WEIGHTED COMPOSITION OPERATORS ON MUSIELAK-ORLICZ SPACES

KULDIP RAJ, AJAY K. SHARMA AND ANIL KUMAR

ABSTRACT. The compact, Fredholm and isometric weighted composition operators are characterized in this paper.

1. Introduction and Preliminaries

Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the set of reals, non-negative reals and the set of natural numbers respectively. Let (G, Σ, μ) be a σ-finite measure space. Denote by $L^0 = L^0(G, \Sigma, \mu)$ the set of all μ-equivalence classes of complex-valued measurasble functions defined on G. A function $M : G \times \mathbb{R} \to [0, \infty)$ is said to be Musielak-Orlicz function if $M(., u)$ is measurable for each $u \in \mathbb{R}$, $M(t, u) = 0$ if and only if $u = 0$ and $M(t, .)$ is convex, even, not identically equal zero and $\frac{M(t,u)}{u} \to 0$ as $u \to 0$ for μ-a.e. $t \in G$. Define on L^0 a convex modular ϱ_M by $\varrho_M(f) = \int_G M(t, f(t))d\mu$ for every $f \in L^0$. By the Musielak-Orlicz space L_M we mean

$$L_M = \Big\{ f \in L^0 : \varrho_M(\lambda|f|) < \infty \ \text{ for some } \ \lambda > 0 \Big\}.$$

Its subspace E_M is defined as

$$E_M = \Big\{ f \in L^0 : \varrho_M(\lambda|f|) < \infty \ \text{ for any } \ \lambda > 0 \Big\}.$$

The space L_M equipped with the Luxemberg norm

$$||f||_M = \inf\Big\{\lambda > 0 : \varrho_M(\frac{|f|}{\lambda}) \leq 1\Big\}$$

is a Banach space (see [10], [11]). For every Musielak-Orlicz function M we define complementary function $M^*(t, v)$ as

$$M^*(t, v) = \sup_{u>0}\Big\{u|v| - M(t, u) : v \geq 0 \text{ and } t \in G \text{ } a.e.\Big\}.$$

It is easy to see that $M^*(t, v)$ is also Musielak-Orlicz function. We say that Musielak-Orlicz function M satisfies Δ_2-condition (write $M \in \Delta_2$) if there

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exists a constant $k > 2$ and a measurable non-negative function f such that $\varrho_M(f) < \infty$ and

$$M(t, 2u) \leq kM(t, u)$$

for every $u \geq f(t)$ and for $t \in G$ a.e. . For more details see ([4],[7],[9],[14]). Throughout this paper we assumed that M satisfies Δ_2-conditions.
If T is a non-singular measurable transformation, then the measure μT^{-1} is absolutely continuous with respect to the measure μ. Hence by Radon Nikodym derivative theorem there exists a positive measurable function f_0 such that $\mu(T^{-1}(E)) = \int_E f_0 d\mu$ for every $E \in \Sigma$. The function f_0 is called the Radon Nikodym derivative of the measure μT^{-1} with respect to the measure μ. It is denoted by $f_0 = \frac{d\mu T^{-1}}{d\mu}$.
Associated with each σ-finite subalgebra $\Sigma_0 \subset \Sigma$, there exists an operator $E = E^{\Sigma_0}$, which is called conditional expectation operator, on the set of all non-negative measurable functions f or for each $f \in L^0(G, \Sigma, \mu)$, and is uniquely determined by the following conditions:

(1) $E(f)$ is Σ_0-measurable, and
(2) if A is any Σ_0-measurable set for which $\int_A f d\mu$ exists, we have $\int_A f d\mu = \int_A E(f) d\mu$.

The expectation operator E has the following properties :

- $E(f.goT) = E(f).(goT)$
- if $f \geq g$ almost everywhere, then $E(f) \geq E(g)$ almost everywhere
- $E(1) = 1$
- $E(f)$ has the form $E(f) = goT$ for exactly one σ- measurable function g.
  In particular $g = E(f)oT^{-1}$ is a well defined measurable function.
- $|E(fg)|^2 \leq (E|f|^2)(E|g|^2)$. This is a Cauchy Schwartz inequality for conditional expectation.
- For $f > 0$ almost everywhere, $E(f) > 0$ almost everywhere.
- If ϕ is a convex function, then $\phi(E(f)) \leq E(\phi(f))$ μ-almost everywhere. For deeper study of properties of E see [8].

Let $w : G \to \mathbb{C}$ be a measurable function and $T : G \to G$ be a non-singular measurable transformation. Then a bounded linear transformation, $S_{w,T} : L_M \to L_M$ defined by

$$(S_{w,T}f)(t) = w(t)f(T(t)),$$

for every $t \in G$ and for every $f \in L_M$, is called a weighted composition operator induced by the pair (w, T) . If we take $w(t) = 1$, the constant one function on G, we write $S_{w,T}$ as C_T and call it a composition operator induced by T. In case $T(t) = t$ for some $t \in G$, we write $S_{w,T}$ as M_w and call it a multiplication operator induced by w.
By $N^{|f_0|}(w, \epsilon)$ we means the set $\Big\{t \in G : E\big[M\big(I \circ T^{-1}(t), w(t)|y|\big)\big] f_0(t) \geq$

$M(t, \epsilon|y|)$ for $y \in \mathbb{C}\Big\}$. The set $\overline{\{t \in G : w(t) \neq 0\}}$ is called support of w and we shall write it as supp$w$. By $B(L_M)$, we denote the set of all bounded linear operator L_M into itself.
The study of compact weighted composition operators on L^p-spaces ($1 \leq p < \infty$) initiated by Takagi [16] in 1992. He also determined the spectra of these operators. In 1993, Takagi [17] characterized the weighted composition operators on $C(X)$ and proved that a weighted composition operator is Fredholm if and only if it is invertible. The same equivalence is true for weighted composition operator on $L^p(\mu)$ spaces $1 \leq p < \infty$, where μ is non-atomic measure. In 1994, Campbell and Hornor [1] used a localized conditional expectation operator to characterized subnormality for the adjoint of a weighted composition operator. In 1996, Hornor and Jamison [5] investigated the criteria for the hypernormality, cohyponormality and normality of weighted composition operators acting on Hilbert spaces of vector-valued functions. For more details on composition and weighted composition operators see ([2], [3], [6], [12], [13], [15], [18]) and references therein.
The main purpose of this paper is to characterize the boundedness, compactness, invertibility, Fredholmness and isometry of the weighted composition operators defined on the Musielak-Orlicz function spaces.

2. Weighted composition Operators

The main purpose of this section is to characterize boundedness and compactness of weighted substitution operator on Musielak-Orlicz spaces. Before proving the compactness of weighted substitution operator, we have also given the necessary and sufficient condition for a weighted substitution to be bounded away from zero.

Theorem 2.1. *Let $w : G \to \mathbb{C}$ and $T : G \to G$ be two mappings. Then $S_{w,T} : L_M \to L_M$ is a bounded operator if and only if there exists $K > 0$ such that*

$$E\Big[M\big(I \circ T^{-1}(t)\big), w(t)|y|\Big] f_0(t) \leq M(t, K|y|) \tag{2.1}$$

for every $y \in \mathbb{C}$ and for μ-almost all $t \in G$.

Proof. Suppose the condition (2.1) is true. Then for every $f \in L_M$, we have

$$\begin{aligned}
\int_G M\Big(t, \frac{|(S_{w,T}f)(t)|}{K||f||}\Big) d\mu(t) &= \int_G M\Big(t, \frac{|w(T(t))f(T(t))|}{K||f||}\Big) d\mu(t) \\
&= \int_G E\Big[M\Big(I \circ T^{-1}(t), \frac{|w(t)f(t)|}{K||f||}\Big)\Big] f_0(t) d\mu(t) \\
&\leq \int_G M\Big(t, \frac{|f(t)|}{||f||}\Big) d\mu(t) \leq 1.
\end{aligned}$$

Therefore

$$||S_{w,T}f|| \le K||f||.$$

Conversely, suppose that the conditions (2.1) is not true. Then for every positive integer k, there exists a measurable set $G_k \subset G$ and some $y_k \in \mathbb{C}$ such that

$$E\big[M(I \circ T^{-1}(t), w(t)|y_k|)\big] f_0(t) \ge M(t, K|y_k|)$$

for μ-almost every $t \in G_k$. Choose a measurable subset F_k of G_k such that $\chi_{F_k} \in L_M$. Let $f_k = y_k \chi_{F_k}$. Then

$$\begin{aligned}
\int_G M\Big(t, \frac{|Kf_k(t)|}{||S_{w,T}f_k||}\Big) d\mu(t) &= \int_{F_k} M\Big(t, \frac{|K(y_k)|}{||S_{w,T}f_k||}\Big) d\mu(t) \\
&\le \int_{F_k} E\Big[M\Big(I \circ T^{-1}(t), \frac{|w(t)(y_k)|}{||S_{w,T}f_k||}\Big)\Big] f_0(t) d\mu(t) \\
&= \int_G E\Big[M\Big(I \circ T^{-1}(t), \frac{|w(t)f_k|}{||S_{w,T}f_k||}\Big)\Big] f_0(t) d\mu(t) \\
&= \int_G M\Big(t, \frac{|(S_{w,T}f_k)(t)|}{||S_{w,T}f_k||}\Big) d\mu(t) \\
&\le 1.
\end{aligned}$$

Thus

$$||S_{w,T}f_k|| \ge K||f_k||.$$

This contradicts the boundedness of $S_{w,T}$. Hence the condition (2.1) must be true.

Theorem 2.2. *Let $w : G \to \mathbb{C}$ be a measurable function and $T : G \to G$ be a measurable transformation. Then $S_{w,T} : L_M \to L_M$ is bounded away from zero if and only if*

$$E\Big[M\Big(I \circ T^{-1}(t), w(t)|y|\Big)\Big] f_0(t) \ge M(t, \delta|y|) \tag{2.2}$$

for each $t \in G$ and $y \in \mathbb{C}$.

Proof. We first suppose that the condition (2.2) is true. Then for every $f \in L_M$, we have

$$\begin{aligned}
\int_G M\Big(t, \frac{|\delta f(t)|}{||S_{w,T}f||}\Big) d\mu(t) &\le \int_G E\Big[M\Big(I \circ T^{-1}(t), \frac{|w(t)f(t)|}{||S_{w,T}f||}\Big)\Big] d\mu(t) \\
&= \int_G M\Big(t, \frac{|(S_{w,T}f)(t)|}{||S_{w,T}f||}\Big) d\mu(t) \\
&\le 1.
\end{aligned}$$

Therefore $||S_{w,T}f|| \ge \delta||f||$ for all $f \in L_M$. This shows that $S_{w,T}$ is bounded away from zero.

Conversely, suppose that the condition (2.2) is not true. Then for every integer k, there exists $y_k \in \mathbb{C}$ and a measurable subset G_k such that

$$E\Big[M(I \circ T^{-1}(t), w(t)|y_k|)\Big] f_0(t) \le M\Big(t, \frac{1}{k}|y_k|\Big).$$

Choose a measurable subset F_k of G_k such that $\chi_{F_k} \in L_M$. Let $f_k = k\chi_{F_k}$. Then

$$\begin{aligned}\int_G M\Big(t, \frac{|k(S_{w,T}f_k)(t)|}{||f_k||}\Big) d\mu(t) &= \int_{F_k} E\Big[M\Big(I \circ T^{-1}(t), \frac{|kw(t)(y_k)|}{||f_k||}\Big)\Big] d\mu(t) \\ &< \int_{F_k} M\Big(t, \frac{|y_k|}{||f_k||}\Big) d\mu(t) \\ &= \int_G M\Big(t, \frac{|f_k(t)|}{||f_k||}\Big) d\mu(t) \le 1.\end{aligned}$$

Hence

$$||S_{w,T}f_k|| \le \frac{1}{k}||f_k||,$$

which shows that $S_{w,T}$ is not bounded away from zero. Hence the condition of the theorem must be true.

Theorem 2.3. *Suppose $S_{w,T} \in B(L_M)$. Then $S_{w,T}$ is compact if and only if the space $L_M\Big(N^{|f_0|}(w, \epsilon)\Big)$ is finite dimensional for each $\epsilon > 0$, where*

$$L_M(N^{|f_0|}(w, \epsilon)) = \Big\{f \in L_M : f(t) = 0, \forall\ t \notin N^{|f_0|}(w, \epsilon)\Big\}.$$

Proof. We first suppose that the space $L_M\Big(N^{|f_0|}(w, \frac{1}{n})\Big)$ is finite dimensional for each $n = 1, 2, 3, \cdots$. Define

$$w_n(t) = \begin{cases} w(t), & \text{if} \quad t \in N^{|f_0|}(w, \frac{1}{n}) \\ 0, & \text{if} \quad t \notin N^{|f_0|}(w, \frac{1}{n}) \end{cases}.$$

Then $S_{w_n,T}$ is a compact operator for each $n \in \mathbb{N}$. Then

$$\begin{aligned}&\int_G M\Big(t, \frac{|n(S_{w_n,T} - S_{w,T})f(t)|}{||f||}\Big) d\mu(t) \\ &= \int_G E\Big[M\Big(I \circ T^{-1}(t), \frac{|n(w_n(t) - w(t))f(t)|}{||f||}\Big)\Big] f_0(t) d\mu(t) \\ &= \int_{(N^{|f_0|}(w, \frac{1}{n}))'} E\Big[M\Big(I \circ T^{-1}(t), \frac{|nw(t)f(t)|}{||f||}\Big)\Big] f_0(t) d\mu(t) \\ &< \int_{(N^{|f_0|}(w, \frac{1}{n}))'} M\Big(t, \frac{|f(t)|}{||f||}\Big) d\mu(t) \\ &\le \int_G M\Big(t, \frac{|f(t)|}{||f||}\Big) d\mu(t) \le 1.\end{aligned}$$

Thus

$$||(S_{w_n,T} - S_{w,T})f|| \leq \frac{1}{n}||f|| \to 0 \text{ as } n \to \infty.$$

Hence $S_{w,T}$ is a compact operator.

Conversely, suppose that $L_M\left(N^{|f_0|}(w,\epsilon)\right)$ is an infinite dimensional for some $\epsilon > 0$. Then the closed unit ball of $L_M\left(N^{|f_0|}(w,\epsilon)\right)$ is not compact. Therefore there exists a bounded sequence $\{f_n\}$ in the closed unit ball of $L_M\left(N^{|f_0|}(w,\epsilon)\right)$ such that it has a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ for which $||f_{n_k} - f_{n_j}|| \geq \delta$ for some $\delta > 0$. Now

$$\begin{aligned}
1 &\geq \int_G M\Big(t, \frac{|(S_{w,T}f_{n_k} - S_{w,T}f_{n_j})(t)|}{\epsilon||S_{w,T}f_{n_k} - S_{w,T}f_{n_j}||}\Big)d\mu(t) \\
&= \int_G E\Big[M\Big(I \circ T^{-1}(t), \frac{|w(t)f_{n_k}(t) - w(t)f_{n_j}(t)|}{||S_{w,T}f_{n_k} - S_{w,T}f_{n_j}||}\Big)\Big]f_0(t)d\mu(t) \\
&> \int_{N^{|f_0|}(w,\epsilon)} M\Big(t, \frac{|\epsilon(f_{n_k}(t) - f_{n_j}(t))|}{||S_{w,T}f_{n_k} - S_{w,T}f_{n_j}||}\Big)d\mu(t) \\
&= \int_G M\Big(t, \frac{|\epsilon(f_{n_k}(t) - f_{n_j}(t))|}{||S_{w,T}f_{n_k} - S_{w,T}f_{n_j}||}\Big)d\mu(t).
\end{aligned}$$

Hence

$$||S_{w,T}f_{n_k} - S_{w,T}f_{n_j}|| \geq \epsilon||f_{n_k} - f_{n_j}|| \geq \epsilon\delta.$$

This proves that $\{S_{w,T}f_n\}$ cannot have convergent subsequence. Therefore $S_{w,T}$ is not compact. Hence the space $L_M\left(N^{|f_0|}(w,\epsilon)\right)$ must be finite dimensional.

Corollary 2.4. *Let $S_{w,T} \in B(L_M)$. Then $S_{w,T}$ is compact if and only if $S_{w,T} = 0$.*

3. Invertible and Isometric Weighted substitution Operators on Musielak-Orlicz spaces

In this section we investigate a necessary and sufficient condition for a weighted composition operator to be invertible and then we make the use of it to characterize Fredholm weighted composition operators. We also make an effort to characterize the isometric weighted composition operator on Musielak-Orlicz spaces.

Theorem 3.1. *Let $S_{w,T} \in B(L_M)$. Then $S_{w,T}$ is invertible if and only if*
(i) T is invertible and
(ii) there exists $\delta > 0$ such that

$$E\Big[M(I \circ T^{-1}(t), w(t)|y|)\Big]f_0(t) \geq M(t, \delta|y|)$$

for μ-almost all $t \in G$ and $y \in \mathbb{C}$.

Proof. Suppose $S_{w,T}$ is invertible. Then clearly T is surjective. If T is not surjective, we can find a measurable subset $F \subset G \setminus T(G)$ for which $\chi_F \in L_M$. We see that $S_{w,T}\chi_F = 0$ which proves that $S_{w,T}$ has non-trivial kernel. Hence T must be surjective. Similarly if T is not injective, then $S_{w,T}$ has not dense range. Since by Hahn-Banach theorem there exists $0 \neq g* \in (L_M)^*$ such that $g^*(S_{w,T}f) = 0$ for all $f \in L_M$. Now by the Representation theorem for linear functionals there exists $g \in L_{M^*}$ such that $g^*(h) = \int h.gd\mu$. Thus

$$(S^*_{w,T}g^*)(f) = g^*(S_{w,T}f) = 0.$$

This proves in view of

$$(\ker S_{w,T})^* = (\overline{\text{Ran} S_{w,T}})^{\perp} \neq \{0\}$$

that $\text{Ran}S_{w,T}$ is not dense. This contradicts that $S_{w,T}$ has dense range. Hence T must be injective. Thus T is invertible. Also $S_{w,T}$ is bounded away from zero. Therefore the condition (2.2) is satisfied.
Conversely, if the condition (i) and (ii) hold, then $S_{w,T}$ is bounded away from zero and has dense range. Hence $S_{w,T}$ is invertible.

Theorem 3.2. *Let $S_{w,T} \in B(L_M)$. Then $S_{w,T}$ is Fredholm if and only if $S_{w,T}$ is invertible.*

Proof. If $S_{w,T}$ is invertible, then clearly $S_{w,T}$ is Fredholm. Conversely, suppose that $S_{w,T}$is Fredholm. Then $\ker S_{w,T}$ is finite dimensional. We know that $\ker S_{w,T}$ is either zero dimensional or infinite dimensional. Hence $\ker S_{w,T} = 0$ which shows that $w \circ T \neq 0$ and T is surjective. Next, if $(\overline{\text{Ran} S_{w,T}})^{\perp} \neq \{0\}$, then there exists a bounded linear functional $0 \neq g^* \in (L_M)^*$ such that $g^*(S_{w,T}f) = 0$ that is $(S^*_{w,T}g^*)(f) = 0$. By the Representation theorem for functionals there exists $g \in L_{M^*}$ such that $g^*(S_{w,T}f) = \int S_{w,T}f.gd\mu = 0$. Let $F = \text{supp}\, g$ and $\{F_n\}$ be a sequence of disjoint measurable sets such that $\cup_{n=1}^{\infty} F_n = F$ and $\chi_{F_n} \in L_M$. Take $g^*_n = g^*\chi_{F_n}$. Clearly $S^*_{w,T}g^*_n = 0$ for all $n = 1, 2, 3, \cdots$. This proves that $\text{Ran}S_{w,T}$ is infinite co-dimensional, which is a contradiction. Hence $(\overline{\text{Ran} S_{w,T}})^{\perp} = \{0\}$ i.e. $\text{Ran}S_{w,T}$ is dense. Since $\text{Ran}S_{w,T}$ is closed. Therefore there exists $\delta > 0$ such that

$$E\Big[M(I \circ T^{-1}(t), w(t)|y|)\Big] f_0(t) \geq M(t, \delta|y|)$$

for μ-almost all $t \in G$ and for all $y \in \mathbb{C}$. This proves that $S_{w,T}$ is invertible.

Theorem 3.3. *Let $S_{w,T} \in B(L_M)$ and $E\Big[M(I \circ T^{-1}(t), w(t)|y|)\Big] f_0(t) = M(t, |y|)$ for μ-almost all $t \in G$ and $y \in \mathbb{C}$. Then $S_{w,T}$ is an isometry if and only if $|\theta| = 1$ a.e. .*

Proof. Suppose $|\theta(t)| = 1$ for μ-almost all $t \in G$. Then $f \in L_M$, we have

$$\begin{aligned}\int_G M\Big(t, \frac{|(S_{w,T}f)(t)|}{||f||}\Big) d\mu(t) &= \int_G M\Big(t, \frac{|w(t)f(t)|}{||f||}\Big) d\mu(t) \\ &= \int_G E\Big[\Big(I \circ T^{-1}(t), \frac{|w(t)f(t)|}{||f||}\Big)\Big] f_0(t) d\mu(t) \\ &= \int_G M\Big(t, \frac{|f(t)|}{||f||}\Big) d\mu(t) \leq 1.\end{aligned}$$

Therefore

$$(3.1) \qquad ||S_{w,T}f|| \leq ||f||.$$

Again

$$\begin{aligned}\int_G M\Big(t, \frac{|f(t)|}{||S_{w,T}f||}\Big) d\mu(t) &= \int_G M\Big(t, \frac{|w(t)f(t)|}{||S_{w,T}f||}\Big) d\mu(t) \\ &= \int_G E\Big[\Big(I \circ T^{-1}(t), \frac{|w(t)f(t)|}{||S_{w,T}f||}\Big)\Big] f_0(t) d\mu(t) \\ &= \int_G M\Big(t, \frac{|(S_{w,T}f)(t)|}{||S_{w,T}f||}\Big) d\mu(t) \leq 1.\end{aligned}$$

Therefore

$$(3.2) \qquad ||S_{w,T}f|| \geq ||f||.$$

Hence

$$||S_{w,T}f|| = ||f||.$$

This proves that $S_{w,T}$ is an isometry.
Conversely, suppose that the condition of the theorem is not true. Then Case I : $|\{t \in G : |\theta(t) < 1\}| > 0$ implies that there is $\epsilon > 0$ such that the set $F = \{t \in G : |\theta(t) < 1 - \epsilon\}$ is of positive measure for some $\epsilon > 0$. We can choose a subset A of F such that $\chi_A \in L_M$. Now

$$\begin{aligned}&\textstyle\int_G M\Big(t, \frac{|(S_{w,T}\chi_A)(t)|}{(1-\epsilon)||\chi_A||}\Big) d\mu(t) \\ &\qquad = \int_A E\Big[M\Big(I \circ T^{-1}(t), \frac{|w(t)\chi_A(t)|}{(1-\epsilon)||\chi_A||}\Big)\Big] f_0(t) d\mu(t) \\ &\qquad = \int_G M\Big(t, \frac{|w(t)\chi_A(t)|}{(1-\epsilon)||\chi_A||}\Big) d\mu(t) \\ &\qquad \leq 1.\end{aligned}$$

Therefore

$$||S_{w,T}\chi_A|| \leq (1-\epsilon)||\chi_A||$$

which is a contradiction.
Case II : $|\{t \in G : |\theta(t) > 1\}| > 0$. It is similar as in the case I.

Thus, the condition of the theorem must be true, i.e. $|\theta(t)| = 1$ a.e.

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School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J&K, India

E-mail address: kuldeepraj68@rediffmail.com

E-mail address: kuldipraj68@gmail.com

E-mail address: aksju_76@yahoo.com