On a new Stirling's series

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Abstract

In this paper, we deduced the following new Stirling series

$$n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n exp\left(\frac{1}{12n+1}\left[1+\frac{1}{12n}\left(1+\frac{2/5}{n}+\frac{29/150}{n^2}-\frac{62/2625}{n^3}-\frac{9173/157500}{n^4}+....\right)^{-1}\right]\right),$$

which is faster than the classical Stirling's series.

Key Words: Stirling' series, Stirling' formula, speed of convergence, approximation formulas.

1 Introduction.

It is well known that the Stirling's formula

$$n! \sim \sqrt{2n\pi}\,(n/e)^n \tag{1}$$

is the most known approximation of $n!$. It is used in many applications, especially in statistics and probability. A number of upper and lower bounds for $n!$ have been obtained by various authors [6], [7]. Most bounds are of the form

$$\sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\alpha_n} < n! < \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\beta_n}, \tag{2}$$

2000 Mathematics Subject Classification: 41A60, 41A25, 57Q55, 33B15, 26D07.

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where α_n and β_n tend to zero through positive values. P. R. Beesack [3] deduced that α_n and β_n satisfy

$$\alpha_n - \alpha_{n+1} < \sum_{k=1}^{\infty} \frac{1}{2k+1} \frac{1}{(2n+1)^{2k}} < \beta_n - \beta_{n+1}. \tag{3}$$

Mansour et al concluded the double inequality (2) with different proof [5] and presented a new family of upper bounds of $n!$, which is differ from the known Stirling's series [1]

$$\frac{B_2}{1.2.n} + \frac{B_4}{3.4.n^3} + \frac{B_6}{5.6.n^5} + \ldots, \tag{4}$$

where the numbers B_i's are called the Bernoulli numbers and are defined by

$$B_0 = 1, \qquad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \qquad n \geq 2. \tag{5}$$

Also, we concluded a $q-$analogy of the double inequality (2) and we presented some double inequalities of the $q-$factorial [4].

E. Artin [2] showed that the sequence $\mu(n) = \ln \frac{n! e^n}{n^n \sqrt{2\pi n}}$ lies between any two successive partial sums of the Stirling's series. Also, H. Robbins [11] showed that the sequence $\mu(n)$ satisfies

$$\frac{1}{12n+1} < \mu(n) < \frac{1}{12n}, \qquad n \geq 2. \tag{6}$$

C. Mortici [10] search the best approximation of the form

$$n! \sim \sqrt{2n\pi} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n+\frac{a}{n+b}}} = \gamma_n, \tag{7}$$

where a and b are real parameters. He showed that the best approximation for (7) is obtained for $a = 2/5$ and $b = 0$ and presented the following new Stirling's series

$$n! \sim \sqrt{2n\pi} \left(\frac{n}{e}\right)^n exp \frac{1}{12n + \frac{2/5}{n + \frac{53/210}{n + \frac{195/371}{n + \frac{22999/22737}{\cdots}}}}}, \tag{8}$$

as a continued fraction, which is faster than classical Stirling's series.

2 New double inequalities of $n!$.

Firstly, consider the following function for $a \in \mathbb{R}$

$$T_n = (n+1/2)\log(1+1/n) - 1 - \left(\frac{1+\frac{1}{an}}{12n+1} - \frac{1+\frac{1}{a(n+1)}}{12(n+1)+1} \right).$$

Then

$$\frac{d^2}{dn^2}T_n = \frac{1}{2an^3(n+1)^3(12n+1)^3(12n+13)^3}\left[-8788 + 169(13a-2172)n\right.$$

$$+143(-40164+611a)n^2 - 72(367260+83a)n^3 + 432(-134940+1213a)n^4$$

$$\left.+6912(-10212+347a)n^5 + 41472(-1092+67a)n^6 + 995328(-12+a)n^7\right].$$

If $a > 13$, then $\frac{d^2}{dn^2}T_n > 0$ and hence the function T_n is convex for $n \geq 13$. But $\lim_{n\to\infty} T_n = 0$, then

$$T_n > 0, \qquad \forall\, a > 13;\ n \geq 13.$$

Hence the sequence $\alpha_n = \frac{1+\frac{1}{an}}{12n+1}$ for $a > 13$ tends to zero through positive values and satisfies

$$\alpha_n - \alpha_{n+1} < (n+1/2)\log(1+1/n) - 1 = \sum_{k=1}^{\infty} \frac{1}{2k+1}\frac{1}{(2n+1)^{2k}} \qquad \forall\, a > 13.$$

Also, if $a < 12$, then $\frac{d^2}{dn^2}T_n < 0$ and hence the function T_n is concave for $n \geq 1$. But $\lim_{n\to\infty} T_n = 0$, then

$$T_n < 0, \qquad \forall\, a < 12;\ n \geq 1.$$

Hence the sequence $\beta_n = \frac{1+\frac{1}{bn}}{12n+1}$ for $b < 12$ tends to zero through positive values and satisfies

$$(n+1/2)\log(1+1/n) - 1 = \sum_{k=1}^{\infty} \frac{1}{2k+1}\frac{1}{(2n+1)^{2k}} < \beta_n - \beta_{n+1} \quad \forall\, 0 < b < 12.$$

Theorem 1. *The factorial $n!$ satisfies the double inequality*

$$\sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1+\frac{1}{na}}{12n+1}} < n! < \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1+\frac{1}{nb}}{12n+1}}, \qquad 0 < b < 12;\ a > 13 \tag{9}$$

where $n \geq 13$ in the left hand side and $n \geq 1$ in the right hand side.

The double inequality (9) can be improved by choosing two positive sequences a_n and b_n such that $a_n \to 13$; $b_n \to 12$ as n tends to infinity, a_n is decrease monotonically and b_n is increase monotonically.

Theorem 2. *The factorial $n!$ satisfies the double inequality*

$$\sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1+\frac{1}{nf_n}}{12n+1}} < n! < \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1+\frac{1}{ng_n}}{12n+1}}, \tag{10}$$

where the two sequences $f_n \to 13$; $g_n \to 12$ as $n \to \infty$ through positive values, f_n is monotonically decreasing, g_n is monotonically increasing, $n \geq 13$ in the left hand side and $n \geq 1$ in the right hand side.

3 Some new approximation formulas of $n!$.

In view of Theorem 2, we will discuss two approximation formulas of $n!$ and we will deduce a double inequality of $\Gamma(x)$. Firstly, we will study the best approximation of the formula

$$n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1+\frac{1}{12\left(1+\frac{h}{n}\right)n}}{12n+1}}, \tag{11}$$

where h is a real parameter. In what follows, we need the following result, which represents a powerful tool to measure the rate of convergence.

Lemma 3.1. *If $(w_n)_{n\geq 1}$ is convergent to zero and there exists the limit*

$$\lim_{n\longrightarrow\infty} n^k(w_n - w_{n+1}) = l \in \mathbb{R} \tag{12}$$

with $k > 1$, then there exists the limit:

$$\lim_{n\longrightarrow\infty} n^{k-1}w_n = \frac{l}{k-1}.$$

This Lemma was first used by C. Mortici for constructing asymptotic expansions, or to accelerate some convergences [8], [9]. By using Lemma (3.1), clearly the sequence $(w_n)_{n\geq 1}$ converges more quickly when the value of k satisfying (12) is larger.
To measure the accuracy of the approximation formula (11), define the sequence $(w_n)_{n\geq 1}$ by the relation

$$n! = \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1+\frac{1}{12\left(1+\frac{h}{n}\right)n}}{12n+1}} e^{w_n}; \qquad n = 1, 2, 3, \dots . \tag{13}$$

This approximation formula will be better as $(w_n)_{n\geq 1}$ converges faster to zero. Using the relation (13), we get

$$w_n = \ln n! - \ln\sqrt{2\pi} - (n+1/2)\ln n + n - \frac{1+\frac{1}{12(1+\frac{h}{n})n}}{12n+1}$$

and hence

$$w_n - w_{n+1} = (n+1/2)\ln(1+1/n) - 1 - \frac{13+144h^2+168n+144n^2+12h(13+24n)}{12(h+n)(1+h+n)(1+12n)(13+12n)}.$$

Then

$$w_n - w_{n+1} = \frac{5}{84n^6} - \frac{1}{15n^5} + \frac{3}{40n^4} - \frac{1}{12n^3} + \frac{1}{12n^2}$$
$$-\frac{13+144h^2+168n+144n^2+12h(13+24n)}{12(h+n)(1+h+n)(1+12n)(13+12n)} + O(n^{-7})$$

and

$$\lim_{n\longrightarrow\infty} n^4(w_n - w_{n+1}) = \frac{1}{240}(5h-2).$$

Now, we get the following result about the rate of convergence of w_n :

Lemma 3.2. *The rate of convergence of the sequence w_n is equal to n^{-4} if $h = 2/5$, since*

$$\lim_{n\longrightarrow\infty} n^4 w_n = \frac{-29}{21600}. \tag{14}$$

Then the fast sequence w_n appears for $h = 2/5$ and hence the best approximation of the formula (11) is

$$n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1+\frac{1}{12\left(1+\frac{2/5}{n}\right)n}}{12n+1}}. \tag{15}$$

In the next step, we will discuss the best approximation of the formula

$$n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1+\frac{1}{12\left(1+\frac{2/5}{n}+\frac{d}{n^2}\right)n}}{12n+1}}. \tag{16}$$

To measure the accuracy of the approximation formula (16), define the sequence $(v_n)_{n\geq 1}$ by the relation

$$n! = \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1+\frac{1}{12\left(1+\frac{2/5}{n}+\frac{d}{n^2}\right)n}}{12n+1}} e^{v_n}; \qquad n = 1,2,3,\dots . \tag{17}$$

Then we get the following result about the rate of convergence of v_n:

Lemma 3.3. *The rate of convergence of the sequence v_n is equal to n^{-5} if $d = 29/150$, since*

$$\lim_{n\longrightarrow\infty} n^5 v_n = \frac{31}{189000}. \tag{18}$$

Then the fast sequence v_n appears for $d = 29/150$ and hence the best approximation of the formula (16) is

$$n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1+\frac{1}{12\left(1+\frac{2/5}{n}+\frac{29/150}{n^2}\right)n}}{12n+1}} = \sigma_n. \tag{19}$$

Theorem 3. *For $x \geq 2$,*

$$\sqrt{2x\pi}\left(\frac{x}{e}\right)^x e^{\frac{1+\frac{1}{12\left(1+\frac{2/5}{x}+\frac{29/150}{x^2}\right)x}}{12x+1}} < \Gamma(x+1) < \sqrt{2x\pi}\left(\frac{x}{e}\right)^x e^{\frac{1+\frac{1}{12\left(1+\frac{2/5}{x}\right)x}}{12x+1}}. \tag{20}$$

Proof. E. Artin [2] showed that

$$\mu(x) = \ln\frac{\Gamma(x+1)e^x}{x^x\sqrt{2\pi x}}$$

lies between any two successive partial sums of the Stirling's series, then

$$\sum_{k=1}^{4}\frac{B_{2k}}{2k(2k-1)x^{2k-1}} < \mu(x) < \sum_{k=1}^{5}\frac{B_{2k}}{2k(2k-1)x^{2k-1}}.$$

But

$$\sum_{k=1}^{4}\frac{B_{2k}}{2k(2k-1)x^{2k-1}} - \frac{1+\frac{1}{12(1+\frac{2/5}{x}+\frac{29/150}{x^2})x}}{12x+1}$$
$$= \frac{1488x^5+3074x^4-3768x^3-2494x^2-1224x-87}{5040x^7(12x+1)(150x^2+60x+29)},$$

then we get

$$\mu(x) > \frac{1+\frac{1}{12(1+\frac{2/5}{x}+\frac{29/150}{x^2})x}}{12x+1}, \qquad x \geq 2.$$

Similarly,

$$\sum_{k=1}^{5}\frac{B_{2k}}{2k(2k-1)x^{2k-1}} - \frac{1+\frac{1}{12(1+\frac{2/5}{x})x}}{12x+1}$$

$$= \frac{-13398x^7 + 6996x^6 + 3828x^5 - 5676x^4 - 2871x^3 + 8202x^2 + 4060x + 280}{166320x^9(5x+2)(12x+1)},$$

then we get

$$\mu(x) < \frac{1 + \frac{1}{12\left(1+\frac{2/5}{x}\right)x}}{12x+1}, \qquad x \geq 2.$$

□

Corollary 3.4. *If n is positive integer, then*

$$\sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1+\frac{1}{12\left(1+\frac{2/5}{n}+\frac{29/150}{n^2}\right)n}}{12n+1}} < n! < \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{1+\frac{1}{12\left(1+\frac{2/5}{n}\right)n}}{12n+1}}, \quad n > 1. \tag{21}$$

Our new formula σ_n (Eq.19) is much stronger than the Mortici formula γ_n (Eq.7). The rate of convergence of each of them is equal to n^{-5} and they define lower bounds, but

$$\gamma_n < \sigma_n,$$

since

$$\frac{1}{12n+\frac{2/5}{n}} - \frac{1 + \frac{1}{12\left(1+\frac{2/5}{n}+\frac{29/150}{n^2}\right)n}}{12n+1}$$

$$= \frac{-29}{(12n+1)(30n^2+1)(150n^2+60n+29)} < 0.$$

Also, we can obtain some further improvements of the above formulas (15) and (19) by using the same technique. For instance, the formula

$$n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n exp\left(\frac{1}{12n+1}\theta_n\right),$$

where

$$\theta_n = 1 + \frac{1}{12n}\left(1 + \frac{2/5}{n} + \frac{29/150}{n^2} - \frac{62/2625}{n^3} - \frac{9173/157500}{n^4} + \frac{1563/43750}{n^5}\right.$$

$$\left. + \frac{9035351/165375000}{n^6} - \frac{81698486/1136953125}{n^7}\right)^{-1}$$

has a rate of convergence equal to n^{-10}. This procedure will give us an easy technique to construct the new Stirling's series

$$n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n exp\left(\frac{1}{12n+1}\left[1+\frac{1}{12n}\left(1+\frac{2/5}{n}+\frac{29/150}{n^2}\right.\right.\right.$$

$$\left.\left.\left. -\frac{62/2625}{n^3}-\frac{9173/157500}{n^4}+....\right)^{-1}\right]\right), \quad (22)$$

which is faster than the classical Stirling's series.

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