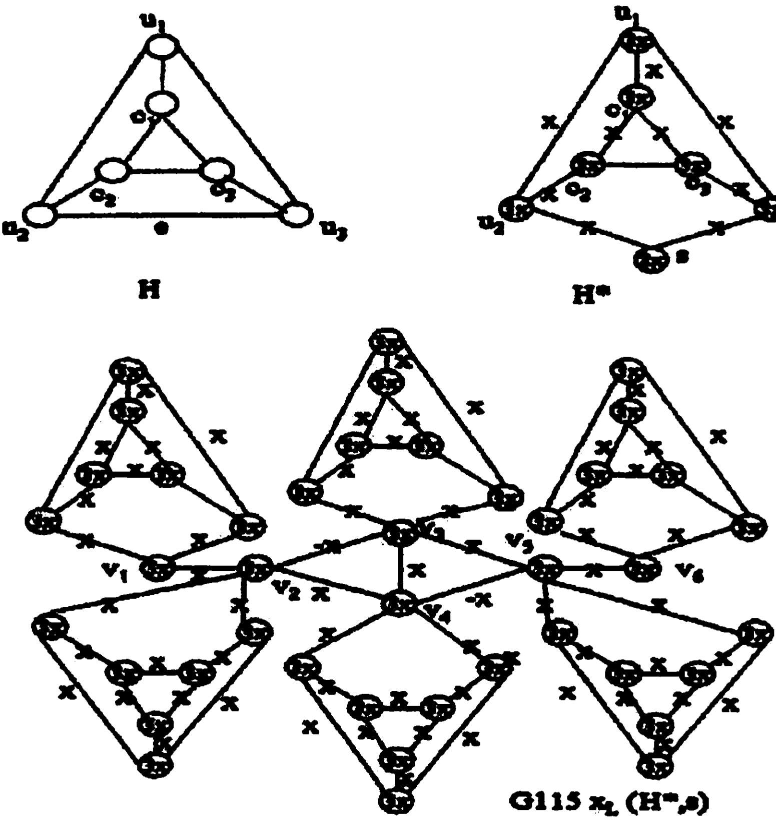


Figure 16.

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