# Chromatic Uniqueness of A Family of $K_4$-Homeomorphs

H. Roslan[1], S. Catada-Ghimire[2]

[1] *Department of Mathematics*
*Faculty of Science and Technology*
*University Malaysia Terengganu*
*21030 Kuala Terengganu, Terengganu, Malaysia*

[2] *School of Mathematical Sciences*
*Universiti Sains Malaysia, 11800 Penang, Malaysia*

[1] *E-mail: hroslan@umt.edu.my*
[2] *E-mail: aspa777@gmail.com*

## ABSTRACT

We discuss the chromaticity of one family of $K_4$-homeomorphs with exactly two non-adjacent paths of length two, where the other four paths are of length greater than or equal to three. We also give a sufficient and necessary condition for the graphs in the family to be chromatically unique.

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## 1 Introduction

All graphs considered here are simple graphs. For such a graph $G$, let $P(G,\lambda)$ (or simply $P(G)$) denote the chromatic polynomial of $G$. Two graphs $G$ and $H$ are chromatically equivalent (or simply $\chi$−equivalent), denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$ (or simply $P(G) = P(H)$). A graph $G$ is chromatically unique (or simply $\chi$−unique) if for any graph $H$ such that $H \sim G$, we have $H \cong G$, i.e, $H$ is isomorphic to $G$.

Graphs derived from the same graph are referred to as homeomorphic. A $K_4$-homeomorphic graph or simply $K_4$-homeomorph, denoted by $K_4(a,b,c,d,e,f)$, is obtained by subdividing the six edges of the complete graph with four vertices, $K_4$, into $a,b,c,d,e,f$ paths, respectively (see Figure 1). Each subdivided edge is called a path and the number of subdivisions is its length. So far, the chromaticity of $K_4$-homeomorphs with girth $g$, where $3 \le g \le 7$ and $K_4$-homeomorphs with at least two paths of length one has been completed (see [?], [?], [?], [?], [?], [?], [?]). Recently, Peng in [?] has published her result on the chromaticity of $K_4(1,3,3,d,e,f)$ with girth seven. Catada-Ghimire S. et al. [?] have studied the chromaticity of $K_4(3,3,4,d,e,f)$ with girth ten. Dong et al. in [?], after summarizing the results which have been obtained so far on the chromaticity of $K_4$−homeomorphs, posted two tasks to tackle. First, study the chromaticity of $K_4$−homeomorphs with exactly two paths of length greater than or equal to two and then study the chromaticity of $K_4$−homeomorphs with exactly one path of length one. Motivated by such challenge, Catada-Ghimire S. et al. [?] discussed the chromaticity of a family of $K_4$−homeomorphs with exactly two adjacent paths of length two, exactly one path of length one and three paths of length at least three, that is, $K_4(a,2,2,d,1,f)$, where $a \geq 3$, $d \geq 3$, $f \geq 3$. With the same motivation, we shall discuss in this paper the chromaticity of one family of $K_4$-homeomorphs with exactly two non-adjacent paths of length two, where the other four paths are of length greater than or equal to three, that is, $K_4(a,b,2,d,2,f)$, where $a \geq 3, b \geq 3, d \geq 3, f \geq 3$ (as shown in Figure 2). The result in this paper is significant in the completion of the study of $K_4$-homeomorphs with exactly two paths of length two and as a consequence, the chromaticity of some families of such graphs with girth greater than seven can be established.

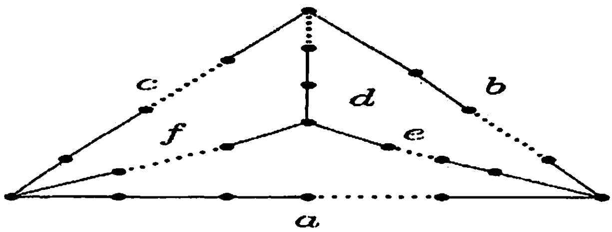


Figure 1. $K_4(a,b,c,d,e,f)$

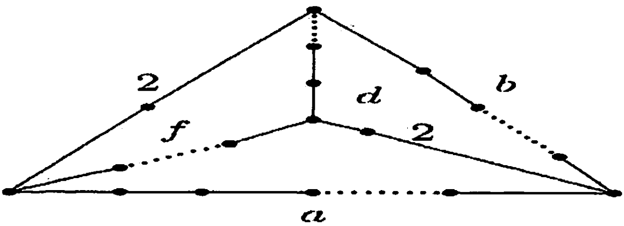


Figure 2. $K_4(a,b,2,d,2,f)$

## 2 Preliminary results and Notations

In this section, we give some known results used in the sequel.

**Lemma 2.1** *Assume that G and H are $\chi$−equivalent. Then the following statements are proven to be true.*

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ *(see [?]);*

(2) *Let $g(G)$ and $g(H)$ denote the girths of G and H, respectively. Then $g(G) = g(H)$. Moreover, G and H have the same number of cycles with length equal to their girth (see [?]);*

(3) *If G is a $K_4$-homeomorph, then H must itself be a $K_4$-homeomorph (see [?]);*

(4) *Let $G = K_4(a,b,c,d,e,f)$ and $H = K_4(a',b',c',d',e',f')$, then*

  (i) *min $\{a,b,c,d,e,f\}$ = min $\{a',b',c',d',e',f'\}$ and the number of times that this minimum occurs in the list $\{a,b,c,d,e,f\}$ is equal to the number of times that this minimum occurs in the list $\{a',b',c',d',e',f'\}$ (see [?]);*

  (ii) *if $\{a,b,c,d,e,f\} = \{a',b',c',d',e',f'\}$ as multisets, then $H \cong G$ (see [?]).*

**Lemma 2.2** *(Ren [?]) Let $G = K_4(a,b,c,d,e,f)$ when exactly three of $a,b,c,d,e,f$ are the same. Then G is not $\chi$−unique if and only if G is isomorphic to $K_4(r,r,r-2,1,2,r)$, $K_4(r,r-2,r,2r-2,1,r)$, $K_4(t,t,1,2t,t+2,t)$, $K_4(t,t,1,2t,t-1,t)$, $K_4(t,t+1,t,2t+1,1,t)$, $K_4(1,t,1,t+1,3,1)$ or $K_4(1,1,t,2,t+2,1)$, where $r \geq 3$, $t \geq 2$.*

**Lemma 2.3** *(Ren and Zhang [?] and Li [?]) The graph $K_4(a,b,c,d,e,f)$ is $\chi$−unique if exactly four numbers among $a,b,c,d,e,f$ are the same.*

**Lemma 2.4** *(Whitehead, Jr. and Zhao [?]) The graph $K_4(a,b,c,d,e,f)$ is $\chi$−unique if the positive integers $a,b,c,d,e,f$ assume no more than two distinct values.*

**Lemma 2.5** *(Li [?] and Whitehead and Zhao [?] ]) The chromatic polynomial of $G = K_4(a,b,c,d,e,f)$ is*

$$P(G,\lambda) = \tfrac{1}{\lambda^2}(-1)^m w[w^{m-1} + Q(G,w) - (w+1)(w+2)],$$

*where $w = 1-\lambda$, $m = |E(G)|$ and $Q(G,w) = -(w^{a+f+c} + w^{a+b+e} + w^{b+c+d} + w^{d+e+f} + w^{a+d} + w^{b+f} + w^{c+e}) + (1+w)(w^a + w^b + w^c + w^d + w^e + w^f)$. $Q(G,w)$ or simply $Q(G)$ is called the essential polynomial of G.*

**Lemma 2.6** *(Li [?]) Two $K_4$-homeomorphs with the same order are $\chi$-equivalent if and only if they have the same essential polynomial.*

The following are notations used in the sequel.

$L_t^+$ = set of the exponents of the positive terms in the left hand side of Equation $(t)$,
$L_t^-$ = set of the exponents of the negative terms in the left hand side of Equation $(t)$,

$R_t^+$ = set of the exponents of the positive terms in the right hand side of Equation $(t)$,
$R_t^-$ = set of the exponents of the negative terms in the right hand side of Equation $(t)$,
$-R_t$ = set of the negative terms in the right hand side of Equation $(t)$,
$R_t$ = set of the positive terms in the right hand side of Equation $(t)$,
$-L_t$ = set of the negative terms in the left hand side of Equation $(t)$,
$L_t$ = set of the positive terms in the left hand side of Equation $(t)$.

Elements can be repeated in a set.

# 3 Main result

We now present the main result in the following theorem.

**Theorem 3.1** *$K_4$-homeomorphs $K_4(a,b,2,d,2,f)$, as shown in Figure 2, where $\min\{a,b,d,f\} \geq 3$, is $\chi$-unique if and only if it is not isomorphic to $K_4(3,4,2,4,2,6)$ or $K_4(3,4,2,4,2,8)$ or $K_4(3,4,2,8,2,4)$. Moreover, each of the following sets is $\chi$-equivalence class:*
$\{K_4(3,4,2,4,2,6), K_4(3,5,4,2,2,5)\}$,
$\{K_4(3,4,2,4,2,8), K_4(3,4,2,7,5,2)\}$,
$\{K_4(3,4,2,8,2,4), K_4(3,4,2,5,7,2)\}$.

**Proof.** If there exists a graph $H$ such that $H \sim G$, by Lemmas 2.1 and 2.6, we know that $H$ is a $K_4$-homeomorph and

$$|E(G)| = |E(H)|, Q(G) = Q(H) \tag{1}$$

Let $H = K_4(i_1, j_1, k_1, l_1, m_1, n_1)$. We define $R' = \{i_1, j_1, k_1, l_1, m_1, n_1\}$ and $R'' = \{i_1+1, j_1+1, k_1+1, l_1+1, m_1+1, n_1+1\}$.

Let $G = K_4(a,b,2,d,2,f)$, where $\min\{a,b,d,f\} \geq 3$. For a clearer presentation of the proof, we shall have G of the form $K_4(i,j,2,k,2,l)$. Without loss of generality, let $i = \min\{i,j,k,l\}$, where $i \geq 3$. By Lemma 2.1(4(i)), $\min\{i_1, j_1, k_1, l_1, m_1, n_1\} = 2$ and exactly two paths among $i_1, j_1, k_1, l_1, m_1, n_1$ are of length two while the other four paths are of length greater than or equal to three. Then Equation (1) yields

$$4+i+j+k+l = i_1+j_1+k_1+l_1+m_1+n_1 \tag{2}$$

$$\begin{aligned}
&-w^{i+l+2} - w^{i+j+2} - w^{j+2+k} - w^{k+2+l} - w^{i+k} - w^{j+l} - w^4 + w^i + w^j \\
&+w^2 + w^k + w^2 + w^l + w^{i+1} + w^{j+1} + w^3 + w^{k+1} + w^3 + w^{l+1} = \\
&-w^{i_1+n_1+k_1} - w^{i_1+j_1+m_1} - w^{j_1+k_1+l_1} - w^{l_1+m_1+n_1} - w^{i_1+l_1} - w^{j_1+n_1} \\
&-w^{k_1+m_1} + w^{i_1} + w^{j_1} + w^{k_1} + w^{l_1} + w^{m_1} + w^{n_1} + w^{i_1+1} + w^{j_1+1} + \\
&w^{k_1+1} + w^{l_1+1} + w^{m_1+1} + w^{n_1+1}
\end{aligned} \tag{3}$$

where $R_3^+ = R' \cup R''$.

We shall now find non-isomorphic graphs $G$ and $H$. In Equation (3), the positive terms (resp. negative terms) in the left hand side can only be cancelled by the negative terms (resp. positive terms) in the left hand side or by the positive terms (resp. negative terms) in the right hand side. We know that the two shortest paths of $H$ are of length two, thus, $2, 2 \in R'$ and $3, 3 \in R''$, where $R_3^+ = R' \cup R''$. Consider the term $-w^4 \in -L_3$. It can easily be shown that if $-w^4 = -w^{i_1+l_1}$ or $-w^4 = -w^{j_1+n_1}$ or $-w^4 = -w^{k_1+m_1}$ then $G \cong H$. Since none of the terms in $-R_3$ can be equal to $-w^4$, where $G$ is not isomorphic to $H$, one of the terms in $L_3$ must be equal to the said term. There are six cases to be considered. Without loss of generality, we shall consider only conditions where at most two pairs of terms in $L_3$ and $-L_3$ are equal. Tables 1-2 show the possible values of exponents in $L_3^+$ in relation to the values of some exponents in $L_3^-$ for each case.

Case 1. Claim: $2 < i \leq j \leq k \leq l$.

Subcase 1.1 If $l = 4$ then with our claim $2 < i \leq j \leq k \leq l$, we have $G = K_4(4,4,2,4,2,4)$ or $G = K_4(3,4,2,4,2,4)$ or $G = K_4(3,3,2,3,2,4)$. Thus, $G$ is $\chi$−unique by Lemmas 3.3, 3.4 and 3.2. Since $H \sim G$, $H \cong G$. Under the same condition, i.e., $l = 4$, $G$ can also be of the form $K_4(3,3,2,4,2,4)$. In this case, each term in $L_3^-$ can neither be 3 nor 5 and there can be only one exponent belonging to the said set which is equal to 4. Hence, we have $\{i, j, i+1, j+1, k+1, l+1\} \subset R_3^+$, where $i = j = 3$, $i+1 = j+1 = 4$, $k+1 = l+1 = 5$, $l = 4$. By Equation (2), we have $R' = \{i, 2, 2, j, k, l\} = \{3,2,2,3,4,4\} = \{i_1, j_1, k_1, l_1, m_1, n_1\}$, where $R'' = \{4,3,3,4,5,5\} = \{i_1+1, j_1+1, k_1+1, l_1+1, m_1+1, n_1+1\}$ and $R_3^+ = R' \cup R''$. Therefore, we have $\{i, 2, 2, j, k, l\} = \{i_1, j_1, k_1, l_1, m_1, n_1\}$. By Lemma 3.1(4(ii)), $H \cong G$. Note that throughout the proof in this paper, whenever $\{i, 2, 2, j, k, l\} = \{i_1, j_1, k_1, l_1, m_1, n_1\}$ occurs, we mean that the two equivalent sets are multisets.

Subcase 1.2 If $l+1 = 4$, i.e., $l = 3$ then by our claim $2 \leq i \leq j \leq k \leq l$, $G = K_4(3,3,2,3,2,3)$. By Lemmas 3.3 and 3.4, $G$ is $\chi$−unique. Since $H \sim G$, $H \cong G$.

Subcase 1.3 If $k = 4$ then by our claim $2 < i \leq j \leq k \leq l$, $G = K_4(4,4,2,4,2,l)$ or $G = K_4(3,4,2,4,2,l)$ or $G = K_4(3,3,2,4,2,l)$, where $l > 4$ (Note that when $l = 4$ we obtain the same results as in Subcase 1.1).

Subcase 1.3.1 Suppose $G = K_4(4,4,2,4,2,l)$, where $l > 4$. Then by Lemma 3.2, $G$ is $\chi$−unique. Since $H \sim G$, $H \cong G$.

Subcase 1.3.2 Suppose $G = K_4(3,4,2,4,2,l)$, where $l > 4$. Then $g(G) = 9$. None of the exponents in $L_3^-$ can be equal to 3 or 5 and there is only one exponent in the said set equal to 4, i.e., $k = 4 \in L_3^-$. Thus, we have $\{i, i+1, j, j+1, k+1\} \subset R_3^+$. Note that $i+1 = 4$. By Equation (2), $R' = \{2, 2, i, j+1, k+1, l-2\} = \{2,2,3,5,5,l-2\} = \{i_1, j_1, k_1, l_1, m_1, n_1\}$, where $R'' = \{3,3,4,6,6,l-1\}$. By Lemma 3.1(2), $g(H) = 9$ and $H$ must have one cycle of length 9. There are two possible shortest cycles of $H$, namely, $(2,2,5)$ and when $l = 6$, $(3,2,l-2) = (3,2,4)$. Hence, we have the following non-isomorphic cases of $H$: $K_4(3,5,l-2,2,2,5)$, where l

| Subcase | $i+j+2$ | $j+2+k$ | $i+k$ | 4 |
|---|---|---|---|---|
| 1.1 | | | | $l$ |
| 1.2 | | | | $l+1$ |
| 1.3.1 | | | $l+1$ | $k$ |
| 1.3.2 | | $l+1$ | | $k$ |
| 1.3.3 | $l+1$ | | | $k$ |
| 1.4 | | | | $k+1$ |
| 1.5.1, 1.5.5 | | | $l+1$ | $j$ |
| 1.5.2, 1.5.6 | | $l+1$ | | $j$ |
| 1.5.3, 1.5.7 | $l+1$ | | | $j$ |
| 1.5.4, 1.5.8 | $k+1$ | | | $j$ |

Table 1: $2 < i \leq j \leq k \leq l$.

is at least 6; $K_4(3,5,5,2,2,l-2)$, $K_4(3,2,l-2,5,2,5)$, where $l > 6$; and $K_4(3,2,4,5,5,2)$. Let us consider $H = K_4(3,5,l-2,2,2,5)$ with $G = K_4(3,4,2,4,2,l)$. Substitute the corresponding values in Equation (3). After simplification, we have:

$$-w^7 - w^{l+4} + w^4 + w^l + w^5 + w^{l+1} = -w^{10} - w^l - w^{l-2} + w^6 + w^{l-1} + w^6.$$

Clearly, $l = 6$. Therefore, we obtain a solution, where $G \cong K_4(3,4,2,4,2,6)$ and $H \cong K_4(3,5,4,2,2,5)$.

*Subcase 1.3.3* Suppose $G = K_4(3,3,2,4,2,l)$, where $l > 4$. Then we can assume $l+1 = i+k$, i.e., $l = 6$ or $l+1 = j+2+k$, i.e., $l = 8$ or $l+1 = i+j+2$, i.e., $l = 7$. We know that $g(G) = 8$ and $G$ has only one cycle of length 8. Since $k, l+1 \in L_3^-$, we have $k+1, i, i+1, j, j+1, l, l+1, 2, 2, 3, 3 \in R_3^+$. By Equation (2), $R' = \{i, j, k+1, l-1, 2, 2\}$. By Lemma 3.1(2), we have the following non-isomorphic cases of $H$: $K_4(3,3,2,5,l-1,2)$, $K_4(3,3,2,5,2,l-1)$, $K_4(3,3,2,2,5,l-1)$. None of these cases of $H$ satisfies Equation (3) for $l = 6,7,8$. Note that the same conclusion can be deduced if this subcase is solved following the procedure in Subcase 1.3.2.

*Subcase 1.4* If $k+1 = 4$, i.e., $k = 3$ then by our claim $2 < i \leq j \leq k \leq l$, $G = K_4(3,3,2,3,2,l)$, where $l \geq 3$. By Lemmas 3.2, 3.3 and 3.4, $G$ is $\chi$-unique. Since $H \sim G$, $H \cong G$.

*Subcase 1.5* If $j = 4$ then $i = 3$ or $i = 4$. We can assume $l+1 = i+k$ or $l+1 = j+2+k$ or $l+1 = i+j+2$ or $k+1 = i+j+2$. When $j$, $l+1 \in L_3^-$, we have $j+1, l, i, i+1, k, k+1, 2, 2, 3, 3 \in R_3^+$. Thus, $R' = \{i, j+1, k, l-1, 2, 2\}$. When $j$, $k+1 \in L_3^-$, we have $j+1, l, i, i+1, k, l, l+1, 2, 2, 3, 3 \in R_3^+$. Thus, $R' = \{i, j+1, k-1, l, 2, 2\}$.

In Subcases 1.5.1 - 1.5.4, we assume $j = 4$ and $i = 3$. By our claim $2 < i \leq j \leq k \leq l$, we have $k \geq 4$. Hence, $G = K_4(3,4,2,k,2,l)$, where

$g(G) = 9$ and $G$ has only one cycle of length 9.

*Subcase 1.5.1* Let $l+1 = i+k$. Then $l = k+2$, $G = K_4(3,4,2,k,2,k+2)$, where $k \geq 4$. We have $R' = \{i, j+1, k, l-1, 2, 2\} = \{3,5,k,k+1,2,2\}$. We can consider $(5,2,2)$ for $k > 4$ or $(3,4,2)$ for $k = 4$ as the shortest cycle of $H$. The following are the non-isomorphic cases of H: $K_4(5,2,2,k,k+1,3)$, $K_4(5,2,2,3,k+1,k)$ for $k > 4$; $K_4(5,2,2,k+1,k,3)$ for $k \geq 4$; $K_4(3,4,2,5,2,5)$, $K_4(3,4,2,2,5,5)$, $K_4(3,4,2,5,5,2)$. We obtain a solution where $G \cong K_4(3,4,2,4,2,6)$ and $H \cong K_4(5,2,2,5,4,3)$, the same result we have in Subcase 1.3.2, by considering the only case here which satisfies Equation (3), that is, $H = K_4(5,2,2,k+1,k,3)$ with $G = K_4(3,4,2,k,2,k+2)$, where $k = 4$.

*Subcase 1.5.2* Let $l+1 = j+2+k$. Then $l = k+5$, $G = K_4(3,4,2,k,2,k+5)$ and $R' = \{i, j+1, k, l-1, 2, 2\} = \{3,5,k,k+4,2,2\}$, where $k \geq 4$. None of the following cases of $H$ satisfies Equation (3): $K_4(5,2,2,k,k+4,3)$, $K_4(5,2,2,3,k+4,k)$ for $k > 4$; $K_4(5,2,2,k+4,k,3)$ for $k \geq 4$; $K_4(3,4,2,5,2,8)$, $K_4(3,4,2,2,5,8)$, $K_4(3,4,2,8,5,2)$, $K_4(3,4,2,8,2,5)$.

*Subcase 1.5.3* Let $l+1 = i+j+2$. Then $l = 8$, $G = K_4(3,4,2,k,2,8)$, where $k \geq 4$. We have $R' = \{i, j+1, k, l-1, 2, 2\} = \{3,5,k,7,2,2\}$. The following are the non-isomorphic cases of H: $K_4(5,2,2,k,7,3)$, $K_4(5,2,2,3,7,k)$ for $k > 4$; $K_4(5,2,2,7,k,3)$ for $k \geq 4$; $K_4(3,4,2,5,2,7)$, $K_4(3,4,2,2,5,7)$, $K_4(3,4,2,7,5,2)$, $K_4(3,4,2,7,2,5)$. Therefore, we obtain a solution where $G \cong K_4(3,4,2,4,2,8)$ and $H \cong K_4(3,4,2,7,5,2)$.

*Subcase 1.5.4* Let $k+1 = i+j+2$. Then $k = 8$, $G = K_4(3,4,2,8,2,l)$, where $l \geq 8$. We have $R' = \{i, j+1, k-1, l, 2, 2\} = \{3,5,7,l,2,2\}$. Among the three non-isomorphic cases of $H$: $K_4(5,2,2,l,3,7)$, $K_4(5,2,2,7,l,3)$, $K_4(5,2,2,3,l,7)$, the only case of $H$ which satisfies Equation (3) is $K_4(5,2,2,3,l,7)$, where $G = K_4(3,4,2,8,2,l)$. We arrive at the following equation after substituting the corresponding values and simplifying:

$$-w^{l+5} - w^{11} - w^{l+4} + w^9 = -w^{l+7} - w^7 - w^8 - w^{l+2} + w^6.$$

Clearly, $l = 4$. But this is a contradiction to our assumption $l \geq 8$.

In the following Subcases 1.5.5 - 1.5.8, we consider $j = i = 4$. By our claim $2 \leq i \leq j \leq k \leq l$, $k \geq 4$. If $k = 4$ then by Lemma 2.2, $G$ is $\chi$-unique. Since $H \sim G$, $G \cong H$. We now consider $k > 4$. We have $G = K_4(4,4,2,k,2,l)$, $g(G) = 10$ and $G$ has only one cycle of length 10.

*Subcase 1.5.5* Let $l+1 = i+k$. Then $l = k+3$, $G = K_4(4,4,2,k,2,k+3)$ and $R' = \{i, j+1, k, l-1, 2, 2\} = \{4,5,k,k+2,2,2\}$, where $k > 4$. The only possibility to have cases of $H$ such that $g(H) = 10$ is when $k = 6$. But none of the following non-isomorphic cases of $H$ satisfies Equation (3): $K_4(6,2,2,4,5,8)$, $K_4(6,2,2,5,4,8)$, $K_4(6,2,2,8,4,5)$.

*Subcase 1.5.6* Let $l+1 = j+2+k$. Then $l = k+5$, $G = K_4(4,4,2,k,2,k+5)$ and $R' = \{i, j+1, k, l-1, 2, 2\} = \{4,5,k,k+4,2,2\}$, where $k > 4$. None of the following non-isomorphic cases of $H$ (when $k = 6$) satisfies Equation (3): $K_4(6,2,2,4,5,10)$, $K_4(6,2,2,5,4,10)$, $K_4(6,2,2,10,4,5)$.

*Subcase 1.5.7* Let $l+1 = i+j+2$. Then $l = 9$, $G = K_4(4,4,2,k,2,9)$ and $R' = \{i, j+1, k, l-1, 2, 2\} = \{4,5,k,8,2,2\}$, where $k > 4$. None the following non-isomorphic cases of $H$ (when $k = 6$) satisfies Equation (3): $K_4(6,2,2,4,5,8)$, $K_4(6,2,2,5,4,8)$, $K_4(6,2,2,8,4,5)$.

| Subcase | $i+l+2$ | $i+j+2$ | $j+l$ | 4 |
|---|---|---|---|---|
| 2.1 | | | | $k$ |
| 2.2 | | | | $k+1$ |
| 2.3 | | | | $l$ |
| 2.4 | | | | $l+1$ |
| 2.5.1, 2.5.5 | | | $k+1$ | $j$ |
| 2.5.2, 2.5.6 | | $k+1$ | | $j$ |
| 2.5.3, 2.5.7 | $k+1$ | | | $j$ |
| 2.5.4, 2.5.8 | | $l+1$ | | $j$ |

Table 2: $2 < i \leq j \leq l < k$.

*Subcase 1.5.8* Let $k+1 = i+j+2$. Then $k = 9$, $G = K_4(4,4,2,9,2,l)$ and $R' = \{i, j+1, k-1, l, 2, 2\} = \{4, 5, 8, l, 2, 2\}$, where $l \geq 9$. Thus, $g(H) \neq 10$.

*Case 2.* Claim: $2 < i \leq j \leq l < k$.

*Subcase 2.1* If $k = 4$ then $l = j = i = 3$, i.e., $G = K_4(3,3,2,4,2,3)$. By Lemma 3.2, $G$ is $\chi$−unique. Since $H \sim G$, $G \cong H$.

*Subcase 2.2* If $k+1 = 4$ then $k = 3, i = j = l = 2$. This contradicts our claim $2 < i \leq j \leq l < k$.

*Subcase 2.3* If $l = 4$ then by our claim $2 < i \leq j \leq l < k$, $G = K_4(4,4,2,k,2,4)$ or $G = K_4(3,4,2,k,2,4)$ or $G = K_4(3,3,2,k,2,4)$, where $k > 4$. Since $l = 4$, i.e., $l \in L_3^-$ and no other elements of $L_3^-$ can be equal to 3 or 4 or 5, we have $l+1, i, i+1, j, j+1, 2, 2, 3, 3 \in R_3^+$. So, $R' = \{l+1, i, j, k-1, 2, 2\}$.

*Subcase 2.3.1* Let $G = K_4(4,4,2,k,2,4)$. Then by Lemma 3.2, $G$ is *chi*−unique. Since $H \sim G$, $G \cong H$.

*Subcase 2.3.2* Let $G = K_4(3,4,2,k,2,4)$. Then $R' = \{l+1, i, j, k-1, 2, 2\} = \{5, 3, 4, k-1, 2, 2\}$, where $k > 4$, $g(G) = 9$ and $G$ has two cycles of length 9. With $(3,4,2)$ and $(5,2,2)$ as the shortest cycles of $H$, we have the following non-isomorphic cases for $H$: $K_4(3,4,2,5,k-1,2)$, where $k > 5$; $K_4(3,4,2,2,k-1,5)$, where $k \geq 5$. When $k = 5$, $(k-1,3,2) = (4,3,2)$ can be one of the two shortest cycles of $H$, thus, $H$ can be $K_4(4,3,2,5,4,2)$ or $K_4(4,3,2,4,2,5)$. However, if $k = 6$ then $(k-1,2,2) = (5,2,2)$ can be one of the two shortest cycles of $H$, thus, $H$ can be $K_4(5,2,2,4,5,3)$ or $K_4(5,2,2,3,5,4)$. Consider $H = K_4(3,4,2,5,k-1,2)$ with $G = K_4(3,4,2,k,2,4)$. Substitute the corresponding values in Equation (3). After simplification, we have:

$$-w^9 - w^9 - w^{k+3} + w^{k+1} = -w^7 - w^{11} - w^{k+1} + w^{k-1}.$$

Clearly, $k = 8$. Therefore, we obtain a solution where $G \cong K_4(3,4,2,8,2,4)$ and $H \cong K_4(3,4,2,5,7,2)$.

*Subcase* 2.3.3 Let $G = K_4(3,3,2,k,2,4)$. Then $R' = \{l+1, i, j, k-1, 2, 2\} = \{5,3,3,k-1,2,2\}$, where $k > 4$, $g(G) = 8$ and $G$ has only one cycle of length 8. Considering $(3,3,2)$ as the shortest cycle of $H$, we have the following non-isomorphic cases of $H$: $K_4(3,3,2,5,k-1,2)$, $K_4(3,3,2,5,2,k-1)$, $K_4(3,3,2,2,k-1,5)$, $K_4(3,3,2,k-1,2,5)$, where $k > 4$; $K_4(3,3,2,2,5,k-1)$, $K_4(3,3,2,k-1,5,2)$, where $k > 5$. None of these cases of $H$ satisfies Equation (3). If $k = 5$ then $(k-1,2,2) = (4,2,2)$ can be the shortest cycle of $H$. In such condition, $\{i,j,k,l,2,2\} = \{i_1+1, j_1+1, k_1+1, l_1+1, m_1+1, n_1+1\}$. By Lemma 3.1(4(ii)), $G \cong H$.

*Subcase* 2.4 If $l+1 = 4$ then $l = i = j = l = 3$, where $k > 3$. By Lemma 3.2, $G$ is $\chi$−unique. Since $H \sim G$, $H \cong G$.

*Subcase* 2.5 If $j = 4$ then $i = 3$ or $i = 4$. We can assume $k+1 = j+l$ or $k+1 = i+j+2$ or $k+1 = i+l+2$ or $l+1 = i+j+2$. If $j, k+1 \in L_3^-$ then $j+1, k, i, i+1, l, l+1, 2, 2 \in R_3^+$. Thus, $R' = \{i, j+1, k-1, l, 2, 2\}$. If $j, l+1 \in L_3^-$ then $j, j+1, k, i, i+1, l, 2, 2 \in R_3^+$. Thus, $R' = \{i, j+1, k, l-1, 2, 2\}$. In Subcases 2.5.1-2.5.4, we consider $j = 4$, $i = 3$, $G = K_4(3,4,2,k,2,l)$, where $l \geq 4$ and $k > l$. Thus, $g(G) = 9$. If $l = 4$ then $G$ has two cycles of length 9. If $l > 4$ then $G$ has only one cycle of length 9.

*Subcase* 2.5.1 Let $k+1 = j+l$. Then $k = l+3$, $G = K_4(3,4,2,l+3,2,l)$ and $R' = \{i, j+1, k-1, l, 2, 2\} = \{3,5,l+2,l,2,2\}$. We have the following non-isomorphic cases of $H$: $K_4(5,2,2,3,l,l+2)$, $K_4(5,2,2,l,3,l+2)$, where $l \geq 4$; $K_4(5,2,2,l+2,3,l)$, where $l > 4$. Equation (3) is not satisfied by any of these cases of $H$ for all values of $l \geq 4$.

*Subcase* 2.5.2 Let $k+1 = i+j+2$. Then $k = 8$, $G = K_4(3,4,2,8,2,l)$ and $R' = \{i, j+1, k-1, l, 2, 2\} = \{3,5,7,l,2,2\}$. $H$ can be $K_4(5,2,2,l,3,7)$ or $K_4(5,2,2,3,l,7)$ or $K_4(5,2,2,7,3,l)$. Consider $H = K_4(5,2,2,3,l,7)$ with $G = K_4(3,4,2,8,2,l)$. Using Equation (3) we get the following after simplification:

$$-w^{l+5} - w^{11} - w^{l+4} + w^8 + w^9 = -w^{l+7} - w^{l+2} + w^6.$$

Evidently, Equation (3) is satisfied for the value of $l = 4$. Therefore, we obtain a solution where $G \cong K_4(3,4,2,8,2,4)$ and $H \cong K_4(5,2,2,3,4,7)$, the same result we have in Subcase 2.3.2.

*Subcase* 2.5.3 Let $k+1 = i+l+2$. Then $k = l+4$, $G = K_4(3,4,2,l+4,2,l)$ and $R' = \{i, j+1, k-1, l, 2, 2\} = \{3,5,l+3,l,2,2\}$. We have the following non-isomorphic cases for $H$: $K_4(5,2,2,l,3,l+3)$, $K_4(5,2,2,3,l,l+3)$, $K_4(5,2,2,l+3,3,l)$. Consider $H = K_4(5,2,2,3,l,l+3)$ with $G = K_4(3,4,2,l+4,2,l)$. Using Equation (3) we get the following after simplification:

$$-w^9 - w^{l+4} = -w^7 - w^8 - w^{l+5} - w^{l+2} + w^{l+3} + w^6.$$

Clearly, Equation (3) is satisfied for the value of $l = 4$. Therefore, we obtain a solution where $G \cong K_4(3,4,2,8,2,4)$ and $H \cong K_4(5,2,2,3,4,7)$, the same result we have in Subcases 2.3.2 and 2.5.2.

*Subcase* 2.5.4 Let $l+1 = i+j+2$. Then $l = 8$, $G = K_4(3,4,2,k,2,8)$ and $R' = \{i, j+1, k, l-1, 2, 2\} = \{3,5,k,7,2,2\}$, where $k > 8$. The non-isomorphic cases of $H$ are as follows: $K_4(5,2,2,k,3,7)$, $K_4(5,2,2,3,7,k)$, $K_4(5,2,2,7,3,k)$. Equation (3) is not satisfied by any of these cases of $H$.

In Subcases 2.5.5-2.5.8, we assume $j = 4$ and $i = 4$. Thus, $G = K_4(4,4,2,k,2,l)$, where $l \geq 4$ and $k > l$. We have $g(G) = 10$. If $l = 4$ then $G$ is $\chi-unique$ by Lemma 3.2. Let us now consider $l > 4$. $G$ has only one cycle of length 10.

<u>*Subcase* 2.5.5</u> Let $k+1 = j+l$. Then $k = l+3$, $G = K_4(4,4,2,l+3,2,l)$ and $R' = \{i, j+1, k-1, l, 2, 2\} = \{4, 5, l+2, l, 2, 2\}$, where $l > 4$. There are no possible cases of $H$ such that $g(H)$=10.

<u>*Subcase* 2.5.6</u> Let $k+1 = i+j+2$. Then $k = 9$, $G = K_4(4,4,2,9,2,l)$ and $R' = \{i, j+1, k-1, l, 2, 2\} = \{4, 5, 8, l, 2, 2\}$, where $l > 4$. Thus, $g(H) \neq 10$.

<u>*Subcase* 2.5.7</u> Let $k+1 = i+l+2$. Then $k = l+5$, $G = K_4(4,4,2,l+5,2,l)$ and $R' = \{i, j+1, k-1, l, 2, 2\} = \{4, 5, l+4, l, 2, 2\}$, where $l > 4$. There are no possible cases of $H$ such that $g(H)$=10.

<u>*Subcase* 2.5.8</u> Let $l+1 = i+j+2$. Then $l = 9$, $G = K_4(4,4,2,k,2,9)$ and $R' = \{i, j+1, k, l-1, 2, 2\} = \{4, 5, k, 8, 2, 2\}$, where $k > 9$. It is not possible for $H$ to have girth 10.

We can follow the same procedure, as shown in Cases 1-2, for the remaining four cases, that is, when $2 < i \leq k \leq j \leq l$, $2 < i \leq k \leq l < j$, $2 < i \leq l \leq j \leq k$, and $2 < i \leq l \leq k < j$, respectively. We note that all solutions are gained from the first two cases and are repeatedly obtained in the next four cases. The reader may refer to [?] for the details of the proof in Cases 3-6. We now summarize our results from what we have shown in this paper and from our technical report [?] as indicated in each of the following equivalence classes:

$$K_4(3,4,2,4,2,6) \sim K_4(3,5,4,2,2,5)$$
(Cases 1,3,4,6/Subcases 1.3.2,1.5.1,3.3.2(a),4.3.2(a),4.5.1(a),6.3.2(a),6.5.1);

$$K_4(3,4,2,4,2,8) \sim K_4(3,2,4,7,2,5)$$
(Cases 1,3,4/Subcases 1.5.2,3.3.2(c),3.5.4,4.3.2(c));

$$K_4(3,4,2,8,2,4) \sim K_4(5,2,2,3,7,4)$$
(Cases 2,5/Subcases 2.3.2,2.5.2,2.5.3,5.3.2(b),5.3.2(c),5.5.2).

At this point, we have shown that $K_4$-homeomorphs $K_4(a,b,2,d,2,f)$ where $\min\{a,b,d,f\} \geq 3$, is $\chi$-unique if and only if it is not isomorphic to $K_4(3,4,2,4,2,6)$ or $K_4(3,4,2,4,2,8)$ or $K_4(3,4,2,8,2,4)$.
The proof of Theorem 3.1 is now complete. □

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