Eternal Total Domination in Graphs

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Abstract

Eternal domination of a graph requires the vertices of the graph to be protected, against infinitely long sequences of attacks, by guards located at vertices, with the requirement that the configuration of guards induces a dominating set at all times. We study some variations of this concept in which the configuration of guards induce total dominating sets. We consider two models of the problem: one in which only one guard moves at a time and one in which all guards may move simultaneously. A number of upper and lower bounds are given for the number of guards required.

Keywords: eternal domination; total domination; clique covering
AMS Subject Classification Number 2000: 05C69

1 Introduction

We consider finite, simple graphs, and unless stated otherwise, denote the number of vertices of the graph $G = (V, E)$ by n. This paper studies the problem of using guards to defend the vertices of G against a sequence of attacks. At most one guard is located at each vertex. A guard can protect

*Supported by the Natural Sciences and Engineering Research Council of Canada.

the vertex at which its located and can move to a neighboring vertex to defend an attack there. This paper deals with the "eternal" version of the problem in which the sequence of attacks is infinitely long and the configuration of guards induces a dominating set before and after each attack has been defended. Eternal dominating sets have been considered in a number of recent papers such as [1, 5, 11, 12, 17, 18].

In this paper, we introduce the concept of an *eternal total dominating set*, which further requires that each vertex with a guard be adjacent to a vertex with a guard. Total dominating sets are a well-studied variant of dominating sets [9, 14], hence our motivation to extend the notion of eternal domination to total domination.

Several variations of this graph protection problem have been studied, including Roman domination [7, 15], weak Roman domination [8], k-secure sets [4], and eternal m-secure sets [11]. The term Roman domination stems from the problem's ancient origins in Emperor Constantine's efforts to defend the Roman Empire from attackers [16, 21]. Secure domination has been studied previously in [6, 8, 10, 13, 20], for example, and secure total domination has been studied in [2, 19]. One can also consider eternal total domination as a generalization of secure total domination, as the latter deals with attack sequences of length one [2, 19].

We shall compare the sizes of smallest eternal dominating sets, eternal total dominating sets, and other graph parameters such as the clique covering number. We formally define these concepts now. Denote the open and closed neighborhoods of $X \subseteq V$ by $N(X)$ and $N[X]$, respectively, and abbreviate $N(\{x\})$ and $N[\{x\}]$ to $N(x)$ and $N[x]$. For any $D \subseteq V$, we denote by $\langle D \rangle$ the subgraph of G induced by D.

A *dominating set* of G is a set $D \subseteq V$ with the property that for each $u \in V - D$, there exists $x \in D$ adjacent to u. The minimum cardinality amongst all dominating sets is the *domination number* $\gamma(G)$.

A *total dominating set* (*TDS*) of G is a set $D \subseteq V$ with the property that for each $u \in V$, there exists $x \in D$ adjacent to u. The minimum cardinality amongst all total dominating sets is the *total domination number* $\gamma_t(G)$. Note that this parameter is only defined for graphs without isolated vertices.

An *eternal dominating set* (*EDS*) of G is a set D such for each sequence of attacks $R = r_1, r_2, \ldots$ with $r_i \in V$ there exists a sequence $D = D_1, D_2 \ldots$ of dominating sets and a sequence of vertices $s_1, s_2, \ldots$, where $s_i \in D_i \cap N[r_i]$, such that $D_{i+1} = (D_i - \{s_i\}) \cup \{r_i\}$. Note that $s_i = r_i$ is possible. The set D_{i+1} is the set of locations of the guards after the attack at r_i is defended. If $s_i \neq r_i$, we say that the guard at s_i has *moved* to r_i. The minimum cardinality amongst all eternal dominating sets is the *eternal domination number* $\gamma^{\infty}(G)$. If the graph G is to be defended against a single attack r_1, as opposed to an arbitrary sequence of attacks, and the sets $D = D_1$ and D_2 have the properties described above, then D is a *secure dominating set (SDS)*, and the minimum cardinality amongst all secure dominating sets is the *secure domination number* $\gamma_s(G)$.

As discussed in [1, 12], it is often more convenient to model this process as a two-player game: Player 1 chooses D_1 and the vertices $s_1, s_2, \ldots$ while Player 2 chooses the vertices $r_1, r_2, \ldots$ (Player 1 chooses s_i to defend the attack Player 2 makes at r_i.) In other words, the location of an attack can be chosen by the attacker depending on the location of the guards.

The *clique covering number* $\theta(G)$ is the minimum number k of sets in a partition $V = V_1 \cup \cdots \cup V_k$ of V such that the subgraph of G induced by each V_i is complete, i.e., $\theta(G)$ is equal to the chromatic number of the complement $\overline{G}$ of G. We denote the independence number of G by $\alpha(G)$.

Goddard et al. [11] noted that for all graphs G,

$$\alpha(G) \leq \gamma^{\infty}(G) \leq \theta(G). \tag{1}$$

A related upper bound on γ^{∞} is the following.

Theorem 1 [17] *For any graph G,*

$$\gamma^{\infty}(G) \leq \binom{\alpha(G)+1}{2}.$$

It was shown in [12] that this bound is sharp for certain graphs.

An *eternal total dominating set (ETDS)* and a *secure total dominating set (STDS)* of G are defined similarly as an EDS and an SDS, respectively, except that all the sets D_i are total dominating sets. The minimum cardinality amongst all ETDSs and all STDSs are the *eternal total domination number* $\gamma_t^{\infty}(G)$ and the *secure total domination number* $\gamma_{\mathrm{st}}(G)$, respectively. Note that these parameters are only defined for graphs without isolated vertices.

An *m-eternal total dominating set (m-ETDS)* is the same as an eternal total dominating set except that, in response to an attack, we may move as many guards as we wish to neighboring vertices. We call this the "all guards move" model. The minimum cardinality amongst all m-eternal total dominating sets is the *m-eternal total domination number* $\gamma_{\mathrm{mt}}^{\infty}(G)$. An *m-eternal dominating set* is defined similarly and the minimum cardinality amongst all such sets is denoted $\gamma_m^{\infty}(G)$.

Finally, let $\gamma_c^{\infty}(G)$ denote the size of a smallest *eternal connected dominating set (ECDS)*, in which the vertices containing guards induce a connected graph. Denote the all-guards move version of this parameter (the cardinality of a minimum *m-eternal connected dominating set* (*m-ECDS*) by $\gamma_{\rm mc}^{\infty}(G)$. The ordinary connected domination number of G is denoted $\gamma_c(G)$ [14]. Obviously, these parameters are only defined for connected graphs.

For the most part in this paper we shall be concerned with upper bounds on the parameters defined above. It would be an interesting future project to prove lower bounds where appropriate.

2 Preliminaries

We follow the notation and terminology of [14]. The

$$\left.\begin{array}{l} \textit{private neighborhood } \mathrm{pn}(x,X) \\ \textit{external private neighborhood } \mathrm{epn}(x,X) \end{array}\right\}$$

of $x \in X$ relative to X is defined by

$$\left\{\begin{array}{lcl} \mathrm{pn}(x,X) & = & N[x] - N[X - \{x\}] \\ \mathrm{epn}(x,X) & = & \mathrm{pn}(x,X) - \{x\} \end{array}\right.$$

and the vertices in these sets are called, respectively, the

$$\left.\begin{array}{l} \textit{private neighbors} \\ \textit{external private neighbors} \end{array}\right\} \text{ of } x \text{ relative to } X.$$

There exist graphs whose only ETDS is the vertex set of the graph. We now characterize these graphs. Denote the set of leaves of a graph by L, and the set of support vertices (vertices adjacent to leaves) by S.

Proposition 2 *For any graph G, $\gamma_t^{\infty}(G) = n$ if and only if $V - S$ is independent.*

Proof: If $V - S$ is independent, then, as proved in [2], V is the only secure total dominating set and hence also the only ETDS.

Conversely, suppose uv is an edge of G with $u, v \in V - S$ and define $D = V(G) - \{u\}$. Let $D_1 = D$, $D_2 = (D - \{v\}) \cup \{u\}$, $D_i = (D_{i-1} - \{v\}) \cup \{u\}$ if i is even, $D_i = (D_{i-1} - \{u\}) \cup \{v\}$ if i odd. The above sets are all TDSs and so D is an ETDS. □

The condition of Proposition 2 is not necessary for an ECDS to consist of V. For example, if $G = P_6$, then K_2 is a component of $V(P_6) - S$, but $V(P_6)$ is the only ECDS of P_6. A stronger condition than the condition in Proposition 2 is required for a characterization of graphs with $\gamma_c^{\infty} = n$. Let X denote the set of cut-vertices of G.

Proposition 3 *For any graph G, $\gamma_c^\infty(G) = n$ if and only $V - X$ is independent.*

Proof: Consider any $x \in X$ and suppose D is an ECDS of G with $x \notin D$. Then $D \subseteq G - x$, which is disconnected, and since D contains vertices in each component of $G - x$ it follows that D is disconnected, a contradiction. Thus X is a subset of each ECDS of G, and so the guards on vertices in X never move. Therefore, if $V - X$ is independent, then each vertex in $V - X$ needs a guard for protection and it follows that $D = V$.

Conversely, suppose uv is an edge of G with $u, v \in V - X$. We may proceed as in the proof of Proposition 2 to obtain an ECDS $D \subsetneq V$. □

Corollary 4 (*i*) *The set of cut-vertices of any graph is contained in all its eternal connected dominating sets.*

(*ii*) *For any tree T, $\gamma_c^\infty(T) = n$.*

We note that it is easy to see that K_2 is the only connected graph with $\gamma_{\text{mt}}^\infty(G) = n$, and K_1 is the only graph with $\gamma_{\text{mc}}^\infty(G) = n$. To see that these parameters are less than n for all other graphs, it suffices to observe that the values of both parameters are less than n for all trees with at least three vertices, while $\gamma_{\text{mc}}^\infty(K_2) = 1$.

For $n \geq 1$, $\gamma_c^\infty(K_n) = 1$ and for $n \geq 2$, $\gamma_t^\infty(K_n) = 2$. However, all other graphs have an independent set of size at least two, and it is easy to see that $3 \leq \gamma_t^\infty(G) \leq \gamma_c^\infty(G)$ for all connected graphs $G \not\cong K_n$. From [19], we know that $\gamma_t(G) = \gamma_{\text{st}}(G)$ if and only if $\gamma_{\text{st}}(G) = 2$. Combining this with the last observation while noting that $\gamma_{\text{st}}(G)$ is a lower bound for $\gamma_t^\infty(G)$ yields the following.

Fact 5 $\gamma_t(G) = \gamma_t^\infty(G)$ *if and only if* $G = K_n$.

A similar result holds for connected domination.

Proposition 6 $\gamma_c(G) = \gamma_c^\infty(G)$ *if and only if* $G = K_n$.

Proof: Assume $G \not\cong K_n$ and let D be a minimum ECDS of G. From the discussion above, $|D| \geq 3$. It is easy to see that each vertex $v \in V(G) - D$ is adjacent to at least two vertices of D. Let $u \in D$ be a vertex such that $D - \{u\}$ is connected. Then $D - \{u\}$ is a connected dominating set of G. □

Now consider the inequality $\gamma_{\text{mt}}^\infty(G) \leq \gamma_{\text{mc}}^\infty(G)$, which holds if G is not complete. If $G \not\cong K_n$ and $\Delta(G) = n - 1$, then it is easy to see that

$\gamma^{\infty}_{\rm mc}(G) = \gamma^{\infty}_{\rm mt}(G) = 2$. The graphs P_4, C_4, C_5 and C_6 are other graphs with $\gamma^{\infty}_{\rm mt}(G) = \gamma^{\infty}_{\rm mc}(G)$. There are also infinitely many trees T having $\Delta(T) < n-1$ with equality between these two parameters: consider caterpillars – a path with pendant vertices attached to the interior vertices of the path.

Problem 1 *Characterize graphs G with $\gamma^{\infty}_{\rm mt}(G) = \gamma^{\infty}_{\rm mc}(G)$.*

It is interesting to note that $2 \times n$ grid graphs with $n \notin \{1,2,3,4,5,6,8\}$ have $\gamma^{\infty}_{mt} < \gamma^{\infty}_{mc}$. For example, one can verify that $\gamma^{\infty}_{mt}(P_7 \Box K_2) = 6$ whereas $\gamma^{\infty}_{mc}(P_n \Box K_2) = n$.

Problem 2 *It is easy to see that, for $1 \le n \le 6$, $\gamma^{\infty}_{mt}(P_n \Box K_2) = n$. Is it true that, for $n \ge 6$, $\gamma^{\infty}_{mt}(P_n \Box K_2) = 6\lfloor \frac{n}{7} \rfloor + \gamma^{\infty}_{mt}(P_{n \bmod 7} \Box K_2)$.*

We cite C_4, C_5, C_6, and $K_{m,n}$ as graphs with $\gamma_t(G) = \gamma_c(G) = \gamma^{\infty}_{\rm mc}(G)$.

Proposition 7 *If T is a nontrivial tree, then $\gamma^{\infty}_{\rm mc}(T) > \gamma_c(T)$.*

Proof: Suppose to the contrary that $\gamma^{\infty}_{\rm mc}(T) = \gamma_c(T)$. Amongst all minimum m-ECDSs, let D be one such that $\langle D \rangle$ contains a path $P = v_1, v_2, \ldots, v_k$ of maximum length. Trivially, $k \ge \operatorname{diam} G - 1$. Observe that v_1 and v_k have external private neighbors v_0 and v_{k+1}, respectively, otherwise $D - \{v_1\}$ or $D - \{v_k\}$ is a connected dominating set.

Consider an attack at v_0 and let D' be the minimum m-ECDS (and thus a minimum connected dominating set) obtained by defending the attack. Since $v_0 \in \operatorname{epn}(v_1, D)$, the guard on v_1 moves to v_0 to defend the attack. Hence there is some integer q with $1 \le q \le k$ such that the guard on v_i moves to v_{i-1} for each $i = 1, \ldots, q$. If a guard not on P moves to v_q, then in $\langle D' \rangle$, $P' = v_0, v_1, \ldots, v_k$ is a path, contradicting the maximality of P.[1] Hence $v_q \notin D'$. But $\langle D' \rangle$ is connected, so if $q < k$, then the path $v_0, v_1, \ldots, v_{q-1}$ is connected to the path $v_{q+1}, \ldots, v_k$ by some path not containing v_q. This implies that v_q lies on a cycle of T, which is impossible. Thus $q = k$. Note that $v_0 v_{k+1} \notin E(T)$ because T is acyclic. Therefore a guard on some vertex $x \notin V(P)$ moves to a neighbor u of v_{k+1} different from v_k. Since $\langle D' \rangle$ is connected, u is connected to the path $v_0, v_1, \ldots, v_{k-1}$ by some path not containing v_k. But then v_k lies on a cycle of T, a contradiction as before. □

Problem 3 *Characterize graphs G with $\gamma^{\infty}_{\rm mc}(G) = \gamma_c(G)$.*

Problem 4 *Characterize graphs G with $\gamma^{\infty}_{\rm mt}(G) = \gamma_t(G)$.*

[1] This includes the case $q = k$.

3 Total domination and clique covers

In order to survey the landscape, we give several inequality chains and show that some of the inequalities are sharp. The first chain appears in [11]. For all graphs,

$$\gamma \leq \gamma_m^\infty \leq \alpha \leq \gamma^\infty \leq \theta. \tag{2}$$

The graphs with $\gamma = \gamma^\infty$ were characterized in [18] to be those with $\gamma = \theta$. Hence all the equalities may be equalities, for some graphs. Likewise, all the inequalities may be strict for some graphs. In fact, if G is a graph with say $\alpha(G) = 20, \gamma^\infty(G) = 30$ and $\theta(G) = 50$ (which exists, as shown in [18]) and we attach a vertex adjacent to all others, then this new graph has all the parameters in (2) different.

The next chain is obvious: for all graphs,

$$\gamma \leq \gamma_t \leq \gamma_{\mathrm{mt}}^\infty \leq \gamma_t^\infty. \tag{3}$$

Note that $\gamma_t(K_{1,m}) = 2$ and $\gamma_t^\infty(K_{1,m}) = m+1$. Also note that $\gamma(G) < \gamma_t^\infty(G)$ for all graphs G without isolated vertices, because if $\gamma(G) = \gamma_t^\infty$, then $\gamma(G) = \alpha(G)$. But is is easy to see that $\gamma_t^\infty(G) > \alpha(G)$, since no independent set is total dominating. The path P_5 is an example with all the parameters in (3) distinct. There do exist graphs for which $\gamma = \gamma_t = \gamma_{\mathrm{mt}}^\infty$, C_4 being an example. In fact, it is easy to verify that $\gamma_t(C_n) = \gamma_{\mathrm{mt}}^\infty(C_n)$, for all $n \geq 3$.

The third chain is also obvious: for all graphs,

$$\gamma \leq \gamma_m^\infty \leq \gamma_{\mathrm{mt}}^\infty \leq \gamma_t^\infty. \tag{4}$$

Again, P_5 is an example with all the parameters in (4) distinct, while $\gamma_t(K_{m,n}) = \gamma_m^\infty(K_{m,n}) = \gamma_{\mathrm{mt}}^\infty(K_{m,n}) < \gamma^\infty(K_{m,n})$.

It was proved in [19] that if G is connected and $\theta(G) > 1$, then $\gamma_t(G) \leq 2\theta - 2$, and that the bound is sharp. Likewise, there exist many graphs G with $\theta(G) > 1$ and $\gamma_t^\infty(G) = 2\theta$. We can, however, prove a slightly better bound for $\gamma_{\mathrm{mt}}^\infty(G)$.

For any graph G, fix a minimum clique cover $\mathcal{C}$ of G. Construct the *clique cover graph* $\mathcal{C}(G)$ *of* G *with respect to* $\mathcal{C}$ by mapping each clique in $\mathcal{C}$ to a corresponding vertex in $\mathcal{C}(G)$ such that two vertices in $\mathcal{C}(G)$ are adjacent if and only if the corresponding cliques in G have adjacent vertices. For each vertex v of $\mathcal{C}(G)$, let Q_v be the clique in $\mathcal{C}$ corresponding to v. Since $\mathcal{C}$ is a minimum clique cover, no two adjacent vertices v, v' of $\mathcal{C}(G)$ correspond to cliques Q_v and $Q_{v'}$ with $Q_v = Q_{v'} = K_1$.

Theorem 8 *If G is connected and $\theta(G) \geq 2$, then $\gamma_{\mathrm{mt}}^{\infty}(G) \leq 2\theta(G) - 1$. This bound is sharp for all $\theta \geq 2$.*

Proof: The case $\theta(G) = 2$ is trivial, so assume $\theta(G) \geq 3$. Fix a minimum clique cover $\mathcal{C}$ of G and consider a spanning tree T of $\mathcal{C}(G)$, the vertex set of which is also a TDS of $\mathcal{C}(G)$, since $\theta(G) > 1$. Define $\mathcal{V}_1 = \{v \in V(T) : Q_v = K_1\}$ and let a be a support vertex of T, say a is adjacent to the leaf ℓ. For each $v \in V(T)$ we define a set $D_v \subseteq V(G)$ as follows.

Let u_ℓ be a vertex of Q_ℓ adjacent to a vertex w_a of Q_a and define $D_a = \{w_a\}$.

If $\ell \in \mathcal{V}_1$, let $D_\ell = \{u_\ell\}$; otherwise, let w_ℓ be any other vertex of Q_ℓ and define $D_\ell = \{u_\ell, w_\ell\}$.

For all $v \in \mathcal{V}_1 - \{a, \ell\}$, let x_v be the vertex of Q_v, let y_v be any vertex of G adjacent to x_v and define $D_v = \{x_v, y_v\}$.

For all $v \in V(T) - (\mathcal{V}_1 \cup \{a, \ell\})$, if each vertex of Q_v is a vertex $y_{v'}$ for some $v' \in \mathcal{V}_1$, let $D_v = \phi$; if some but not all vertices of Q_v are such a vertex $y_{v'}$, let z_v be a vertex of Q_v not already chosen and define $D_v = \{z_v\}$; and if no vertex of Q_v is such a vertex $y_{v'}$, let y_v, z_v be any two vertices of Q_v and define $D_v = \{y_v, z_v\}$.

Define $D_1 = \bigcup_{v \in V(T)} D_v$. Then D_1 is a TDS of G and $|D_1| \leq 2\theta(G) - 1$. A guard is stationed at each vertex in D_1. For each $v \in V(T) - (\mathcal{V}_1 \cup \{a, \ell\})$ for which z_v is defined, let g_v be the guard stationed at z_v. Also, let g_a (g_ℓ, g'_ℓ, respectively) be the guard stationed at w_a (w_ℓ, u_ℓ, respectively).

Consider an attack at a vertex $r_1 \in V(G) - D_1$.

(*i*) If r_1 belongs to Q_v for $v \in V(T) - \{a, \ell\}$, then $r_1 \in N(z_v)$ and guard g_v moves to r_1; note that $D_2 = (D_1 - \{z_v\}) \cup \{r_1\}$ is a TDS of G.

(*ii*) If r_1 belongs to Q_ℓ, then $Q_\ell \neq K_1$, $r_1 \in N(w_\ell)$ and g_ℓ moves to r_1; note that $D_2 = (D_1 - \{w_\ell\}) \cup \{r_1\}$ is a TDS of G.

(*iii*) The only other attack (at a vertex not in D_1) is at a vertex of Q_a. Then $r_1 \in N(w_a)$, $w_a \in N(u_\ell)$, and, if w_ℓ is defined, $u_\ell \in N(w_\ell)$. Thus g_a moves to r_1, g'_ℓ moves to w_a and g_ℓ moves to u_ℓ if necessary; note that $D_2 = (D_1 - \{u_\ell\}) \cup \{r_1\}$ or $D_2 = (D_1 - \{w_\ell\}) \cup \{r_1\}$ (as appropriate) is a TDS of G.

Hence D_1 defends G against any single attack. Note that in each case $w_a \in D_1 \cap D_2$. Suppose, after i attacks, D_i has defended G against the i^{th} attack, and D_{i+1} is a TDS of G. Consider an attack at a vertex $r_{i+1} \in V(G) - D_{i+1}$.

If r_{i+1} belongs to Q_v for $v \in V(T) - \{a, \ell\}$, then as in (*i*), g_v defends against this attack, regardless of any previous attacks.

If r_{i+1} belongs to Q_ℓ and g_ℓ is stationed at a vertex of Q_ℓ other than u_ℓ, then g_ℓ defends as in (*ii*). If g_ℓ is stationed at u_ℓ, then, reversing the defense in (*iii*), g_ℓ moves to r_{i+1}, g'_ℓ moves from w_a to u_ℓ and g_a moves to w_a.

If r_{i+1} belongs to Q_a and g_a is stationed at w_a, then g_a, g_ℓ and g'_ℓ defend as in (*iii*), while if g_a is stationed at a vertex of Q_a other than w_a, then g_a moves to r_{i+1}.

It follows that each set D_i, $i = 1, \ldots,$ is an ETDS of G.

Since $\gamma_{\mathrm{mt}}^{\infty}(P_4) = 3$, the bound is exact for $\theta = 2$. For $\theta \geq 3$, construct the class $\mathcal{G}_\theta$ of graphs as follows. Let $H = K_m$ for any $m \geq \theta$, and let $F_1, \ldots, F_{\theta-1}$ be disjoint nontrivial complete graphs. Join one vertex u_i of each F_i to some vertex v_i of H so that $\theta - 1$ vertices of H are joined to a vertex not in H. Let $\mathcal{G}_\theta$ be the class of all graphs thus constructed. Note that $\theta(G) = \theta$ for each $G \in \mathcal{G}_\theta$.

Let $G \in \mathcal{G}_\theta$, consider any minimum m-ETDS D of G and suppose $|D| \leq 2\theta - 2$. Since D is a TDS, either $|D \cap V(F_i)| = 2$, or $|D \cap V(F_i)| = 1$ and $\{u_i, v_i\} \subseteq D$. It follows that $|D| = 2\theta - 2$. Let x be a vertex of H that is not adjacent to a vertex of any F_i. Then $x \notin D$, hence $\{u_i, v_i\} \subseteq D$ for at least one i to dominate x. To defend an attack at x, a guard at some v_i moves to x. If the guard at u_i does not move, then u_i is isolated in the resulting set, and if the guard at u_i moves to v_i to ensure that there are no isolated vertices, then at least one vertex in F_i is not dominated. In either case we obtain a contradiction. □

The graphs constructed in the proof of Theorem 8 to show that the bound is sharp are not the only graphs with this property. We characterize this class of graphs in the next theorem. Recall that a *star* is a graph isomorphic to $K_{1,m}$.

Theorem 9 *If G is connected, then $\gamma_{\mathrm{mt}}^{\infty}(G) = 2\theta - 1$ if and only if one of the following conditions holds.*

(*i*) *$\theta(G) = 2$, $\Delta(G) < n - 1$, and in any minimum clique covering of G, there is a vertex that is not adjacent to any vertex in the clique that does not contain it*[2].

(*ii*) *$\theta(G) = k \geq 3$, and for any minimum clique covering C of G, $C(G)$ is a star with center (say) x and leaves $u_1, \ldots, u_{k-1}$, such that*

(a) *$Q_{u_i} \neq K_1$ for each i,*

[2]It can be shown that under these conditions there is a unique minimum clique covering.

(b) *no vertex of Q_x is adjacent to vertices in more than one Q_{u_i},*

(c) *there either exists a vertex $v \in Q_x$ such that $N[v] = Q_x$; or there exist two vertices of Q_x that are adjacent to vertices in Q_{u_i} for some i, but not to a common vertex in Q_{u_i}, and some vertex in Q_{u_i} has no neighbors in Q_x.*

(iii) $\theta(G) = 3$, and for any minimum clique covering C of G, $C(G)$ is a triangle such that (ii) holds for any spanning subtree of $C(G)$.

Proof: (*i*) Suppose $\theta(G) = 2$ and let $C = \{Q_1, Q_2\}$ be a minimum clique cover of G. If $\deg(u) = n-1$ for some vertex u, place one guard on u and another guard on an arbitrary vertex v. Assume $v \in Q_1$. To defend an attack at an unguarded vertex w, the guard on v moves to w if $w \in Q_1$, and if $w \in Q_2$, then the guard on u moves to w while the guard on v moves to u. It is clear that this strategy may be repeated indefinitely to yield a TDS of G. Hence $\gamma_{\mathrm{mt}}^{\infty}(G) = 2$. Now suppose that each vertex in Q_i is adjacent to some vertex in Q_j, $i \neq j$. Place one guard on an arbitrary vertex $u_1 \in Q_1$ and another guard on a neighbor u_2 of u_1 in Q_2. To defend an attack at an unguarded vertex $w \in Q_i$, the guard on u_i moves to w while the guard on u_j, $j \neq i$, moves to a neighbor of w in Q_j. It is clear that this strategy can also be repeated indefinitely.

Hence suppose $\Delta(G) < n-1$, let $u \in Q_1$ be a vertex that is not adjacent to any vertex in Q_2 and let D be any TDS of G that contains u. Then $v \in D$ for some vertex $v \in Q_1 - \{u\}$, otherwise u is isolated in $\langle D \rangle$. But $\deg(v) < n-1$, hence there exists a vertex $w \in Q_2$ such that $\{u, v\}$ does not dominate w. It follows that $\gamma_{\mathrm{mt}}^{\infty}(G) \geq \gamma_t(G) \geq 3$.

(*ii*) Suppose $\theta(G) \geq 3$ and assume firstly that C is a minimum clique cover of G such that $C(G)$ is not a star or a triangle. Then $C(G)$ has a spanning tree T that is not a star and so $\operatorname{diam} T \geq 3$. Hence we may remove an edge of T to obtain two nontrivial trees T_1 and T_2 of order k_1 and k_2, respectively, where $k_1 + k_2 = \theta(G)$. Proceeding as in the proof of Theorem 8 in each T_i separately, we can show that $\gamma_{\mathrm{mt}}^{\infty}(G) \leq 2k_1 - 1 + 2k_2 - 1 = 2\theta(G) - 2$.

Now assume that for any minimum clique covering C of G, $C(G)$ is a star with center x and leaves $u_1, ..., u_{k-1}$. We henceforth abbreviate Q_{u_i} to Q_i. By the proof of Theorem 8, if $Q_i = K_1$ for some i, then $\gamma_{\mathrm{mt}}^{\infty}(G) \leq 2\theta(G) - 2$. Hence we may assume that $Q_i \neq K_1$ for each i.

Suppose that $v \in Q_x$ is adjacent to a vertex $w_i \in Q_i$ for $i = 1, 2$. Place a guard on each of v, w_1 and w_2. Also place a guard on $w_2' \in Q_2$, and for each $i = 3, ..., k-1$, place a guard on two vertices $w_i, w_i' \in Q_i$. Thus $2\theta(G) - 2$ guards are deployed. It is obvious that attacks at vertices of Q_i, $i = 3, ..., k-1$ can be defended indefinitely by moving one of the two guards, neither of which ever moves outside Q_i. To defend an attack at an unguarded vertex y in Q_1, the guards move as follows: $w_2' \to w_2 \to v \to w_1 \to y$. Now to defend an attack at an unguarded vertex z in Q_2, this movement is reversed, except that the guard on w_2 moves to z instead of w_2'. It is obvious how to defend an attack at an unguarded vertex in Q_1 or Q_2 when there are two guards present in the clique. To defend an attack against an unguarded vertex q in Q_x when, without loss of generality, there are two guards (on w_2 and w_2') in Q_2, the guards move as follows: $w_2' \to w_2 \to v \to q$. Now if there is an attack in Q_2, this movement is reversed, and a similar movement of guards defends an attack in Q_1. It follows that $\gamma_{\mathrm{mt}}^{\infty}(G) \leq 2\theta(G) - 2$.

Hence we assume henceforth that each vertex in Q_x is adjacent to vertices in at most one Q_i. Suppose (ii)(c) does not hold. Then each vertex in Q_x is adjacent to a vertex in some Q_i, and if two vertices in Q_x are adjacent to vertices in some Q_i but do not have a common neighbor in Q_i, then $Q_i \subseteq N(Q_x)$. That is, if some vertex in Q_i is not adjacent to any vertex in Q_x, then there is a vertex $w_i \in Q_i$ that dominates all vertices in Q_x with neighbors in Q_i. Define

$$I = \{i : \text{some vertex in } Q_i \text{ is not adjacent to any vertex in } Q_x\} \text{ and}$$
$$J = \{1, ..., k-1\} - I.$$

For each $i \in I$, let w_i be a vertex in Q_i which dominates all the neighbors of Q_i in Q_x, and let $w_i' \in Q_i - \{w_i\}$. For each $j \in J$, let z_j be any vertex in Q_j and $y_j \in Q_x$ a neighbor of z_j. Place a guard on each w_i, w_i', z_j and y_j.

For $i \in I$, defend an attack at a vertex in Q_i by moving the guard on w_i' to the attacked vertex. Defend an attack at a neighbor $v_i \in Q_x$ of w_i by moving the guards on w_i and any other vertex in Q_i to $v_i$and w_i, respectively. Defending subsequent attacks at vertices in Q_i or neighbors of w_i in Q_x is a simple matter; details are omitted. For $j \in J$, defend an attack at a vertex $z_j' \in Q_j$ by moving the guard on z_j to z_j' and the guard on y_j to a neighbor y_j' of z_j' in Q_x (if $y_j' \neq y_j$). Defend an attack at a vertex in Q_x adjacent to a vertex in Q_j, and subsequent attacks of any kind, similarly. Since each vertex in Q_x is adjacent to a vertex in some Q_i, G can be guarded with $2\theta(G) - 2$ guards.

Assume (ii) and (a) - (c) hold for any minimum clique cover of G. If $y \in Q_x$ is adjacent to all vertices in Q_i for some i, let C' be the clique cover with $Q_x' = Q_x - \{y\}$, $Q_i' = Q_i \cup \{y\}$, and $Q_j' = Q_j$ for $j \neq i$. Let z be a vertex in Q_x' adjacent to a vertex in Q_j', $j \neq i$. Since Q_x is a clique, z is adjacent to $y \in Q_x$. But $y \in Q_i'$, so (b) does not hold for z in C', a contradiction. Hence

$$\text{for each } i, \text{ no vertex in } Q_x \text{ is adjacent to all vertices in } Q_i. \qquad (5)$$

Assume firstly that $\mathcal{C}$ is a clique cover such that some $v \in Q_x$ is not adjacent to a vertex in any Q_i. Let D be a TDS of G containing v. If $D \cap Q_i = \varnothing$, then by (5), Q_i is dominated by at least two vertices in Q_x, neither of which is v. If $D \cap Q_i \neq \varnothing$, then either $|D \cap Q_i| \geq 2$ (since $\langle D \rangle$ has no isolated vertices), or $D \cap Q_i = \{w_i\}$ (say), and w_i is adjacent to a vertex in $Q_x \cap D$. By (b), if v_i, v_j are adjacent to vertices in Q_i, Q_j, respectively, then $v_i \neq v_j$ for $i \neq j$. Hence $|D| \geq 2\theta(G) - 1$ and therefore any m-ETDS of G has cardinality at least (and thus exactly) $2\theta(G) - 1$.

Finally, assume that $\mathcal{C}$ is a clique cover such that $v, v' \in Q_x$ are respectively adjacent to $w, w' \in Q_1$ (say), but not to a common vertex in Q_1, and $z \in Q_1$ is not adjacent to any vertex in Q_x. Suppose to the contrary that $\gamma^{\infty}_{\text{mt}}(G) \leq 2\theta(G) - 2$ and let D be an m-ETDS of G containing z. As above, for $i \geq 2$, either $|D \cap Q_i| \geq 2$, or $|D \cap Q_i| = \{w_i\}$ and w_i is adjacent to $v_i \in Q_x \cap D$. Since $z \in D$, $|D \cap Q_1| \geq 2$. If $|D \cap Q_1| \geq 3$, we are done, so suppose $|D \cap Q_1| = 2$. Since v and v' do not have a common neighbor in Q_1 and neither of them is adjacent to z, at least one of them, say v, is not protected by a vertex in $D \cap Q_1$. But v is also not protected by any $v_i \in D$, because if the guard on v_i moves to v, the guard on w_i is either isolated, or moves to v_i, in which case (5) asserts that not all vertices in Q_i are dominated. This contradiction completes the proof of (ii).

(iii) If G has a minimum clique cover $\mathcal{C}$ such that $\mathcal{C}(G)$ is a triangle and for some spanning subtree of $\mathcal{C}(G)$, (ii) does not hold, then G can be guarded by $4 = 2\theta(G) - 2$ guards as described in the different cases in the proof of (ii). Suppose (iii) holds for any minimum clique cover $\mathcal{C}$ of G; say $\mathcal{C} = \{Q_0, Q_1, Q_2\}$. By (ii)(b) there exist distinct vertices $v_i, w_i \in Q_i$, $0 \leq i \leq 2$, such that w_i is adjacent to $v_{i+1 \pmod 3}$. Thus $v_0, w_0, v_1, w_1, v_2, w_2, v_0$ is a 6-cycle, and (ii) also implies that it is an induced 6-cycle.

By considering the three spanning trees of $\mathcal{C}(G)$ separately, it follows from (ii)(c) that (without loss of generality) one of the following three conditions holds:

(I) Each Q_i contains a vertex u_i with $N[u_i] \subseteq Q_i$.

(II) For $i = 0, 1$, Q_i contains a vertex u_i with $N[u_i] \subseteq Q_i$, and Q_2 contains two vertices, say a_2, b_2, that are adjacent to two vertices $a_0, b_0 \in Q_0$ respectively, and no vertex in Q_0 is adjacent to a_2 as well as b_2.

(III) Q_0 contains a vertex u_0 with $N[u_0] \subseteq Q_0$, and for $i = 1, 2$, Q_i contains two vertices, say a_i, b_i, such that a_1, b_1 are adjacent to $a_0', b_0' \in Q_0$ respectively, a_2, b_2 are adjacent to $a_0, b_0 \in Q_0$ respectively, and no vertex in Q_0 is adjacent to a_1 as well as b_1, or to a_2 as well as b_2. Moreover, by (ii)(b), $\{a_0, b_0\} \cap \{a_0', b_0'\} = \varnothing$.

If (I) or (II) holds, it is easy to see that $\gamma_{\mathrm{mt}}^{\infty}(G) = 5$; details are omitted. Assume (III) holds, suppose to the contrary that $\gamma_{\mathrm{mt}}^{\infty}(G) = 4$ and let D be an m-ETDS containing u_0. Then $|D \cap Q_0| \geq 2$. Since no vertex in Q_0 is adjacent to w_1 or v_2, and (by (ii)(b)) no vertex adjacent to Q_0 is also adjacent to both w_1 and v_2, two vertices in $Q_1 \cup Q_2$ are required to totally dominate w_1 and v_2. Hence $|D \cap Q_0| = 2$ and $|D \cap (Q_1 \cup Q_2)| = 2$; say $D \cap (Q_1 \cup Q_2) = \{x, y\}$. Now, at most one of a_0, a_0', b_0, b_0' is in D, so x and y are required to dominate at least three of a_1, b_1, a_2, b_2. But, again by (ii)(b), no vertex in Q_2 is adjacent to a_1 or b_1, and no vertex in Q_1 is adjacent to a_2 or b_2. We may therefore assume without loss of generality that $x \in Q_1 - \{a_1, b_1\}$ and $y \in Q_2 - \{a_2, b_2\}$. We may also assume that $\{a_0, b_0, b_0'\} \cap D = \varnothing$.

The only possible defense of an attack at a_2 requires the guard on y to move to a_2. To avoid becoming isolated, the guard on x moves to y. But then b_1 is not dominated, a contradiction. □

4 Bounds on eternal total domination number

We now show the eternal total domination number is always greater than the eternal domination number. Note that there exist graphs G such as $K_{1,m}$ having $\gamma_t^{\infty}(G) = \gamma^{\infty}(G) + 1$.

Theorem 10 *For all graphs $G = (V, E)$ without isolated vertices, $\gamma_t^{\infty}(G) > \gamma^{\infty}(G)$.*

Proof: Let $D = \{v_1, v_2, \ldots, v_k\}$ be an ETDS. Observe that no vertex in D has an external private neighbor, else an attack at a vertex v that is an external private neighbor of some vertex in D will destroy the total dominating set.

We prove that there is a set $D' \subset D, |D'| = k - 1$ such that D' is an EDS. Observe that $D' = D - \{v_k\}$ is a dominating set, since each vertex in $V - D$ has at least two neighbors in D and each vertex in D has a neighbor in D. Our strategy is to have D' "shadow" D by defending each attack as described below and always maintaining after each attack that the modified set D' is a subset of size $k - 1$ of the modified set D.

Assume that after m attacks that $D_m' \subset D_m, |D_m'| = k - 1$ and D_m' is a dominating set, where the subscripts m indicate that the ETDS D_m and D_m' have evolved over the prior m attacks. Suppose the $m + 1^{\mathrm{st}}$ attack is at v. If $v \in D_m'$, then we have nothing to do. If $v \in D_m, v \notin D_m'$, then v has a neighbor w in D_m such that $w \in D_m'$. Defending the attack with a guard at w maintains $D'_{m+1} \subset D_m = D_{m+1}$. If $v \notin D_m$, there are two cases. If D_m defends with a guard at $y \in D'_m$, then D'_m defends with the same guard and we maintain $D'_{m+1} \subset D_{m+1}$. On the other hand, if D_m defends with a guard at $z \notin D'_m$, then D'_m defends with a guard at u such that $uv \in E$. Such a vertex u exists because no vertex is an eternal private neighbor of any vertex in D'_m. Therefore, we maintain $D'_{m+1} \subset D_{m+1}$ and $|D'_{m+1}| = k-1$. As above, any subset of size $k-1$ of an ETDS of size k is a dominating set. Hence the proof.□

The same does not hold in the "all-guards move" model. That is, there exist graphs G, such as C_4, for which $\gamma^{\infty}_{\mathrm{mt}}(G) = \gamma^{\infty}_{m}(G)$. In fact, there exist infinitely many graphs such that $\gamma^{\infty}_{\mathrm{mc}}(G) = \gamma^{\infty}_{m}(G)$: take a path with $n \geq 2$ vertices, attach a pendant vertex to each interior vertex on the path, and attach at least two pendant vertices to each of the end vertices of the path.

Conjecture 1 *For all connected graphs G with $\Delta(G) < n-1$, $\gamma^{\infty}_{c}(G) > \theta(G)$.*

Note that there exist graphs, such as $K_{m,n}$, for which $\gamma^{\infty}_{\mathrm{mc}}(G) < \theta(G)$.

Theorem 11 *For all graphs $G = (V,E)$ without isolated vertices, $\gamma^{\infty}_{t}(G) \leq \gamma^{\infty}(G) + \gamma(G) \leq 2\gamma^{\infty}(G) \leq 2\theta(G)$.*

Proof: Let A be a minimum dominating set of G and B be a minimum eternal dominating set of G such that $|A \cap B|$ is as small as possible. Note that if $A = B$, then each contains all the vertices in the graph and so we are done, hence we may assume that $A \neq B$.

If $A \cap B = \emptyset$, then we are done by the following strategy: keep a guard at each vertex of the two sets. If an attack occurs at a vertex in either set, do nothing, else move a guard from a vertex in B, using the eternal domination strategy that exists for B. Since each vertex in A is adjacent to at least one vertex in B (since B is a dominating set also) and vice versa, we have a total dominating set.

Suppose $A \cap B \neq \emptyset$. Keep a guard at each vertex of $A \cup B$. Our basic strategy will be to keep the guards of $A - B$ fixed at their initial locations and to move the guards of B, as necessary (when at attack occurs at a vertex without a guard), as per an eternal domination strategy for set B (as above, such a strategy acts as if there were only guards at B). However, we may need some additional guards.

Let $v \in A \cap B$. We use the minimality of $A \cap B$ to prove that any guard in $A \cap B$ never needs to move. Clearly any guard in $A \cap B$ will never need to move to a vertex in $A - B$ (since the guards in $A - B$ never move).

Likewise, no guard initially located at a vertex of $B-A$ ever needs to move to a vertex in $A-B$. Furthermore, no guard in $B-A$ needs to move to a vertex in $A \cap B$ as long as each vertex in $A \cap B$ contains a guard. Thus if a guard at $v \in A \cap B$ moves to some vertex in $V - \{A \cup B\}$, then we have found a new configuration of guards such that $A \cap B$ is smaller. Hence we can defend any sequence of attacks (ignoring the requirement of total domination for the moment) moving only guards in $B-A$.

In order to guarantee that we maintain a total dominating set, we need to ensure each vertex in $A \cap B$ has a neighbor with a guard; it is easy to see that the strategy above ensures that each vertex in $A-B$ and $B-A$ has a neighbor with a guard, since A and B are themselves dominating sets. Thus for each vertex $v \in A \cap B$, keep a guard at a neighbor of v and this guard will never move. Hence we maintain a total dominating set with at most $\gamma^{\infty}(G) + \gamma(G)$ guards.

The last two inequalities are obvious.□

Note that K_n and P_4 are examples where the bound is sharp. We could also ask whether the ratio $\gamma_t^{\infty}/\gamma_t$ can be bounded by a constant. To see that the answer is negative, consider complements of Kneser graphs, which we denote as $G(n,k)$ The vertices of these graphs are all the k-sets drawn from $\{1,2,\ldots,n\}$ with two vertices adjacent if and only if their k-sets have a non-empty intersection. It was shown in [12] that some of these graphs have $\gamma^{\infty} = \binom{\alpha(G)+1}{2}$. It is easy to see that $\alpha(G(n,k)) = \lfloor \frac{n}{k} \rfloor$. Likewise, it is not difficult to see that $G(n,k)$ contains a total dominating set of size at most $2\lfloor \frac{n}{k} \rfloor$. Note that this also implies that there exist graphs for which γ_t^{∞} is much larger than $\gamma_{\mathrm{mt}}^{\infty}$.

Theorem 12 *For all graphs $G=(V,E)$ without isolated vertices, $\gamma_{\mathrm{mt}}^{\infty}(G) \leq 2\gamma(G)$.*

Proof: Let D be a minimum dominating set of G such that $\mathrm{epn}(v,D) \neq \emptyset$ for each $v \in D$. (Such a set exists for all graphs without isolated vertices – see [3].) Place a guard at each $v \in D$ and at $v' \in \mathrm{epn}(v,D)$, define $D' = D \cup \{v' : v \in D\}$ and note that $|D'| = 2\gamma(G)$.

If an attack occurs at a vertex $u \notin D'$, move a guard located at $v \in N(u) \cap D$ to u and move the guard at v' to v, and note that a total dominating set containing D is obtained. This process can obviously be repeated indefinitely. □

The graphs K_n and P_5 are examples where the bound is sharp, and $K_{n,n}$ is an example where $\gamma^{\infty}(K_{n,n}) = n$, $\gamma_t^{\infty}(K_{n,n}) = n+1$ and $\gamma_{\mathrm{mt}}^{\infty}(K_{n,n}) = 2$, for all $n \geq 1$.

5 Paths and Trees

5.1 Paths

Theorem 13 *The eternal total number domination number of P_n is $\lceil 3\frac{n-2}{4}\rceil + 2$ (for $n > 1$).*

Proof: For $1 < n \leq 5$, it is obvious that n guards are required. Assume the path is laid out from left to right. Note that we must keep a guard at all times at both the leftmost and rightmost vertices and there can never be adjacent vertices without guards. For $n > 5$, observe that if we choose to keep two guards fixed at the leftmost two vertices followed by an "empty" vertex, then one must initially have guards on the next three vertices and the only guard of these three that can ever move is the leftmost guard. Alternatively, one could choose to keep three guards fixed at the leftmost three vertices, but it is not difficult to verify that this provides no advantage. The proof follows by a straightforward induction. □

The value of secure total domination number was given in [2] as $\lceil 5\frac{n-2}{7}\rceil + 2$. This is equal to the secure total domination number when $n \leq 19$ and $n \notin \{9, 13, 17\}$.

One can also easily show that $\gamma^{\infty}_{\mathrm{mt}}(P_n) = \lceil \frac{2n}{3} \rceil$, for all $n > 1$.

5.2 Trees

We describe a family of trees that can be partitioned into stars (i.e., $K_{1,m}$'s) of order at least three in a special way. In such a partitioning, the value of m can be different for different $K_{1,m}$'s. Let T be a tree. Fix one vertex v as the root of T and let the *height* of a vertex be its distance from v. If u is a vertex in T of height h, then its *parent* is the unique vertex x of height $h-1$ such that xu is an edge, and its *grandparent* is its parent's parent. A *sibling* of u is any vertex other than u having the same parent as u. We iteratively try to partition the vertices of T into stars of order at least three as follows. Let w be a vertex of maximum height not yet contained in a part such that neither its parent nor its grandparent is contained in any part. Create a new part containing w, its parent, its grandparent, and all siblings of u that are not yet contained in a part. If we reach the root and there remains an additional star containing at least three vertices including the root, then those vertices form a part. If this process terminates with all the vertices of T contained in a part, then we say T has a *perfect partitioning.* The *partition number* of T is the minimum number of parts that can be formed during the partitioning process. A *star-partition* is formed as a result of this process, though it may be the case that a star-partition is not a perfect partitioning; i.e., some vertices may not be contained in any star having at least three vertices.

Theorem 14 *Let T be a tree with at least three vertices and partition number q. Then $\gamma_{\mathrm{mt}}^{\infty}(T) = 2q$ if and only if T has a perfect partitioning.*

Proof: If T has a perfect partitioning, then it is easy to see that $\gamma_{\mathrm{mt}}^{\infty} \leq 2q$ since $\gamma_{\mathrm{mt}}^{\infty}(P_3) = 2$ and $\gamma_{\mathrm{mt}}^{\infty}(K_{1,m}) = 2$.

Now we prove that at least two guards are required in each part at all times. The proof is by induction on q. The case when $q = 1$ is easy to see. Let T be a tree with partition number $q > 1$. Let P be a part of T containing a leaf and let $T' = T - P$. Clearly, P contains at least two guards. Suppose there exists an m-eternal total dominating set with fewer than two guards in some part of T. Since no guard from P can move outside of P, there exists an m-eternal total dominating set of T' with fewer than two guards in some part of T'. This is a contradiction.

For the other direction, we must prove that if T does not have a perfect partitioning, then $\gamma_{\mathrm{mt}}^{\infty}(T) > 2q$. Perform the partitioning algorithm described above. Of course, some vertices will be contained in no part, since T does not have a perfect partitioning. By a similar induction as above, we obtain that $\gamma_{\mathrm{mt}}^{\infty}(T) > 2q$. □

Corollary 15 *Let T be a tree with at least two vertices, partition number q, and c vertices contained in no part. Then $2q + c \geq \gamma_{\mathrm{mt}}^{\infty}(T) \geq 2q$.*

Proof: The lower bound follows from the discussion above. If v is a vertex in no part, there are two cases. If all of v's neighbors are contained in parts, by initially placing a guard on v, we have a total dominating set if we place two guards in each part on two vertices of minimum height. It is easy to see that a total dominating set can be maintained eternally, though the guard on v may have to move to a neighbor. If v has a neighbor u that is not in a part, then the neighborhood of either u or v containing vertices not in parts induces a $K_{1,m}$ and two guards suffice for a m-eternal total dominating set. □.

There exist many trees T with partition number q, but $\gamma_t(T) < 2q$, such as P_9 and P_{12}. It is easy to prove that $\gamma_t(T) \leq 2q$ for all trees T.

Call an arbitrary partitioning of the vertices of a tree into $K_{1,m}$'s a $K_{1,m}$-partitioning (the value of m can be different for different $K_{1,m}$'s). There exist trees without a perfect partitioning that can be partitioned into j $K_{1,m}$'s, even if we require each $K_{1,m}$ to have at least three vertices. This gives an upper bound on $\gamma_{\mathrm{mt}}^{\infty}(T)$ of $2j$ for such trees, which is in some cases better than twice the partition number plus the number of vertices not contained in any part of a star-partition. However, there also exist trees without a perfect partitioning such that twice partition number of the tree plus the number of vertices not contained in any part of a star-partition is equal to $\gamma_{\mathrm{mt}}^{\infty}(T)$ and this is less than twice the number of $K_{1,m}$'s in some $K_{1,m}$-partitionings of the tree. An example of such is the following: connect two claws with an edge joining one of the degree one vertices from each claw, say u and v. Attach a pendant vertex to u.

Proposition 16 *Let T be a tree such that all maximal paths are of length at least ten. Then $\gamma_{\mathrm{mt}}^{\infty}(T) > \gamma_t(T)$.*

Proof: If T is isomorphic to P_n, $n \geq 11$, then the proposition is easy to verify. Otherwise, let P be a shortest maximal path in T (so P has at least eleven vertices). Then $\gamma_t(P)+2 \leq \gamma_{\mathrm{mt}}^{\infty}(P)$, so we can force $\gamma_t(P)+2$ guards to be in P. Furthermore, since P is a shortest maximal path, at most one vertex in P, say v, has degree greater than two in T. A guard at v can dominate at most one vertex in $T-P$. It is clearly not possible to totally dominate the vertices of $T-P$ with $\gamma_t(T-P)-2$ vertices plus v. □

Though we believe the following conjecture is true, we suspect that even stronger statements are likely to be true (i.e., if one weakens the conditions about leaf heights).

Conjecture 2 *There exists a constant c such that for all trees T having all leaves of height at least c, $\gamma_{\mathrm{mt}}^{\infty}(T) > \gamma_t(T)$.*

Acknowledgements We thank the anonymous referee for their careful reading of the paper and their valuable suggestions.

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