An implicit degree Dirac condition for hamiltonian cycles*

Huawei Dai[a], Junqing Cai[b†]

a. Department of Mathematics, Huizhou University,
Huizhou 516007, P.R. China

b. School of Management, Qufu Normal University,
Rizhao, 276826, P.R. China

E-mails: dhwee@hzu.edu.cn, caijq09@163.com

Abstract

In 1989, Zhu, Li and Deng introduced the definition of implicit degree, denoted by $id(v)$, of a vertex v in a graph G and they obtained sufficient conditions for a graph to be hamiltonian with the implicit degrees. In this paper, we prove that if G is a 2-connected graph of order n with $\alpha(G) \le n/2$ such that $id(v) \ge (n-1)/2$ for each vertex v of G, then G is hamiltonian with some exceptions.

Keywords: Implicit degree; Hamilton cycles; Graph

1 Introduction

All the graphs in this paper are undirected and simple. We use the notation and terminology in [4]. In addition, for a graph $G = (V(G), E(G))$, let H be a subgraph of G. $G[H]$ denotes the subgraph of G induced by $V(H)$. $N_H(u) = \{v \in V(H) : uv \in E(G)\}$ and $d_H(u) = |N_H(u)|$ denote the neighborhood and the degree of a vertex $u \in V(G)$ in H, respectively. If there is no fear of confusion, we can use $N(v)$ and $d(v)$ in place of $N_G(v)$ and $d_G(v)$, respectively. Let $N_1(v) = N(v)$ and $N_2(v) = \{u \in V(G) : d(u, v) = 2\}$, where $d(u, v)$ indicates the distance from u to v. A and B being the subsets of $V(G)$, $e(A, B)$ is the number of edges ab of G with $a \in A$ and $b \in B$. We write $e(A, b)$ instead of $e(A, \{b\})$.

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†Corresponding author.

For a cycle (or a path) C in G with a given orientation and a vertex y in C, y^+ and y^- denote the successor and the predecessor of y in C, respectively. Define $y^{+(h+1)} = y^{+(+h)}$ for every integer $h \geq 0$, with $y^{0+} = y$. y^{-h} is defined analogously. And for any $A \subseteq V(C)$, let

$$A^- = \{y : y^+ \in A\} \quad and \quad A^+ = \{y : y^- \in A\}.$$

We denote by $c(G)$ the circumference, i.e. the length of a longest cycle in G. A cycle or a path containing all vertices of G is a Hamilton cycle or a Hamilton path. A graph G of order n is hamiltonian graph if $c(G) = n$.

In order to give the results of this paper, we define some special graphs.

(1) Let $n \geq 7$ be an odd integer. $G \in \mathcal{G}_n$ if and only if $|V(G)| = n$ and the vertex-set of G is the disjoint union of the sets A_1, A_2, B_1, B_2 and $\{a_1, a_2, b\}$ so that

(i) $|A_i \cup B_i| = (n-3)/2, i = 1, 2$;

(ii) $|A_i| \geq 2, i = 1, 2$;

(iii) $G[A_i \cup B_i]$ and $G(A_i \cup \{a_j\})$ are both complete subgraphs of G for $i = 1, 2$ and $j = 1, 2$;

(iv) $e(a_1, a_2) \leq 1$;

(v) $|A_1 \cup A_2| \geq (n-3)/2 - e(a_1, a_2)$; and

(vi) $d(b) = 2$ and the neighbors of b are a_1 and a_2. (Fig.1.)

(2) H is the graph of order 9 depicted in Fig.2.

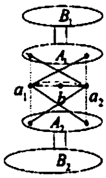

Fig.1

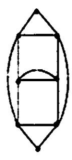
Fig.2

(3) Let $n \geq 9$ be an odd integer. $G \in \mathcal{B}_n$ if and only if $|V(G)| = n$ and the vertex-set of G is the disjoint union of the sets A_1, A_2, B_1, B_2 and $\{a_1, a_2, b\}$ so that they satisfy the above (i),(ii),(iv),(v),(vi) and

(vii) $G[A_i]$ and $G(A_i \cup \{a_j\})$ are both complete subgraphs of G for $i = 1, 2$ and $j = 1, 2$;

(viii) $|A_i| \geq \max\{|N_2(b)| + 2 : b \in B_i\}, i = 1, 2$.

(4)$\mathcal{H}_n = (kK_1 \cup 2K_{\frac{n-1}{2}-k}) \vee K_{k+1}$.

The hamiltonian problem has been studied widely by many researchers. Among the conditions in most significant results, degree conditions or various neighborhood conditions play important roles in sufficient conditions for a graph to be hamiltonian. We have two classic results due to Dirac and Fan respectively.

Theorem 1. *(Dirac [5]) If G is a graph of order $n \geq 3$ such that $d(u) \geq n/2$ for each vertex u in G, then G is hamiltonian. The bound is sharp.*

Theorem 2. *(Fan [7]) Let G be a 2-connected graph of order $n \geq 3$ such that $\max\{d(u), d(v)\} \geq n/2$ for each pair of vertices u and v at distance 2, then G is hamiltonian. The bound is sharp.*

An improvement of Theorem 2 is as follows, where $\alpha(G)$ is the independence number of G.

Theorem 3. *(Benhocine and Wojda [1]) Let G be a 2-connected graph of order $n \geq 3$ with $\alpha(G) \leq n/2$ such that $\max\{d(u), d(v)\} \geq (n-1)/2$ for each pair of vertices u and v at distance 2, then either G is hamiltonian or $G \in \mathscr{G}_n \cup H$.*

In order to generalize Theorems 1 and 2, Zhu, Li and Deng proposed the concept of implicit degrees of vertices in [10].

Definition 1. *(Zhu et al. [10]) Let v be a vertex of a graph G. If $N_2(v) \neq \emptyset$ and $d(v) \geq 2$, then set $k = d(v) - 1$, $m_2 = \min\{d(u) : u \in N_2(v)\}$ and $M_2 = \max\{d(u) : u \in N_2(v)\}$. Suppose $d_1 \leq d_2 \leq \cdots \leq d_{k+1} \leq \cdots$ is the degree sequence of vertices of $N(v) \cup N_2(v)$. Let*

$$d^*(v) = \begin{cases} m_2, & \text{if } d_k < m_2; \\ d_{k+1}, & \text{if } d_{k+1} > M_2; \\ d_k, & \text{if } d_k \geq m_2 \text{ and } d_{k+1} \leq M_2. \end{cases}$$

Then the implicit degree of v, is defined as $id(v) = \max\{d(v), d^*(v)\}$. If $N_2(v) = \emptyset$ or $d(v) \leq 1$, then we define $id(v) = d(v)$.

From the definition of implicit degree, it is clear that $id(v) \geq d(v)$ for every vertex v. Zhu *et al.* [10] gave a sufficient condition for a graph to be hamiltonian.

Theorem 4 *(Zhu et al. [10]) Let G be a 2-connected graph of order n such that $id(u) + id(v) \geq c$ for each pair of nonadjacent vertices u and v in G. Then $c(G) \geq \min\{n, c\}$.*

Corollary 5. *Let G be a 2-connected graph of order n such that $id(u) \geq n/2$ for each $u \in V(G)$. Then G contains a Hamilton cycle.*

In this paper, we study implicit degrees and the hamiltonicity of graphs, and we obtain the following main result.

Theorem 6. *Let G be a 2-connected graph of order n with $\alpha(G) \leq n/2$ such that $id(u) \geq (n-1)/2$ for each $u \in V(G)$. Then G contains a Hamilton cycle or $G \in \mathscr{B}_n \cup H$ or G is a subgraph of $\mathscr{H}_n$.*

2 The proof of Theorem 6

We first have the following lemma that will be used in our proof.

Lemma 1. *(Dirac [6]) Let G be a 2-connected graph of order n, and let $P(a,b)$ be a longest path of G with $d(a)+d(b) \geq c$, then $c(G) \geq \min\{n,c\}$.*

Lemma 2.*(Benhocine and Wojda [1]) If a graph G of order $n \geq 4$ has a cycle C of length $n-1$, such that the vertex x not in $V(C)$ has degree at least $n/2$, then G is hamiltonian.*

Lemma 3. *Let $P = x_1x_2\cdots x_p$ be a path and y_1, y_2 be two vertices not in $V(P)$. If $N_P^-(y_1) \cap N_P(y_2) = \emptyset$ and $x_1y_1 \notin E(G)$, then*

$$d_P(y_1) + d_P(y_2) \leq |V(P)|.$$

Proof. We prove by induction on $|V(P)|$. If $|P| = 1,2,3$, the lemma follows. Suppose that the lemma is true for any path P' with $|P'| < |P|$. Now we let

$$P' = x_2x_3x_4\cdots x_p.$$

If $x_1y_2 \notin E(G)$, we have $d_P(y_1)+d_P(y_2) = d_{P'}(y_1)+d_{P'}(y_2) \leq (|V(P')|+1)-1 = |V(P)|$. Otherwise, $y_ix_2 \notin E(G)$, we have $d_P(y_1)+d_P(y_2) = d_{P'}(y_1)+d_{P'}(y_2)+1 \leq |V(P')|+1 = |V(P)|$. □

Proof of Theorem 6. Let G be a graph satisfying the conditions in Theorem 6 and suppose G is not hamiltonian. By Theorem 4, G contains a cycle of length $n-1$. We choose an cycle C of length $n-1$ such that the degree of the remaining vertex is as large as possible. Let x be the remaining vertex of G not in C. Choosing an arbitrary orientation on C, define $y_1, y_2, \cdots, y_{k+1} (k \geq 1)$ to be the neighbors of x. Since $\{x, y_1, y_2, \cdots, y_{k+1}\}$ is an independent set, $d(x, y_i^+) = 2$ for every $i = 1, 2, \cdots, k+1$. By Lemma 2, $d(x) \leq (n-1)/2$.

If $d(x) = (n-1)/2$, then $\{x, y_1, y_2, \cdots, y_{(n-1)/2}\}$ is an independent set of G with $(n+1)/2$ elements, contradicting $\alpha(G) \leq n/2$.

So we can assume $d(x) < (n-1)/2$. Set $m_2^x = \min\{d(u) : u \in N_2(x)\}$ and $M_2^x = \max\{d(u) : u \in N_2(x)\}$. Suppose $d_1^x \le d_2^x \le \cdots \le d_{k+1}^x \le \cdots$ is the degree sequence of vertices of $N(x) \cup N_2(x)$. Since $N^+(x) \subseteq N_2(x)$, by the definition of implicit degree, we can easily get that $id(x) \neq d_{k+1}^x$. We consider the following two cases.

Case 1. $id(x) = m_2^x$.

Since $d(x, y_i^+) = 2$ for each $i = 1, 2, \cdots, k+1$, we have $d(y_i^+) \ge m_2^x = id(x) \ge (n-1)/2$ for each i.

Since G is not hamiltonian, it is easy to check that

(1) $e(y_1^+, z^+) + e(y_2^+, z) \le 1$ for every $z \in A = \{y_1^+, y_1^{+2}, \cdots, y_1^{+h}\}$, where h is the minimum integer such that $y_1^{+h} = y_2^-$, and

(2) $e(y_1^+, z) + e(y_2^+, z^+) \le 1$ for every $z \in B = \{y_2^+, y_2^{+2}, \cdots, y_2^{+l}\}$, where l is the minimum integer such that $y_2^{+l} = y_1^-$.

As $y_1^+ x \notin E(G)$ and $y_2^+ x \notin E(G)$, (1) and (2) imply

$$\begin{aligned} n-1 &\le d(y_1^+) + d(y_2^+) = \sum_{z \in A} [e(y_1^+, z^+) + e(y_2^+, z)] \\ &\quad + \sum_{z \in B} [e(y_1^+, z) + e(y_2^+, z^+)] + e(y_1^+, y_1) + e(y_2^+, y_2) \\ &\le h + l + 2 = n - 1, \end{aligned}$$

which implies that all the inequalities above are equalities. In particular, $d(y_1^+) = d(y_2^+) = (n-1)/2$ and n is odd.

If $d(x) \ge 3$, we have $e(y_1^+, y_3^+) + e(y_2^+, y_3^{+2}) = 1$. As $y_1^+ y_3^+ \notin E(G)$, we deduce $y_2^+ y_3^{+2} \in E(G)$ and G has a cycle of length $n-1$ avoiding y_3^+ whose degree is at least $(n-1)/2$, contrary to the choice of C. (An analogous argument shows that $y_1^+ y_1^{+3} \in E(G)$ and $y_2^+ y_1^{+2} \notin E(G)$.) So we can assume that $d(x) = 2$ and $h \ge 2, l \ge 2$.

By the choice of C, we can assume that whenever we have a cycle of length $n-1$, then the vertex not in the cycle has degree 2.

Observe that y_1^+ and y_2^+ have degree precisely $(n-1)/2$ and are joined by a Hamilton path P in G, where $P = y_1^+ y_1^{+2} \cdots y_1^{+h} y_2 x y_1 y_2^{+l} y_2^{+(l-1)} \cdots y_2^+$. For convenience, let $P = x_1 x_2 \cdots x_n$, where $x_1 = y_1^+, x_2 = y_1^{+2}$, and so on. We may easily deduce the following useful properties:

(i) $e(x_1, x_{i+1}) + e(x_n, x_i) = 1$ for every $i = 1, 2, \cdots, n-1$;

(ii) If $e(x_1, x_{i+1}) + e(x_n, x_{i-1}) = 2$ for some $i = 2, 3, \cdots, n-1$, then $d(x_i) = 2$. Moreover, by the definition of implicit degree, we have $d(x_{i-2}) \ge id(x_i) \ge (n-1)/2$ and $d(x_{i+2}) \ge id(x_i) \ge (n-1)/2$;

(iii) $x_1x_{n-1} \notin E(G)$ and $x_nx_2 \notin E(G)$.

Since $x_1x_3 = y_1^+y_1^{3+} \in E(G), y_1^+y_1 \in E(G)$ and $y_1^+x \notin E(G)$, only two cases can arise.

Case 1.1. There are i and $j, j \geq i+1$, such that $x_1x_{i-1}, x_1x_{j+1} \in E(G)$ and $x_1x_k \notin E(G)$ for each $k = i, i+1, \cdots, j$.

Choose such i such that i is as small as possible. By (i) and (iii), we have $i \geq 4$ and $j \leq n-3$; by (i) $x_nx_k \in E(G)$ for all $k = i-1, i, \cdots, j-1$; by (ii) $d(x_j) = 2$.

Statement. *If $z_1z_2\cdots z_n$ is a Hamilton path of G such that there are i and $j, i+1 \leq j$, $z_1z_{i-1} \in E(G), z_1z_{j+1} \in E(G), z_1z_k \notin E(G)$ for $k = i, i+1, \cdots, j$ and $d(z_i) \geq (n-1)/2$, then $j = i+1$, $d(z_{i+3}) \geq (n-1)/2$ and $d(z_{i-1}) \geq (n-1)/2$.*

Proof. Suppose $j \geq i+2$ and consider the Hamilton path

$$z_1z_2\cdots z_{i-1}z_nz_{n-1}\cdots z_i.$$

Then (i) gives $e(z_1, z_{j-1}) + e(z_i, z_j) = 1$, hence $z_iz_j \in E(G)$, contrary to $d(z_j) = 2$. The Statement follows. □

Case 1.1.1. $d(x_i) \geq (n-1)/2$.

By the Statement, we have $j = i+1$, $d(x_{i+3}) \geq (n-1)/2$ and $d(x_{i-1}) \geq (n-1)/2$. Let $P' = x_1x_2\cdots x_{i-1}x_nx_{n-1}\cdots x_i$. Since $x_1x_{i-1} \in E(G), x_1x_n \notin E(G), x_1x_{n-1} \notin E(G), x_1x_{i+2} \in E(G)$ and $d(x_n) \geq (n-1)/2$, we have $x_1x_{n-2} \in E(G)$ and $d(x_{n-3}) \geq (n-1)/2$ by the Statement. Moreover, $d(x_{n-1}) = 2$ since $x_ix_n \in E(G)$. Then use P we can obtain $x_nx_{n-3} \notin E(G)$.

If $i+3 < n-2$, then considering the Hamilton path

$$x_{i-1}x_{i-2}\cdots x_1x_{i+2}x_{i+1}x_ix_nx_{n-1}\cdots x_{i+3},$$

to get by (i) $x_{i-1}x_{n-2} \in E(G)$. So taking the Hamilton path

$$x_1x_2\cdots x_{i-1}x_{n-2}x_{n-1}x_nx_ix_{i+1}\cdots x_{n-3},$$

and observing that $x_{n-3}x_n \notin E(G)$ implies by (i) $x_1x_i \in E(G)$, but this contradicts the hypothesis in Case 1.1.

Suppose $i+3 = n-2$ and then i is even. Referring to the Hamilton path

$$x_ix_{i+1}x_{i+2}x_1x_2\cdots x_{i-1}x_{i+5}x_{i+4}x_{i+3},$$

we have by (i), $x_ix_{i+2} \in E(G)$.

Since $x_ix_{i+2} \in E(G), x_1x_i \notin E(G), x_2x_i \notin E(G)$(for $x_{i+3}x_1 \in E(G)$), $x_ix_{i-1} \in E(G)$ and $d(x_1) \geq (n-1)/2$, we have by the Statement $x_ix_3 \in E(G)$ implying $d(x_2) = 2(x_1x_{i+3} \in E(G))$. If $i = 4$, we obtain $n = 9$ and G is isomorphic to H.

Then suppose $i \geq 6$. We have $d(x_4) \geq (n-1)/2$ and $d(x_{i+2}) \geq (n-1)/2$. Taking the Hamilton path

$$x_{i+2}x_{i+1}x_ix_3x_2x_1x_{i+3}x_{i+4}x_{i+5}x_{i-1}x_{i-2}\cdots x_4,$$

by (i) and the fact $d(x_{i+4}) = 2$, we obtain $x_{i+2}x_{i+5} \in E(G)$. A Hamilton cycle is then

$$x_{i+2}x_{i+1}\cdots x_1x_{i+3}x_{i+4}x_{i+5}x_{i+2},$$

a contradiction.

Case 1.1.2. $d(x_i) < (n-1)/2$.

We have $x_{i-2}x_i \notin E(G)$ for $x_nx_{i-1} \in E(G), x_1x_{i-1} \in E(G)$ and G is not hamiltonian. And $d(x_{j-2}) \geq id(x_j) \geq (n-1)/2$ and $d(x_{j+2}) \geq id(x_j) \geq (n-1)/2$ by (ii).

Claim 1. $d(x_{i-2}) < (n-1)/2$.

Proof. Suppose $d(x_{i-2}) \geq (n-1)/2$, by considering the Hamilton path

$$x_{i-2}x_{i-3}\cdots x_1x_{j+1}x_j\cdots x_{i-1}x_nx_{n-1}\cdots x_{j+2},$$

and using the fact that $x_{i-2}x_n \notin E(G)$, we deduce $x_{j+2}x_{i-1} \in E(G)$. Then

$$x_1x_2\cdots x_{i-1}x_{j+2}x_{j+3}\cdots x_nx_ix_{i+1}\cdots x_{j+1}x_1,$$

is a Hamilton cycle of G, a contradiction. □

Claim 2. $j = i + 1$.

Proof. Suppose $j \geq i + 2$, by considering the Hamilton path

$$x_{j-2}x_{j-3}\cdots x_1x_{j+1}x_jx_{j-1}x_nx_{n-1}\cdots x_{j+2},$$

and using the fact $x_1x_{j-2} \notin E(G)$, we deduce $x_{j+2}x_2 \in E(G)$. Then

$$x_{j+1}x_jx_{j-1}\cdots x_ix_nx_{n-1}\cdots x_{j+2}x_2x_3\cdots x_{i-1}x_1x_{j+1}$$

is a Hamilton cycle of G, a contradiction. □

Claim 3. $x_1x_k \in E(G)$ for any $k \leq i - 2$.

Proof. Suppose there exists some $k, (4 \le k \le i-2)$ such that $x_1x_{k-1} \in E(G), x_1x_{k+1} \in E(G)$ and $x_1x_k \notin E(G)$. By (i) $x_nx_{k-1} \in E(G)$ and $x_nx_{k-2} \notin E(G)$; by (ii) $d(x_k) = 2$, thus $d(x_{k+2}) \ge id(x_k) \ge (n-1)/2$ and $d(x_{k-2}) \ge id(x_k) \ge (n-1)/2$. So $x_1x_{k-2} \in E(G)$. We consider the following two case.

(a) $x_1x_{k+2} \notin E(G)$.

By (i), $x_nx_{k+1} \in E(G)$. Since $d(x_{k+2}) \ge (n-1)/2$, by (ii), $x_1x_{k+3} \notin E(G)$. So $x_nx_{k+2} \in E(G)$ and $x_1x_{k+4} \in E(G)$. By the choice of i, we have $i = k+2$, contrary to $d(x_i) < (n-1)/2$. □

(b) $x_1x_{k+2} \in E(G)$.

By (i), $x_nx_{k+1} \notin E(G)$. By considering the Hamilton path

$$x_{k-2}x_{k-3}\cdots x_1x_{k-1}x_kx_{k+1}\cdots x_n$$

and using the fact that $x_nx_{k+1} \notin E(G)$, we deduce $x_{k-2}x_{k+2} \in E(G)$. Then

$$x_{k-2}x_{k-3}\cdots x_1x_{k+1}x_kx_{k-1}x_nx_{n-1}\cdots x_{k+2}x_{k-2}$$

is a Hamilton cycle of G, a contradiction. □

Claim 4. $x_1x_{i+3} \in E(G)$.

Proof. Otherwise, $x_nx_{i+2} \in E(G)$ by (i), then considering the Hamilton path

$$x_1x_2\cdots x_ix_{i+1}x_{i+2}x_nx_{n-1}\cdots x_{i+3}$$

and using the fact $x_1x_{i+1} \notin E(G)$, we deduce $x_ix_{i+3} \in E(G)$. Then

$$x_1x_2\cdots x_{i-1}x_nx_{n-1}\cdots x_{i+3}x_ix_{i+1}x_{i+2}x_1$$

is a Hamilton cycle of G, a contradiction. □

Claim 5. $x_1x_s \notin E(G)$ for any $s = i+4, i+5, \cdots, n$.

Proof. Otherwise, there is some s with $i+4 \le s \le n$ such that $x_1x_s \in E(G)$. Clearly, $s \ne n-1, n$. We choose such s such that s is as small as possible. If $s = i+4$, then considering the Hamilton path

$$x_{i+3}x_{i+2}x_{i+1}x_ix_{i-1}\cdots x_1x_{i+4}x_{i+5}\cdots x_n,$$

and using the fact that $x_{i+1}x_n \notin E(G)$, we deduce $x_ix_{i+3} \in E(G)$. Then

$$x_1x_2\cdots x_{i-1}x_nx_{n-1}\cdots x_{i+3}x_ix_{i+1}x_{i+2}x_1,$$

is a Hamilton cycle of G, a contradiction.

So we assume $i+5 \leq s \leq n-2$. By (i) and (ii), we get $d(x_{s+1}) \geq (n-1)/2$. By considering the Hamilton path

$$x_{i-1}x_{i-2}\cdots x_1x_sx_{s-1}\cdots x_ix_nx_{n-1}\cdots x_{s+1},$$

and using the fact $x_{i-1}x_{i+1} \notin E(G)$, we deduce $x_{i+2}x_{s+1} \in E(G)$. Then

$$x_{i+2}x_{i+1}\cdots x_1x_sx_{s-1}\cdots x_{i+3}x_nx_{n-1}\cdots x_{s+1}x_{i+2},$$

is a Hamilton cycle of G, a contradiction. □

By Claim 5 and (i), we have $e(x_1, \{x_{i+4}, x_{i+5}, \cdots, x_n\}) = 0$ and $e(x_n, \{x_{i-1}, x_i, x_{i+3}, x_{i+4}, \cdots, x_{n-1}\}) = n-i-1$. The degrees of x_1 and x_n impose $i = (n-1)/2$. For every $s \leq i-2$ and $t \geq i+4$, we have $x_sx_t \notin E(G), x_sx_i \notin E(G)$ and $x_tx_{i+2} \notin E(G)$, for $x_sx_{s-1}\cdots x_1x_{s+1}x_{s+2}\cdots x_{t-1}x_n x_{n-1}\cdots x_t$, $x_sx_{s-1}\cdots x_1x_{s+1}x_{s+2}\cdots x_{i-1}x_nx_{n-1}\cdots x_i$, $x_tx_{t+1}\cdots x_nx_{t-1} x_{t-2}\cdots x_{i+3}x_1x_2\cdots x_{i+2}$ are Hamilton paths of G, respectively. We deduce that $\{x_{i-1}, x_i, x_{i+1}x_{i+2}, x_{i+3}\}$ is a cut-set of G. Let $U_1 = \{x_1, x_2, \cdots, x_{i-2}\}$ and $U_2 = \{x_{i+4}, x_{i+5}, \cdots, x_n\}$, we see that $|U_1| = |U_2| = (n-5)/2$.

We can claim $d(x_{i-1}) = d(x_{i+3}) = (n-1)/2$ for

$$x_{i-1}x_{i-2}\cdots x_1x_{i+2}x_{i+1}x_ix_nx_{n-1}\cdots x_{i+3},$$

is a Hamilton path of G.

Claim 6. $e(x_{i-1}, U_2 \setminus \{x_n\}) = 0$. Similarly, $e(x_{i+3}, U_1 \setminus \{x_1\}) = 0$.

Proof. Considering the Hamilton path

$$P = x_{i-1}x_{i-2}\cdots x_1x_{i+2}x_{i+1}x_ix_nx_{n-1}\cdots x_{i+3},$$

and using the fact $d(x_{i+3}) = (n-1)/2$, we have $x_{i-1}x_{i+4} \notin E(G)$. If there exists some $m, i+5 \leq m \leq n-1$ such that $x_{i-1}x_m \in E(G)$, choose such m such that m is as large as possible. Then

$$P' = x_{i-1}x_{i-2}\cdots x_1x_{i+2}x_{i+1}x_ix_nx_{n-1}\cdots x_{m+2}x_{i+3}x_{i+4}\cdots x_mx_1$$

is a cycle of length $n-1$ avoiding x_{m+1}, but $d(x_{m+1}) \geq 3$, a contradiction. So $e(x_{i-1}, U_2 \setminus \{x_n\}) = 0$. Therefore, all the vertices in $U_2 \setminus \{x_n\}$ with degree less than $(n-1)/2$. Similarly, $e(x_{i+3}, U_1 \setminus \{x_1\}) = 0$. □

We can easily check that $d(x_2) < (n-1)/2$ for $x_2x_{i+2} \notin E(G)$ and $x_2x_{i+3} \notin E(G)$. Set $d(x_2) = k'+1, m_2' = \min\{d(x) : x \in N_2(x_2)\}$ and $M_2' = \max\{d(x) : x \in N_2(x_2)\}$. Suppose $d_1' \leq d_2' \leq \cdots \leq d_{k'+2}' \leq \cdots$ be the degree sequence of the vertices of $N_1(x_2) \cup N_2(x_2)$. By the definition of implicit degree and the fact $d(x_{i+3}) = (n-1)/2$, $x_{i+3} \in N_2(x_2)$ and $d(x_{i+2}) < (n-1)/2$, we get that $id(x_2) = d'_{k'} \geq (n-1)/2$. Then there exist at least two vertices in U_2 with degree at least $(n-1)/2$, a contradiction.

Case 1.2. $x_1x_{i-1} \in E(G), x_1x_{i+1} \in E(G)$ and $x_1x_i \notin E(G)$ for some $i \in [4, n-3]$.

By (i), $x_nx_{i-1} \in E(G)$ and $x_nx_{i-2} \notin E(G)$; by (ii), $d(x_i) = 2$, thus $d(x_{i+2}) \geq id(x_i) \geq (n-1)/2$ and $d(x_{i-2}) \geq id(x_i) \geq (n-1)/2$. So $x_nx_{i-3} \notin E(G)$. (otherwise, by (ii), $2 = d(x_{i-2}) \geq (n-1)/2$.)

Choose such i such that i is as small as possible, then $e(x_1, \{x_2, x_3, \cdots, x_{i-1}\}) = i-2$ and $e(x_n, \{x_1, x_2, \cdots, x_{i-2}\}) = 0$.

Considering the Hamilton path

$$x_{i-2}x_{i-3}\cdots x_1x_{i-1}x_i\cdots x_n$$

and noting that $x_nx_i \notin E(G)$ implies by (i), $x_{i-2}x_{i+1} \in E(G)$; but since

$$x_1x_2\cdots x_{i-2}x_{i+1}x_ix_{i-1}x_nx_{n-1}\cdots x_{i+2}$$

is a Hamilton path of G, we must have $x_1x_{i+2} \notin E(G)$. Which implies by (i) $x_nx_{i+1} \in E(G)$ and by (ii) $x_1x_{i+3} \notin E(G)$. Now, we can suppose that $e(x_1, \{x_{i+2}, x_{i+3}, \cdots, x_n\}) = 0$, otherwise Case 1.1 holds. Thus $e(x_n, \{x_{i+1}, x_{i+2}, \cdots, x_{n-1}\}) = n-i-1$. The degree of x_1 and x_n impose $i = (n+1)/2$.

For every $s \leq i-2$ and $t \geq i+2$, we have $x_sx_t \notin E(G)$ for

$$x_sx_{s-1}\cdots x_1x_{s+1}x_{s+2}\cdots x_{t-1}x_nx_{n-1}\cdots x_t$$

is a Hamilton path of G. We deduce that $\{x_{i-1}, x_i, x_{i+1}\}$ is a cut-set of G, and $d(u) \leq (n-1)/2$ for any $u \in V_1 \cup V_2$, where $V_1 = \{x_1, x_2, \cdots, x_{i-2}\}$ and $V_2 = \{x_{i+2}, x_{i+3}, \cdots, x_n\}$. We see that $|V_1| = |V_2| = (n-3)/2$.

Claim 7. *$d(x_{i-1}) \geq (n-1)/2$ and $d(x_{i+1}) \geq (n-1)/2$.*

Proof. Suppose, without loss of generality, $d(x_{i-1}) < (n-1)/2$.

Let $d(x_{i-1}) = s+1$, $m_2^{x_{i-1}} = \min\{d(x) : x \in N_2(x_{i-1})\}, M_2^{x_{i-1}} = \max\{d(x) : x \in N_2(x_{i-1})\}$. Set $d_1^{x_{i-1}} \leq d_2^{x_{i-1}} \leq \cdots \leq d_{s+1}^{x_{i-1}} \leq \cdots$ is a degree sequence of the vertices of $N(x_{i-1}) \cup N_2(x_{i-1})$. Since x_1 is adjacent to all the vertices of $\{x_2, x_3, \cdots, x_{i-1}, x_{i+1}\}$ and x_n is adjacent to all the vertices of $\{x_{i-1}, x_{i+1}, x_{i+2}, \cdots, x_{n-1}\}$, we get that $|N(x_{i-1}) \cup N_2(x_{i-1})| = n-1$. Since all the vertices with degree at least $(n-1)/2$ must be adjacent to x_{i-1} and x_{i+1}, we get that $d(u) < (n-1)/2$ for each $u \in N_2(x_{i-1})$. By the definition of implicit degree, we can easily check that $id(x_{i-1}) \neq m_2^{x_{i-1}}, d_s^{x_{i-1}}$. Therefore, $id(x_{i-1}) = d_{s+1}^{x_{i-1}}$, then $d_{s+1}^{x_{i-1}} > M_2^{x_{i-1}}$, but $|N_2(x_{i-1})| > l$, a contradiction. □

For $j = 1, 2$, V_j can be partitioned into $A_j \cup B_j$ such that $d(a) \geq (n-1)/2$ for each $a \in A_1 \cup A_2$ and $d(b) < (n-1)/2$ for each $b \in B_1 \cup B_2$. Since $x_1, x_{i-2}, x_{i+2}, x_n$ have degree at least $(n-1)/2$, we have $|A_j| \geq 2, j = 1, 2$. Moreover, taking $a \in A_1$, we have

$$\begin{aligned}(n-1)/2 &\leq d(a) \\ &\leq |A_1| - 1 + |B_1| + e(a, \{x_{i-1}, x_{i+1}\}) \\ &\leq |V_1| + 1.\end{aligned}$$

And similarly, $(n-1)/2 \leq |V_2| + 1$. Then $n - 1 \leq |V_1| + |V_2| + 2 = n - 1$, that implies $e(A_1 \cup A_2, \{x_{i-1}, x_{i+1}\}) = 2|A_1 \cup A_2|$.

If $B_1 \cup B_2 = \emptyset$, then $d(u, x_i) = 2$ for any $u \in V_1 \cup V_2$. Therefore, by the definition of implicit degree, we have $d(u) = (n-1)/2$ for any $u \in V_1 \cup V_2$. Then $G \in \mathscr{B}_n$.

So suppose $B_1 \cup B_2 \neq \emptyset$. No vertex of $B_j, j = 1, 2$, is joined to $\{x_{i-1}, x_{i+1}\}$, so $d(x_{i-1}) = d(x_{i+1}) = |A_1| + |A_2| + 1 + e$ where $e = e(x_{i-1}, x_{i+1})$. Since $d(x_{i-1}) = d(x_{i+1}) \geq (n-1)/2$, we get $|A_1| + |A_2| + 1 + e \geq (n-1)/2$, so $|A_1| + |A_2| \geq (n-3)/2 - e$.

Choose a vertex b in $B_1 \cup B_2$, without loss of generality, suppose $b \in B_1$. Let $d(b) = \alpha + 1, m_2^b = \min\{d(u) : u \in N_2(b)\}$ and $M_2^b = \max\{d(u) : u \in N_2(b)\}$. Set $d_1^b \leq d_2^b \leq \cdots \leq d_{\alpha+1}^b \leq \cdots$ be degree sequence of the vertices of $N(b) \cup N_2(b)$. And let $|A_1| = m, |N(b) \cap B_1| = k_1$ and $|N_2(b) \cap B_1| = k_2$. Then $k_1 + k_2 + m = (n-5)/2$ and $\alpha + 1 = k_1 + m$. Since $d(x_{i-1}, b) = 2$ and $d(x_{i-1}) \geq (n-1)/2$, we can easily check that $id(b) \neq d_{\alpha+1}^b$. If $k_2 = 0$, then $G[B_1]$ is complete. If $k_2 \neq 0$, then $id(b) \neq m_2^b$. So $id(b) = d_\alpha^b$, then $d_\alpha^b > m_2^b$ and $d_{\alpha+1}^b \leq M_2^b$. Therefore, $k_1 + k_2 \leq \alpha - 1 = k_1 + m - 2$. Then $k_2 \leq m - 2$. By the arbitrary of b, we have $|A_1| \geq \max\{|N_2(b) \cap B_1| + 2 : b \in B_1\}$. Similarly, $|A_2| \geq \max\{|N_2(b) \cap B_2| + 2 : b \in B_2\}$. Consequently, $G \in \mathscr{B}_n$.

Case 2. $id(x) = d_k$.

Then $d_k > m_2$ and $k \geq 2$. For $i = 1, 2, \cdots, k+1$, let $W_1 = \{y_i : |V(C(y_i, y_{i+1}))| = 1\}$ and $W_2 = \{y_i : |V(C(y_i, y_{i+1}))| \geq 2\}$. Set $|W_1| = w_1, |W_2| = w_2$. Then $w_1 + w_2 = k + 1$. Moreover, $\{y_i^+, y_{i+1}^- : y_i \in W_2\} \subseteq N_2(x)$ and $\{y_i^+ : y_i \in W_1\} \subseteq N_2(x)$, so $|N_2(x)| \geq w_1 + 2w_2$.

By the choice of C, we can get that $d(y_i^+) \leq d(x) < (n-1)/2$ for any $y_i \in W_1$. Since $id(x) = d_k$, there are at least $w_2 + 2$ vertices in $N_2(x)$ with degree at least $id(v) \geq (n-1)/2$.

Claim 8. $w_2 = 2$.

Proof. If $w_2 \leq 1$, since there are at least $u_2 + 2$ vertices in $N_2(x)$ with degree at least $(n-1)/2$, we can easily check that there exists at least one vertex, say y_1, in W_1 such that $d(y_1^+) \geq (n-1)/2$, contrary to the choice of C.

By similar arguments as in Case 1, we can get that there are ai most two vertices in $\{y_i^+ : y_i \in W_2\}$ with degrees at least $id(x)$ and there are ai most two vertices in $\{y_{i+1}^- : y_i \in W_2\}$ with degrees at least $id(x)$. Then $|W_2| \leq 2$. Therefore, $|W_2| = 2$. □

By Claim 8, we assume $W_2 = \{y_i, y_{k+1}\}$. Then $d(y_i^+), d(y_{i+1}^-), d(y_{k+1}^+) \geq id(x)$ and $d(y_1^-) \geq id(x)$.

Claim 9. *$N(u) = N(x)$ for any $u \in W_1^+$.*

Proof. We assume $y_1 \in W_1$. We just need to prove $N(y_1^+) = N(x)$. Let $d(y_1^+) = s+1$. Since $x \in N_2(y_1^+)$, $d(x) < \frac{n-1}{2}$ and G is not hamiltonian, we can get that $id(y_1^+) \neq m_2^{y_1^+}, d_{s+1}^{y_1^+}$. Then $id(y_1^+) = d_s^{y_1^+}$. If there exists some vertex y_t such that $y_t y_1^+ \in E(G)$ and $y_{t+1}y_1^+ \notin E(G)$ or $y_{t-1}y_1^+ \notin E(G)$, then by similar argument as in Claim 12, we can get that $d(y_t^+) \geq id(y_1^+) \geq \frac{n-1}{2}$, a contradiction. Since $y_1^+ y_1 \in E(G)$, we have $y_1^+ y_s \in E(G)$ for each $s = 2, 3, \cdots, i$.

If $y_1^+ y_{i+1} \notin E(G)$, we can get that $y_1^+ y_t \notin E(G)$ for each $t = i+2, i+3, \cdots, k+1$. Then we can get that there is a vertex y_t^+ with $1 \leq t \leq i-1$ with $d(y_t^+) \geq id(y_1^+) \geq \frac{n-1}{2}$, a contradiction. So $y_1^+ y_{i+1} \in E(G)$. Similarly, $y_1^+ y_t \in E(G)$ for each $i+2, i+3, \cdots, k+1$. Therefore, $N(y_1^+) = N(x)$. □

Claim 10. *$N(x) \subseteq N(u)$ for any $u \in \{y_i^+, y_{i+1}^-, y_{k+1}^+, y_1^-\}$.*

Proof. By symmetry, we just prove $N(x) \subseteq N(y_i^+)$. Considering the Hamilton path $P = y_i^+ y_i^{+2} \cdots y_{k+1} x y_i y_i^- y_i^{-2} \cdots y_{k+1}^+$ and using the fact $d(y_i^+) \geq \frac{n-1}{2}$ and $d(y_{k+1}^+) \geq \frac{n-1}{2}$, we deduce $d(y_i^+) = d(y_{k+1}^+) = \frac{n-1}{2}$. Since $y_s^+ y_{k+1}^+ \notin E(G)$ for any $y_s \in W_1$ and $x y_{k+1}^+ \notin E(G)$, we have $N(x) \backslash \{y_{i+1}\} \subseteq N(y_i^+)$.

By Claim 9, $y_{k+1}^+ y_{k+1}^{+2} \cdots y_1 x y_{k+1} y_{k+1}^- \cdots y_{i+1} y_1^+ y_1^{+2} \cdots y_{i+1}^-$ is a Hamilton path, then $y_{k+1}^+ y_{i+1}^- \notin E(G)$. Then by using P, we get that $y_i^+ y_{i+1} \in E(G)$. Therefore $N(x) \subseteq N(y_i^+)$. □

Let $C_1 = C[y_i^+, y_{i+1}^-], C_2 = C[y_{k+1}^+, y_1^-]$ and $C_3 = C[y_{i+1}, y_{k+1}] \cup C[y_1, y_i]$. By Claim 10, $d_{C_3}(y_i^+) = d_{C_3}(y_{i+1}^-) = d_{C_3}(y_{k+1}^+) = d_{C_3}(y_1^-) = k+1$. Since G is non-hamiltonian, we have $N_{C_2}^+(y_{k+1}^+) \cap N_{C_2}(y_i^+) = \emptyset$, by Lemma 4, we can get that $d_{C_2}(y_i^+) + d_{C_2}(y_j^+) \leq |C_2| - 1$. Similarly, $d_{C_1}(y_i^+) + d_{C_1}(y_j^+) \leq |C_1| - 1$, $d_{C_2}(y_{i+1}^-) + d_{C_2}(y_{j+1}^-) \leq |C_2| - 1$ and $d_{C_1}(y_{i+1}^-) + d_{C_1}(y_{j+1}^-) \leq |C_1| - 1$. By the above inequalities, we get

$$2(n-1) \leq d_C(y_i^+) + d_C(y_j^+) + d_C(y_{i+1}^-) + d_C(y_{j+1}^-)$$

$$\begin{aligned} &\leq 4(k+1)+2(|C_1|-1)+2(|C_2|-1) \\ &\leq 2(n-1), \end{aligned}$$

which implies that all the inequalities are equalities. If there exists some vertex $x_s \in V(C_1)$ such that $y_{k+1}^+ x_s \in E(G)$, then $y_1^- x_s^-, y_1^- x_s^+, y_1^- y_i^+$, $y_1^- y_i^{2+} \notin E(G)$ and $y_{i+1}^- x_s^- \notin E(G)$. By Lemma 4, we can get that $d_{C_1}(y_{i+1}^-) + d_{C_1}(y_1^-) < |C_1| - 1$, a contradiction. Hence, $N_{C_1}(y_{k+1}^+) = \emptyset$. Similarly, we can get that $N_{C_1}(y_1^-) = \emptyset, N_{C_2}(y_i^+) = \emptyset$ and $N_{C_2}(y_{i+1}^-) = \emptyset$. Hence, $d_{C_1}(y_i^+) = |V(C_1)| - 1$ and $d_{V(C_2)}(y_{k+1}^+) = |V(C_2)| - 1$. Since $d(y_i^+) = \frac{n-1}{2}$ and $d(y_{k+1}^+) = \frac{n-1}{2}$, we can get that $|V(C_1)| = |V(C_2)| = \frac{n-1}{2} - k$. Therefore, we can get that G is the subgraph of $\mathscr{H}_n$.

Then Theorem 6 holds. □

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