IDENTITIES ON THE BERNOULLI AND THE EULER NUMBERS AND POLYNOMIALS

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Abstract In this paper we give some interesting identities on the Bernoulli and the Euler numbers and polynomials by using reflection symmetric properties of Euler and Bernoulli polynomials. To derive our identities, we investigate some properties of the fermionic p-adic integrals on $\mathbb{Z}_p$.

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1. Introduction and preliminaries

Let p be a fixed odd prime number. Throughout this paper, the symbols $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$ and $\mathbb{C}_p$ denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers. The p-adic norm on $\mathbb{C}_p$ is normalized so that $|p|_p = p^{-1}$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$(1) \qquad I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad (\text{see } [7]).$$

From (1), we have

$$(2) \qquad I_{-1}(f_1) = -I_{-1}(f) + 2f(0), \ (\text{see } [7,9]),$$

where $f_1(x) = f(x+1)$.
Let us take $f(x) = e^{xt}$. Then, by (2), we get

$$(3) \qquad \int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},$$

where E_n are the ordinary Euler numbers (see [1-12]).
From the same method of (3), we can also derive the following equation:

$$(4) \qquad \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},$$

where $E_n(x)$ are called the n-th Euler polynomials(see [1-12]).
By (3) and (4), we get

$$(5) \qquad \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n, \text{ and } \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x).$$

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From (2), we have

$$(6)\qquad \int_{\mathbb{Z}_p}(x+1)^n d\mu_{-1}(x)+\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x)=2\delta_{0,n},$$

where the symbol $\delta_{0,n}$ is the Kronecker symbol.
Thus, by (5) and (6), we get

$$(7)\qquad (E+1)^n+E_n=2\delta_{0,n} \text{ (see [7,9])}.$$

From (1), we can derive the following equation:

$$(8)\qquad \int_{\mathbb{Z}_p}(1-x+x_1)^n d\mu_{-1}(x_1)=(-1)^n\int_{\mathbb{Z}_p}(x+x_1)^n d\mu_{-1}(x_1).$$

By (5) and (8), we see that

$$(9)\qquad E_n(1-x)=(-1)^n E_n(x).$$

Thus, by (7), we get $E_n(2)=(-1)^n E_n(-1)$.
From (7), we have

$$(10)\qquad E_n(2)=2-E_n(1)=2+E_n-2\delta_{0,n}.$$

The Bernoulli polynomials $B_n(x)$ are defined by

$$(11)\qquad \frac{t}{e^t-1}e^{xt}=e^{B(x)t}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \text{ (see [13])},$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$.

In the special case, $x=0$, $B_n(0)=B_n$ is called the n-th Bernoulli number. By (11), we easily see that

$$(12)\qquad B_n(x)=\sum_{l=0}^{n}\binom{n}{l}x^{n-l}B_l=(B+x)^n.$$

Thus, by (11) and (12), we get reflection symmetric formula for the Bernoulli polynomials as follows:

$$(13)\qquad B_n(1-x)=(-1)^n B_n(x),$$

and

$$(14)\qquad B_0=1, (B+1)^n-B_n=\delta_{1,n} \text{ (see [3,9])}.$$

From (13)and (14), we can also derive the following identity:

$$(15)\qquad (-1)^n B_n(-1)=B_n(2)=n+B_n(1)=n+B_n+\delta_{1,n}.$$

In this paper we investigate some properties of the fermionic p-adic integrals on $\mathbb{Z}_p$. By using these properties, we give some new identities on the Bernoulli and the Euler numbers which are useful in studying combinatorics.

2. Identities on the Bernoulli and the Euler numbers

Let us consider the following fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned} I_1 &= \int_{\mathbb{Z}_p} B_n(x)d\mu_{-1}(x) = \sum_{l=0}^{n}\binom{n}{l}B_{n-l}\int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) \\ &= \sum_{l=0}^{n}\binom{n}{l}B_{n-l}E_l, \text{ for } n \in \mathbb{Z}_+ = \mathbb{N}\cup\{0\}. \end{aligned} \tag{16}$$

On the other hand, by (13) and (14), we get

(17)

$$\begin{aligned} I_1 &= (-1)^n\int_{\mathbb{Z}_p} B_n(1-x)d\mu_{-1}(x) = (-1)^n\sum_{l=0}^{n}\binom{n}{l}B_{n-l}\int_{\mathbb{Z}_p}(1-x)^l d\mu_{-1}(x) \\ &= (-1)^n\sum_{l=0}^{n}\binom{n}{l}B_{n-l}(-1)^l E_l(-1) = (-1)^n\sum_{l=0}^{n}\binom{n}{l}B_{n-l}E_l(2) \\ &= (-1)^n\sum_{l=0}^{n}\binom{n}{l}B_{n-l}(2+E_l-2\delta_{0,l}) \\ &= 2(-1)^n B_n(1) + (-1)^n\sum_{l=0}^{n}\binom{n}{l}B_{n-l}E_l + 2(-1)^{n+1}B_n \\ &= 2(-1)^n(B_n+\delta_{1,n}) + (-1)^n\sum_{l=0}^{n}\binom{n}{l}B_{n-l}E_l + 2(-1)^{n+1}B_n. \end{aligned}$$

Equating (16) and (17), we obtain the following theorem.

Theorem 1. *For $n \in \mathbb{Z}_+$, we have*

$$(1+(-1)^{n+1})\sum_{l=0}^{n}\binom{n}{l}B_{n-l}E_l = 2(-1)^n\delta_{1,n}.$$

In particular,

$$\sum_{l=0}^{2n+1}\binom{2n+1}{l}B_{2n+1-l}E_l = -\delta_{0,n}.$$

By using the reflection symmetric property for the Euler polynomials, we can also obtain some interesting identities on the Euler numbers.
Now, we consider the fermionic p-adic integral on $\mathbb{Z}_p$ for the polynomials as follows:

$$\begin{aligned} I_2 &= \int_{\mathbb{Z}_p} E_n(x)d\mu_{-1}(x) = \sum_{l=0}^{n}\binom{n}{l}E_{n-l}\int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) \\ &= \sum_{l=0}^{n}\binom{n}{l}E_{n-l}E_l, \text{ for } n \in \mathbb{Z}_+. \end{aligned} \tag{18}$$

On the other hand, by (7), (9) and (10), we get

(19)

$$\begin{aligned}
I_2 &= (-1)^n \int_{\mathbb{Z}_p} E_n(1-x)d\mu_{-1}(x) = (-1)^n \sum_{l=0}^{n} \binom{n}{l} E_{n-l} \int_{\mathbb{Z}_p} (1-x)^l d\mu_{-1}(x) \\
&= (-1)^n \sum_{l=0}^{n} \binom{n}{l} E_{n-l}(-1)^l E_l(-1) = (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l} E_l(2) \\
&= (-1)^n \sum_{l=0}^{n} \binom{n}{l} E_{n-l}(2 + E_l - 2\delta_{0,l}) \\
&= 2(-1)^n E_n(1) + (-1)^n \sum_{l=0}^{n} \binom{n}{l} E_{n-l} E_l + 2(-1)^{n+1} E_n \\
&= 2(-1)^n (2\delta_{0,n} - E_n) + (-1)^n \sum_{l=0}^{n} \binom{n}{l} E_{n-l} E_l + 2(-1)^{n+1} E_n.
\end{aligned}$$

Equating (18) and (19), we obtain the following theorem.

Theorem 2. *For $n \in \mathbb{Z}_+$, we have*

$$(1 + (-1)^{n+1}) \sum_{l=0}^{n} \binom{n}{l} E_{n-l} E_l = 4(-1)^{n+1} E_n + 4\delta_{0,n}.$$

Let us consider the fermionic p-adic integral on $\mathbb{Z}_p$ for the product of $B_n(x)$ and $E_n(x)$ as follows:

(20)

$$\begin{aligned}
I_3 &= \int_{\mathbb{Z}_p} B_m(x) E_n(x) d\mu_{-1}(x) \\
&= \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} B_{m-k} E_{n-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_{-1}(x) \\
&= \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} B_{m-k} E_{n-l} E_{k+l}.
\end{aligned}$$

On the other hand, by (9) and (13), we get

(21)

$$\begin{aligned}
I_3 &= \int_{\mathbb{Z}_p} B_m(x)E_n(x)d\mu_{-1}(x) \\
&= (-1)^{n+m}\int_{\mathbb{Z}_p} B_m(1-x)E_n(1-x)d\mu_{-1}(x) \\
&= (-1)^{n+m}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}E_{n-l}\int_{\mathbb{Z}_p}(1-x)^{k+l}d\mu_{-1}(x) \\
&= (-1)^{n+m}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}E_{n-l}(2+E_{k+l}-2\delta_{0,k+l}) \\
&= 2(-1)^{n+m}B_m(1)E_n(1)+(-1)^{n+m}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}E_{n-l}E_{k+l} \\
&\quad -2(-1)^{m+n}B_mE_n.
\end{aligned}$$

By (20) and (21), we easily see that

(22)

$$\begin{aligned}
&(1+(-1)^{n+m+1})\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}E_{n-l}E_{k+l} \\
&= 2(-1)^{m+n}(\delta_{1,m}+B_m)(2\delta_{0,n}-E_n)+2(-1)^{m+n+1}B_mE_n \\
&= 2(-1)^{m+n+1}B_mE_n+4(-1)^{m+n}\delta_{1,m}\delta_{0,n}+2(-1)^{m+n+1}\delta_{1,m}E_n \\
&\quad +4B_m(-1)^{m+n}\delta_{0,n}+2(-1)^{m+n+1}B_mE_n.
\end{aligned}$$

Therefore, by (22), we obtain the following theorem.

Theorem 3. *For $m, n \in \mathbb{Z}_+$, we have*

$$\begin{aligned}
&(1+(-1)^{n+m+1})\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}E_{n-l}E_{k+l} \\
&= 4(-1)^{m+n+1}B_mE_n+2(-1)^{m+n+1}\delta_{1,m}E_n+4(-1)^{m+n}\delta_{1,m}\delta_{0,n} \\
&\quad +4B_m(-1)^{m+n}\delta_{0,n}.
\end{aligned}$$

Corollary 4. *For $m, n \in \mathbb{Z}_+$, we have*

$$\sum_{k=0}^{2m}\sum_{l=0}^{2n-1}\binom{2m}{k}\binom{2n-1}{l}B_{2m-k}E_{2n-1-l}E_{k+l} = 2B_{2m}E_{2n-1}.$$

Let us consider the fermionic p-adic integral on $\mathbb{Z}_p$ for the product of Bernoulli polynomials and Bernstein polynomials. For $n, k \in \mathbb{Z}_+$, with $0 \le k \le n$, $B_{k,n}(x) = \binom{k}{n}x^k(1-x)^{n-k}$ are called the Bernstein polynomials of degree n, see [9]. It is easy to show that $B_{k,n}(x) = B_{n-k,n}(1-x)$.

$$
\begin{aligned}
(23) \qquad I_4 &= \int_{\mathbf{Z}_p} B_m(x) B_{k,n}(x) d\mu_{-1}(x) \\
&= \binom{n}{k} \sum_{l=0}^{m} \binom{m}{l} B_{m-l} \int_{\mathbf{Z}_p} x^{k+l}(1-x)^{n-k} d\mu_{-1}(x) \\
&= \binom{n}{k} \sum_{l=0}^{m} \sum_{j=0}^{n-k} \binom{m}{l} \binom{n-k}{j} (-1)^j B_{m-l} \int_{\mathbf{Z}_p} x^{k+l+j} d\mu_{-1}(x) \\
&= \binom{n}{k} \sum_{l=0}^{m} \sum_{j=0}^{n-k} \binom{m}{l} \binom{n-k}{j} (-1)^j B_{m-l} E_{k+l+j}.
\end{aligned}
$$

On the other hand, by (13) and (23), we get

(24)

$$
\begin{aligned}
I_4 &= (-1)^m \int_{\mathbf{Z}_p} B_m(1-x) B_{n-k,n}(1-x) d\mu_{-1}(x) \\
&= (-1)^m \binom{n}{k} \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{m}{l} \binom{k}{j} (-1)^j B_{m-l} \int_{\mathbf{Z}_p} (1-x)^{n-k+l+j} d\mu_{-1}(x) \\
&= (-1)^m \binom{n}{k} \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{m}{l} \binom{k}{j} (-1)^j B_{m-l} (2 - 2\delta_{0,n-k+l+j} + E_{n-k+l+j}) \\
&= 2(-1)^m \binom{n}{k} B_m(1) \delta_{0,k} + 2(-1)^{m+1} \binom{n}{k} B_m \delta_{k,n} \\
&\quad + (-1)^m \binom{n}{k} \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{m}{l} \binom{k}{j} (-1)^j B_{m-l} E_{n-k+l+j}.
\end{aligned}
$$

Equating (23) and (24), we see that

$$
\begin{aligned}
(25) \qquad & \sum_{l=0}^{m} \sum_{j=0}^{n-k} \binom{m}{l} \binom{n-k}{j} (-1)^j B_{m-l} E_{k+l+j} \\
&= 2(-1)^m B_m(1) \delta_{0,k} + 2(-1)^{m+1} B_m \delta_{k,n} \\
&\quad + (-1)^m \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{m}{l} \binom{k}{j} (-1)^j B_{m-l} E_{n-k+l+j}.
\end{aligned}
$$

Thus, from (25), we obtain the following theorem.

Theorem 5. *For $m, n \in \mathbb{N}$, we have*

$$
\begin{aligned}
& \sum_{l=0}^{2m} \sum_{j=0}^{n} \binom{2m}{l} \binom{n}{j} (-1)^j B_{2m-l} E_{l+j} \\
&= 2B_{2m} + \sum_{l=0}^{2m} \binom{2m}{l} B_{2m-l} E_{n+l}.
\end{aligned}
$$

In particular,

$$\sum_{l=0}^{2m}\sum_{j=0}^{2n}\binom{2m}{l}\binom{2n}{j}(-1)^j B_{2m-l}E_{l+j} = 2B_{2m} - mE_{2n+2m-1}.$$

Finally, we consider the fermionic p-adic integral on $\mathbb{Z}_p$ for the product of Euler polynomials and Bernstein polynomials as follows:

$$\begin{aligned}(26)\qquad I_5 &= \int_{\mathbb{Z}_p} E_m(x)B_{k,n}(x)d\mu_{-1}(x)\\ &= \binom{n}{k}\sum_{l=0}^{m}\binom{m}{l}E_{m-l}\int_{\mathbb{Z}_p} x^{k+l}(1-x)^{n-k}d\mu_{-1}(x)\\ &= \binom{n}{k}\sum_{l=0}^{m}\sum_{j=0}^{n-k}\binom{m}{l}\binom{n-k}{j}(-1)^j E_{m-l}\int_{\mathbb{Z}_p} x^{k+l+j}d\mu_{-1}(x)\\ &= \binom{n}{k}\sum_{l=0}^{m}\sum_{j=0}^{n-k}\binom{m}{l}\binom{n-k}{j}(-1)^j E_{m-l}E_{k+l+j}.\end{aligned}$$

On the other hand, by (9) and (23), we get

(27)

$$\begin{aligned} I_5 &= (-1)^m\int_{\mathbb{Z}_p} E_m(1-x)B_{n-k,n}(1-x)d\mu_{-1}(x)\\ &= (-1)^m\binom{n}{k}\sum_{l=0}^{m}\sum_{j=0}^{k}\binom{m}{l}\binom{k}{j}(-1)^j E_{m-l}\int_{\mathbb{Z}_p}(1-x)^{n-k+l+j}d\mu_{-1}(x)\\ &= (-1)^m\binom{n}{k}\sum_{l=0}^{m}\sum_{j=0}^{k}\binom{m}{l}\binom{k}{j}(-1)^j E_{m-l}(2+E_{n-k+l+j}-2\delta_{0,n-k+l+j})\\ &= 2(-1)^m\binom{n}{k}E_m(1)\delta_{0,k} - 2(-1)^m\binom{n}{k}E_m\delta_{k,n}\\ &\quad +(-1)^m\binom{n}{k}\sum_{l=0}^{m}\sum_{j=0}^{k}\binom{m}{l}\binom{k}{j}(-1)^j E_{m-l}E_{n-k+l+j}\end{aligned}$$

Equating (26) and (27), we obtain

$$\begin{aligned}(28)\qquad &\sum_{l=0}^{m}\sum_{j=0}^{n-k}\binom{m}{l}\binom{n-k}{j}(-1)^j E_{m-l}E_{k+l+j}\\ &= 2(-1)^m\binom{n}{k}E_m(1)\delta_{0,k} - 2(-1)^m\binom{n}{k}E_m\delta_{k,n}\\ &\quad +(-1)^m\binom{n}{k}\sum_{l=0}^{m}\sum_{j=0}^{k}\binom{m}{l}\binom{k}{j}(-1)^j E_{m-l}E_{n-k+l+j}.\end{aligned}$$

Therefore, by (28), we obtain the following theorem.

Theorem 6. *For $m, n \in \mathbb{N}$, we have*

$$\sum_{l=0}^{2m-1}\sum_{j=0}^{n}\binom{2m-1}{l}\binom{n}{j}(-1)^j E_{2m-1-l}E_{l+j} = 2E_{2m-1} - \sum_{l=0}^{2m-1}\binom{2m-1}{l}E_{2m-1-l}E_{n+l}.$$

Moreover,

$$\sum_{l=0}^{2m-1}\sum_{j=0}^{2n}\binom{2m-1}{l}\binom{2n}{j}(-1)^j E_{2m-1-l}E_{l+j} = 2E_{2m-1} - E_{2m+2n-1}.$$

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