Fault tolerant path-embedding in locally twisted cubes

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Abstract The locally twisted cube LTQ_n is an important variation of hypercube and possesses many desirable properties for interconnection networks. In this paper, we investigate the problem of embedding paths in faulty locally twisted cubes. We prove that a path of length l can be embedded between any two distinct vertices in $LTQ_n - F$ for any faulty set $F \subset V(LTQ_n) \cup E(LTQ_n)$ with $|F| \leq n - 3$ and any integer l with $2^{n-1} - 1 \leq l \leq |V(LTQ_n - F)| - 1$ for any integer $n \geq 3$. The result is tight with respect to the two bounds on path length l and faulty set size $|F|$ for a successful embedding.

Keywords: Interconnection network; Locally twisted cube; Fault-tolerant; Path

1 Introduction

An interconnection network can be represented by a connected graph $G = (V, E)$, where the vertex set $V = V(G)$ represents the set of processing elements and the edge set $E = E(G)$ represents the set of communication channels, respectively. In this paper, we use graphs and interconnection networks (networks for short) interchangeably.

The embedding problem, which maps a guest graph into a host graph, is an important topic in recent years. Many graph embeddings take paths, cycles, trees, and meshes as guest graphs [2, 3, 4, 9, 10, 12, 13, 15, 16], because they are the architectures widely used in parallel computing systems. In particular, paths are probably the most common structure of graph embedding in parallel computing since paths are often used to model linear arrays [1, 7, 11].

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Failures are inevitable when a large network is put in use. It is very important to study graph embedding in the case where some vertices and/or edges in the host graphs have become faulty. It is desirable to find an embedding of a guest graph into a host graph where all faulty elements have been removed. This is called fault-tolerant embedding.

It is well known that the hypercube network Q_n is one of the most popular interconnection networks. As an important variant of Q_n, the locally twisted cube, LTQ_n, proposed by Yang et al. [14], has many properties superior to Q_n. One advantage is that the diameter of locally twisted cubes is only about half of the diameter of hypercubes. The variously desirable properties of LTQ_n have been extensively investigated in the literature [4, 5, 6, 15].

In this paper, we study embedding of paths of different lengths between any two vertices in faulty locally twisted cube. We prove that there is a path of length l between any two distinct vertices in $LTQ_n - F$ for any faulty set $F \subset V(LTQ_n) \cup E(LTQ_n)$ with $|F| \leq n-3$ and any integer l with $2^{n-1} - 1 \leq l \leq |V(LTQ_n - F)| - 1$ for any integer $n \geq 3$.

The rest part of this paper is organized as follows. In Section 2, the structure of LTQ_n is elaborated, and some definitions and notations are introduced. In Section 3, some properties of LTQ_n are derived. Finally, we investigate the fault-tolerant path-embedding in LTQ_n in Section 4.

2 Preliminaries

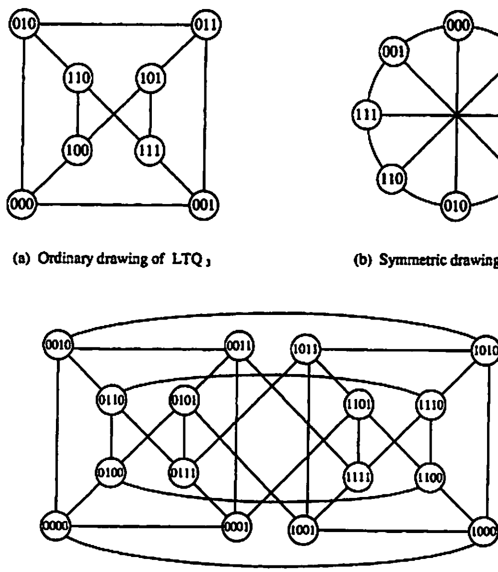
Let $G = (V, E)$ be a connected graph. Two vertices u and v are adjacent if $(u, v) \in E$. A path $P = \langle u, u_1, \ldots, v\rangle$ is a sequence of adjacent vertices, in which all the vertices are distinct except possible $u = v$. The length of a path is the number of edges on the path. The path P between u and v is called a uv-path. Let $P = \langle u, P(u, x), x, y, P(y, v), v\rangle$ be a uv-path of length at least two, where $P(u, x)$ is the subpath of P from u to x and $P(y, v)$ is the subpath of P from y to v. A path P forms a cycle C if the length of P is at least 3 and $u = v$. The distance between u and v is the length of a shortest uv-path, denoted as $d_G(u, v)$. The diameter of G, denoted as $D(G)$, is the maximum distance between any two vertices. A path (cycle) which contains each vertex in G exactly once is called a Hamiltonian path (cycle). A graph G is Hamiltonian if there is a Hamiltonian cycle in G, and a graph G is Hamiltonian connected if there is a hamiltonian path between any two distinct vertices in G.

We now recall the two definitions of the locally twisted cube proposed by Yang et al. in [14].

Definition 1 Let $n \geq 2$. The n-dimensional locally twisted cube, denoted by LTQ_n, is defined recursively as follows.

(1) LTQ_2 is a graph consisting of four vertices labelled with 00, 01,10 and 11, respectively,
connected by four edges (00,01), (00,10),(01,11) and (10,11).

(2) For $n \geq 3$, LTQ_n is built from two disjoint copies of LTQ_{n-1} according to the following steps. Let $0LTQ_{n-1}$ (respectively, $1LTQ_{n-1}$) denote the graph obtained by prefixing the label of each vertex of one copy of LTQ_{n-1} with 0 (respectively, 1). Each vertex $u = 0u_{n-1}\dots u_2u_1$ of $0LTQ_{n-1}$ is connected with the vertex $1(u_{n-1} \oplus u_1)\dots u_2u_1$ of $1LTQ_{n-1}$ by an edge, where "$\oplus$" represents the modulo 2 addition.



(a) Ordinary drawing of LTQ$_3$ (b) Symmetric drawing of LTQ$_3$

(c) LTQ$_4$

Figure 1. LTQ$_n$ for n=3,4

Figure 1 illustrates LTQ_3 and LTQ_4. According to Definition 1, we can denote $LTQ_n = L \odot R$, where $L = 0LTQ_{n-1}$ and $R = 1LTQ_{n-1}$. The edges between L and R are called crossed edges and denoted by $E_c = (u_L, u_R)$, where $u_L \in L$ and $u_R \in R$.

Let $\{0,1\}^n$ denote the set of all binary strings of length n. The locally twisted cubes can also be equivalently defined with the following non-recursive fashion.

Definition 2 Let $n \geq 2$. The n-dimensional locally twisted cube LTQ_n is a graph with $\{0,1\}^n$ as the vertex set. Two vertices $u = u_n \dots u_2u_1$ and

$v = v_n \dots v_2 v_1$ are adjacent if and only if one of the following conditions are satisfied.

(1) There is an integer $3 \le k \le n$ such that
   (a) $u_k \ne v_k$;
   (b) $u_{k-1} = v_{k-1} \oplus u_1$;
   (c) all the remaining bits of u and v are identical.

(2) There is an integer $k \in \{1, 2\}$ such that u and v differ only in the k-th bit.

From the above definitions, LTQ_n is an n-regular graph, and the labels of any two adjacent vertices of LTQ_n differ in at most two successive bits. It is known that the diameter of LTQ_n is $\lceil \frac{n+3}{2} \rceil$ for $n \ge 5$ and $D(LTQ_3) = 2$ [14].

An edge in LTQ_n is called 2-dimensional edge if its end-vertices differ in the 2-th position. Let E_2 be the set of 2-dimensional edges. Clearly, E_2 is a perfect matching of LTQ_n and there are exactly 2^{n-1} 2-dimensional edges in LTQ_n. By Definition 2, if (u_L, v_L) is a 2-dimensional edge in L, then (u_R, v_R) is also a 2-dimensional edge in R, where (u_L, u_R), (v_L, v_R) are two crossed edges of LTQ_n.

3 Properties

In this section, some properties of LTQ_n are established, which are useful to construct fault-free paths in LTQ_n described in the next section.

Lemma 1 [8] Let $n \ge 3$ and $F \subset V(LTQ_n) \cup E(LTQ_n)$, then $LTQ_n - F$ is Hamiltonian if $|F| \le n-2$, and Hamiltonian-connected if $|F| \le n-3$.

Lemma 2 [6] For any two different vertices u and v in LTQ_n ($n \ge 3$), there exists a uv-path of length l with $d(u,v) + 2 \le l \le 2^n - 1$.

Lemma 3 For any two different vertices u and v in LTQ_3, there exists a uv-path of length l with $3 \le l \le 7$.

Proof. For any $u, v \in V(LTQ_3)$ with $u \ne v$. Since $D(LTQ_3) = 2$, then $d(u,v) = 2$ if u is not adjacent to v. Thus, by Lemma 2, there exists a uv-path of length l with $d(u,v) + 2 \le l \le 7$. We only need to prove the lemma for $d(u,v) = 2$ and $l = 3$. Since LTQ_3 is a vertex symmetric graph shown in Figure 1(b), we only need to verify the lemma for $u = 000$, and $v \in \{001, 111, 110, 010\}$. Paths of length 3 between u and v are as follows. $\langle 000, 010, 011, 001 \rangle$; $\langle 000, 100, 101, 111 \rangle$; $\langle 000, 001, 111, 110 \rangle$; $\langle 000, 100, 110, 010 \rangle$.

The lemma holds. ∎

Lemma 4 For any $l \in \{4, 5, 7\}$, and $x \in V(LTQ_3)$, there are at least three 2-dimensional edges that lie in a cycle C of length l in $LTQ_3 - x$.

Proof. Since LTQ_3 is a vertex symmetric graph shown in Figure 1(b), we may assume $x = 000$. For any integer $l \in \{4,5,7\}$, in what follows, we will provide the cycles of length l that contain at least three 2-dimensional edges (labelled by underlines) in $LTQ_3 - x$.
$\langle \underline{001,011}, \underline{101,111}, 001\rangle$; $\langle \underline{110,100}, 101, 111, 110\rangle$; $\langle \underline{001,011}, 010, 110, 111, 001\rangle$;
$\langle \underline{111,101}, \overline{011,010}, 110, \overline{111}\rangle$; $\langle \underline{110,100}, 101, 011, \overline{010,110}\rangle$;
$\langle \underline{001,011}, 010, \underline{110,100}, \underline{101,111}, 001\rangle$. ∎

Lemma 5 For any $l \in \{6,7\}$, and $u, v \in V(LTQ_3)$ with $u \neq v$, there exist at least two 2-dimensional edges of LTQ_3 that lie in a path of length l between u and v in LTQ_3.

Proof. In view of the vertex symmetry of LTQ_3 shown in Figure 1(b), we only need to consider the case for $u = 000$ and $v \in \{001, 111, 110, 010\}$. For any $l \in \{6,7\}$, all uv-paths of required length, whose 2-dimensional edges are labeled by underlines, are constructed as follows.

The paths of different lengths between 000 and 001 are listed as follows:

$P_6 = \langle 000, \underline{100, 110}, \underline{111, 101}, \underline{011, 001}\rangle$.

$P_7 = \langle 000, \underline{100, 110}, 010, 011, \underline{101, 111}, 001\rangle$.

The paths of different lengths between 000 and 111 are listed as follows:

$P_6 = \langle 000, \underline{100, 110}, 010, 011, \underline{101, 111}\rangle$.

$P_7 = \langle \underline{000, 010}, \underline{110, 100}, 101, 011, 001, 111\rangle$.

The paths of different lengths between 000 and 110 are listed as follows:

$P_6' = \langle 000, 001, \underline{111, 101}, 011, 010, 110\rangle$.

$P_6'' = \langle 000, 100, 101, \underline{011, 001}, 111, 110\rangle$.

$P_7 = \langle \underline{000, 010}, \underline{011, 001}, \underline{111, 101}, \underline{100, 110}\rangle$.

The paths of different lengths between 000 and 010 are listed as follows:

$P_6 = \langle 000, 001, \underline{111, 101}, \underline{100, 110}, 010\rangle$.

$P_7' = \langle 000, 100, 101, \underline{011, 001}, 111, 110, 010\rangle$.

$P_7'' = \langle 000, 001, 111, \underline{110, 100}, 101, 011, 010\rangle$. ∎

Lemma 6 Let $F \subset V(LTQ_4) \cup E(LTQ_4)$ and $|F| \leq 1$. For any two different vertices u and v in $V(LTQ_4 - F)$, there exists a uv-path of length l with $7 \leq l \leq |V(LTQ_4 - F)| - 1$.

Proof. If $|F| = 0$, by Lemma 2, the lemma holds. As a result, we suppose $|F| = 1$. Recall that $LTQ_4 = L \odot R$, where $L = 0LTQ_3$ and $R = 1LTQ_3$. We have the following two cases.

Case 1 $F = \{x\} \subset V(LTQ_4)$. Without loss of generality, we may assume $x \in V(L)$. Three subcases are further considered:

Subcase 1.1 $u, v \in V(L) - \{x\}$.

Suppose that $7 \leq l \leq 14$. We can write $l = (l_0 - 2) + (l_1 + 2)$ where $4 \leq l_0 \leq 7$ and $3 \leq l_1 \leq 7$. We can mark the faulty vertex x as temporarily fault-free. By Lemma 3, there exists a path P_0 of length l_0 between u and v in the amended L. If P_0 passes the vertex x, then we denote P_0 as $P_0 = \langle u, P_0(u,y), y, x, z, P_0(z,v), v\rangle$ (Notice that u and y or z and v may

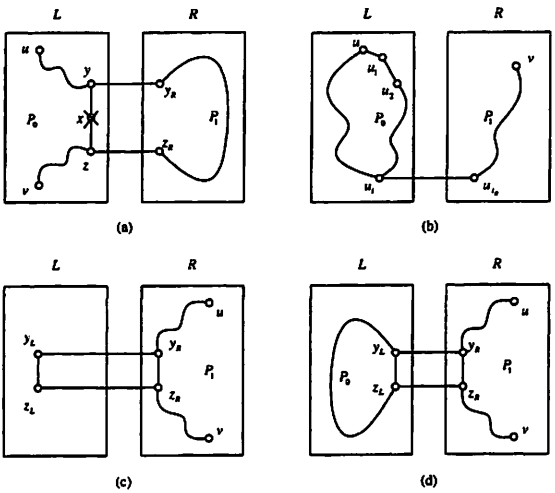


Figure 2 Illustration for Lemma 6
A straight line represents an edge and a curve line represents a path between two vertices

be the same vertex); otherwise, we select a vertex a ($a \notin \{u,v\}$) in P_0 and denote P_0 as $P_0 = \langle u, P_0(u,y), y, a, z, P_0(z,v), v\rangle$. We use y_R and z_R to denote the neighbors of y and z in R, respectively. Since $F \cap R = \emptyset$, there exists a path P_1 of length l_1 between y_R and z_R in R. Therefore, $P = \langle u, P_0(u,y), y, y_R, P_1, z_R, z, P_0(z,v), v\rangle$ is a uv-path of length l in $LTQ_4 - F$ (See Figure 2(a)).

Subcase 1.2 $u \in V(L) - \{x\}$, $v \in V(R)$.

Suppose that $7 \le l \le 14$. We can write $l = i + l_1 + 1$ where $3 \le i \le 6$ and $3 \le l_1 \le 7$. By Lemma 1, there exists a cycle $C = \langle u, u_1, u_2, \ldots, u_6, u\rangle$ of length 7 in $L - F$. There are two different paths of length i starting from u in the cycle, one of which is $\langle u, u_1, \ldots, u_i\rangle$ and another is $\langle u, u_6, \ldots, u_{7-i}\rangle$. It is clear that $u_i \neq u_{7-i}$ for any $3 \le i \le 6$. Let $v_i \in \{u_i, u_{7-i}\}$ such that $v_{i_R} \neq v$. Without loss of generality, let $v_i = u_i$, where u_{i_R} is the neighbor of u_i in R. By Lemma 3, there exists a path P_1 of length l_1 between u_{i_R} and v in R. Therefore, $P = \langle u, u_1, \ldots, u_i, u_{i_R}, P_1, v\rangle$ is a uv-path of length l in $LTQ_4 - F$ (See Figure 2(b)).

Subcase 1.3 $u, v \in R$. By Lemma 3, there exists a path P_1 of length $3 \le l_1 \le 7$ between u and v in R. Hence, we only need to consider the case for $8 \le l \le 14$.

Suppose that $8 \le l \le 9$. We can write $l = l_1 - 1 + 3$, where $l_1 \in \{6, 7\}$. By Lemma 5, there exist at least two 2-dimensional edges (y_R, z_R) and (y'_R, z'_R) that lie in paths of length l_1 between u and v in R. We may assume (y_R, z_R) lie in uv-path P_1 of length l_1 in R such that $x \notin \{y_L, z_L\}$.

Then $P = \langle u, P_1(u, y_R), y_R, y_L, z_L, z_R, P_1(z_R, v), v\rangle$ is a uv-path of length l in $LTQ_4 - F$ (See Figure 2(c)).

Suppose that $10 \leq l \leq 14$. We can write $l = l_0 + l_1 + 1$, where $l_0 \in \{3,4,6\}$ and $l_1 \in \{6,7\}$. By Lemma 4, there is at least three 2-dimensional edges that lie in cycles C of length $l_0 + 1$ in $L - x$. By Lemma 5, there is at least two 2-dimensional edges that lie in paths of length l_1 in R. Since there exists four 2-dimensional edges in L and R, respectively. Then, there exists a 2-dimensional edge (y_L, z_L) in C of length $l_0 + 1$ in $L - \{x\}$ such that (y_R, z_R) is the 2-dimensional edge in P_1 of length l_1 in R, where (y_L, y_R), (z_L, z_R) are the crossing edges. Denote $P_1 = \langle u, P(u, y_R), y_R, z_R, P(z_R, v), v\rangle$ and $P_0 = C - (y_L, z_L)$ is a $y_L z_L$-path of length l_0. Therefore, $\langle u, P(u, y_R), y_R, y_L, P_0, z_L, z_R, P(z_R, v), v\rangle$ is a uv-path of length l in $LTQ_4 - F$ (See Figure 2(d)).

Case 2 $F \subset E(L)$ or $F \subset E(R)$. Without loss of generality, we may assume $F \subset E(L)$.

Let $e = (x, y)$ be the faulty edge. By Lemma 1, there exists a path of length 15 between any two different vertex in $LTQ_4 - \{e\}$. Hence, we only need to consider the case for $7 \leq l \leq 14$. Three subcases are further considered:

Subcase 2.1 $u, v \in V(L)$.

Suppose that $7 \leq l \leq 14$, we can write $l = (l_0 - 1) + l_1 + 2$, where $3 \leq l_0 \leq 7$ and $3 \leq l_1 \leq 7$. We can mark the faulty edge $e = (x, y)$ as temporarily fault-free. By Lemma 3, there exists a path P_0 of length l_0 between u and v in the amended L. If the path P_0 passes the faulty edge (x, y), we denote P_0 as $P_0 = \langle u, P_0(u, x), x, y, P_0(y, v), v\rangle$; otherwise, we select an edge (a, b) in P_0 and denote P_0 as $P_0 = \langle u, P(u, a), a, b, P_0(b, v), v\rangle$. Since R is fault-free, there exists a path P_1 of length l_1 between x_R and y_R or a_R and b_R in R. Therefore, $P = \langle u, P_0(u, x), x, x_R, P_1, y_R, y, P_0(y, v), v\rangle$ or $P = \langle u, P_0(u, a), a, a_R, P_1, b_R, b, P_0(b, v), v\rangle$ is a uv-path of length l in $LTQ_4 - F$.

Subcase 2.2 $u \in V(L), v \in V(R)$.

Suppose that $7 \leq l \leq 14$, we can write $l = i + l_1 + 1$, where $i \in \{3,5,6\}$ and $3 \leq l_1 \leq 7$. By Lemma 1, there exists a Hamiltonian cycle $C = \langle u, u_1, u_2, \ldots, u_7, u\rangle$ in $L - F$. There are two different paths of length i starting from u in the cycle, one of which is $\langle u, u_1, \ldots, u_i\rangle$ and another is $\langle u, u_7, \ldots, u_{8-i}\rangle$. It is clear that $u_i \neq u_{8-i}$ for any $i \in \{3,5,6\}$. Let $v_i \in \{u_i, u_{8-i}\}$ such that $v_{i_R} \neq v$. Without loss of generality, let $u_i = v_i$, where u_{i_R} is the neighbor of u_i in R. By Lemma 3, there exists a path P_1 of length l_1 between u_{i_R} and v in R. Therefore, $P = \langle u, u_1, \ldots, u_i, u_{i_R}, P_1, v\rangle$ is a uv-path of length l in $LTQ_4 - F$.

Subcase 2.3 $u, v \in V(R)$. By Lemma 3, there exists a path P_1 of length l_1 between u and v in R with $3 \leq l_1 \leq 7$. Hence, we only need to consider the case for $8 \leq l \leq 14$.

Suppose that $8 \le l \le 9$, we can write $l = (l_1 - 1) + 3$, where $l_1 \in \{6, 7\}$. By Lemma 5, there exist at least two 2-dimensional edges (y_R, z_R) and (y'_R, z'_R) that lie in paths of length l_1 between u and v in R. We may assume (y_R, z_R) lie in uv-path P_R of length l_1 in R such that $e \neq (y_L, z_L)$. Then $P = \langle u, P_R(u, y_R), y_R, y_L, z_L, z_R, P_R(z_R, v), v\rangle$ is a uv-path of length l in $LTQ_4 - F$.

Suppose that $10 \le l \le 14$. We can write $l = l_0 + l_1 + 1$, where $l_0 \in \{3, 4, 6\}$ and $l_1 \in \{6, 7\}$. By Lemma 4, there is at least three 2-dimensional edges that lie in cycles C of length $l_0 + 1$ in $L - e$. By Lemma 5, there exists a path $P_1 = \langle u, P(u, y), y, z, P(z, v), v\rangle$ of length l_1 in R, such that (y, z) is a 2-dimensional edge and (y_L, z_L) lies in the cycle C of length $l_0 + 1$ in $L - e$. Then $P(y_L, z_L) = C - (y_L, z_L)$ is a y_Lz_L-path of length l_0. Therefore, $\langle u, P(u, y), y, y_L, P(y_L, z_L), z_L, z, P(z, v), v\rangle$ is a uv-path of length l in $LTQ_4 - F$.

Case 3 $F \subset E_c$. Two subcases are further considered:

Subcase 3.1 $u, v \in V(L)$ or $u, v \in V(R)$. Without loss of generality, we may assume $u, v \in V(L)$. By Lemma 3, there exists a uv-path P of length l in L with $3 \le l \le 7$.

Suppose that $8 \le l \le 15$. We can write $l = l_0 + l_1 + 1$, where $4 \le l_0 \le 7$ and $3 \le l_1 \le 7$. By Lemma 3, there is a uv-path P_0 of length l_0 in L. There must exist an edge (y, z) in P_0 such that $F \notin \{(y, y_R), (z, z_R)\}$. By Lemma 3, there is a y_Rz_R-path P_1 of length l_1 in R. Therefore, $\langle u, P_0(u, y), y, y_R, P_1(y_R, z_R), z_R, z, P_0(z, v), v\rangle$ is a uv-path of length l in $LTQ_4 - F$.

Subcase 3.2 $u \in V(L), v \in V(R)$.

Suppose that $7 \le l \le 15$. We can write $l = l_0 + l_1 + 1$, where $3 \le l_0 \le 7$ and $3 \le l_1 \le 7$. There is a fault-free edge $(u_L, v_R) \in E_c$ such that $u_L \neq u$ and $v_R \neq v$. By Lemma 3, there exist a uu_L-path P_L of length l_0 in L and a v_Rv-path P_R of length l_1 in R. Then $P = \langle u, P_L(u, u_L), u_L, v_R, P_R(v_R, v), v\rangle$ is a uv-path of length l in $LTQ_4 - F$. ∎

Lemma 7 [4] Let x, y, u, v be four distinct vertices in LTQ_n, where $n \ge 4$. There exist two vertex-disjoint paths P_1 and P_2 such that: (1) P_1 connects u to v, (2) P_2 connects x to y, and (3) $V(P_1) \cup V(P_2) = V(LTQ_n)$.

Lemma 8 For $n \ge 3$ and $l \in \{2^n - 2, 2^n - 1\}$, any two different vertices $u, v \in V(LTQ_n)$, there exist at least 2^{n-2} 2-dimensional edges of LTQ_n that lie in a path of length l between u and v in LTQ_n.

Proof. We prove the lemma by induction on n. By Lemma 5, the lemma holds for $n = 3$. Assume the Lemma is true for $n - 1$. We now consider LTQ_n for $n \ge 4$. We identify the following two cases:

Case 1 $u, v \in V(L)$ or $u, v \in V(R)$. Without loss of generality, we may assume $u, v \in V(L)$.

Suppose that $l = (l_0 - 1) + l_1 + 2$, where $l_0 \in \{2^{n-1} - 2, 2^{n-1} - 1\}$ and $l_1 = 2^{n-1} - 1$. By the induction hypothesis, there is at least 2^{n-3}

2-dimensional edges of L that lie in paths of length l_0 between u and v in L. For any uv-path P_0 of length l_0, there exists an edge $(a,b) \notin E_2$ in P_0, then $(a_R, b_R) \notin E_2$. By the induction hypothesis, there is at least 2^{n-3} 2-dimensional edges of R that lie in paths of length l_1 between a_R and b_R in R. The paths of length l_0 combining with the paths of length l_1 form the uv-paths of length l which contained at least 2^{n-2} 2-dimensional edges.

Case 2 $u \in V(L), v \in V(R)$.

Suppose that $l = l_0 + l_1 + 1$ where $l_0 \in \{2^{n-1} - 2, 2^{n-1} - 1\}$ and $l_1 = 2^{n-1} - 1$. There is a vertex $a \neq u$ in L such that $a_R \neq v$ in R. By the induction hypothesis, there is at least 2^{n-3} 2-dimensional edges of L that lie in paths of length l_0 between u and a in L and at least 2^{n-3} 2-dimensional edges of R that lie in paths of length l_1 between a_R and v in R. The paths of length l_0 combined with the paths of length l_1 form the uv-paths of length l which containing at least 2^{n-2} 2-dimensional edges. ∎

4 Main result

Theorem For any integer $n \geq 3$, $F \subset V(LTQ_n) \cup E(LTQ_n)$ with $|F| \leq n-3$, and any integer l with $2^{n-1} - 1 \leq l \leq |V(LTQ_n - F)| - 1$, there is a path of length l between any two distinct vertices in $LTQ_n - F$.

Proof. We prove the theorem by induction on n. By Lemma 3 and Lemma 6, the theorem holds for $n \in \{3, 4\}$. Assume that the theorem is true for LTQ_{n-1} with $n \geq 5$. We now consider LTQ_n. We denote $F^L = F \cap L$, $F^R = F \cap R$, $F_v = F \cap V(LTQ_n)$, $F_e = F \cap E(LTQ_n)$, $f_v = |F_v|$, $f_v^L = |F_v \cap V(L)|$, $f_v^R = |F_v \cap V(R)|$. Without loss of generality, we may assume $|F^L| \geq |F^R|$. Let u and v be any two different vertices in $LTQ_n - F$. By Lemma 1, we only need to prove that there is a uv-path of length l in $LTQ_n - F$, for any integer l with $2^{n-1} - 1 \leq l \leq |V(LTQ_n - F)| - 2$. We will construct the desired paths according to the following two cases.

Case 1 $|F^L| \leq n-4$, then $|F^R| \leq n-4$. We further consider the following subcases:

Subcase 1.1 $u, v \in V(L - F^L)$ or $u, v \in V(R - F^R)$. Without loss of generality, we may assume $u, v \in V(L - F^L)$.

Suppose that $2^{n-1} - 1 \leq l \leq 2^n - f_v - 1$. We can write $l = l_0 + l_1 + 1$ where $2^{n-2} - 1 \leq l_0 \leq 2^{n-1} - f_v^L - 1$ and $2^{n-2} - 1 \leq l_1 \leq 2^{n-1} - f_v^R - 1$. By the induction hypothesis, there exists a fault-free uv-path P_0 of length l_0 in L. Since $l_0 \geq 2^{n-2} - 1$, there must exist an edge (a,b) on the path $P_0 = \langle u, P_0(u,a), a, b, P_0(b,v), v\rangle$ such that the two crossed edges (a, a_R) and (b, b_R) are fault-free. Suppose to the contrary that there does not exist such an edge. Then there are at least $\lceil (2^{n-2} - 1)/2 \rceil = 2^{n-3}$ faults outside L. However, $2^{n-3} > n-3$ for $n \geq 5$, a contradiction. Since $|F^R| \leq |F^L| \leq n-4$ and $2^{n-2} - 1 \leq l_1 \leq 2^{n-1} - f_v^R - 1$, by the induction hypothesis, there exists a fault-free $a_R b_R$-path P_1 of length l_1 in R. Then $\langle u, P_0(u,a), a, a_R, P_1, b_R, b, P_0(b,v), v\rangle$ is a uv-path of length l in $LTQ_n - F$ (See Figure 3(a)).

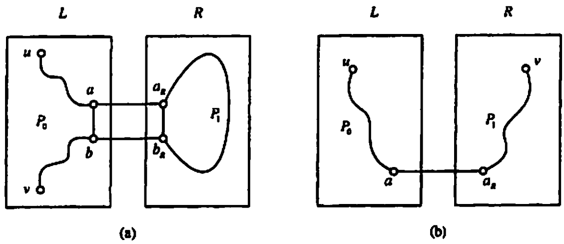


Figure 3 Illustration for Case 1 of Theorem1
A straight line represents an edge and a curve line represents a path between two vertices

Subcase 1.2 $u \in V(L - F^L), v \in V(R - F^R)$.

Since $|F| \le n-3$ and there are 2^{n-1} crossed edges between L and R, there exists a fault-free crossed edge (a_L, a_R) in LTQ_n where $a_L \ne u$ and $a_R \ne v$. For $2^{n-1} - 1 \le l \le 2^n - f_v - 1$, we can write $l = l_0 + l_1 + 1$ where $2^{n-2} - 1 \le l_0 \le 2^{n-1} - f_v^L - 1$ and $2^{n-2} - 1 \le l_1 \le 2^{n-1} - f_v^R - 1$. By the induction hypothesis, there exist a fault-free ua_L-path P_0 of length l_0 in L and a fault-free $a_R v$-path P_1 of length l_1 in R. Then $\langle u, P_0(u,a_L), a_L, a_R, P_1(a_R,v), v\rangle$ is a uv-path of length l in $LTQ_n - F$ (See Figure 3(b)).

Case 2 $|F^L| = n-3$, then $F^L = F$.

We have the following subcases:

Subcase 2.1 $u, v \in V(L)$. We have the following two subcases:

Subcase 2.1.1 $F_v \ne \emptyset$.

Suppose that $2^{n-1} - 1 \le l \le 2^n - f_v - 1$. We can write $l = (l_0 - 2) + l_1 + 2$, where $2^{n-2} - 1 \le l_0 \le 2^{n-1} - f_v$ and $\lceil \frac{n+3}{2} \rceil + 2 \le 2^{n-2} \le l_1 \le 2^{n-1} - 1$. Let $x \in F_v$. We can mark the faulty vertex x as temporarily fault-free. Since $|F - \{x\}| = n-4$, by the inductive hypothesis, there exists a fault-free path P_0 of length l_0 between u and v in the amended L. If the path P_0 passes the faulty vertex x, we denote P_0 as $P_0 = \langle u, P_0(u,a), a, x, b, P_0(b,v), v\rangle$; otherwise, we select a vertex c in P_0 and denote P_0 as $P_0 = \langle u, P_0(u,a), a, c, b, P_0(b,v), v\rangle$. By Lemma 2, there exists a path P_1 of length l_1 between a_R and b_R in R. Therefore, there exists a uv-path $P = \langle u, P_0(u,a), a, a_R, P_1, b_R, b, P_0(b,v), v\rangle$ of length l in $LTQ_n - F$ (See Figure 4(a)).

Subcase 2.1.2 $F_v = \emptyset$. Then $F^L = F_e$.

Suppose that $2^{n-1} - 1 \le l \le 2^n - f_v - 1$, we can write $l = (l_0 - 1) + l_1 + 2$, where $2^{n-2} - 1 \le l_0 \le 2^{n-1} - f_v - 1$ and $\lceil \frac{n+3}{2} \rceil + 2 \le 2^{n-2} \le l_1 \le 2^{n-1} - 1$. Let $(x,y) \in F_e$. We can mark the faulty edge (x,y) as temporarily fault-free. Since $|F - \{(x,y)\}| = n-4$, by the inductive hypothesis, there

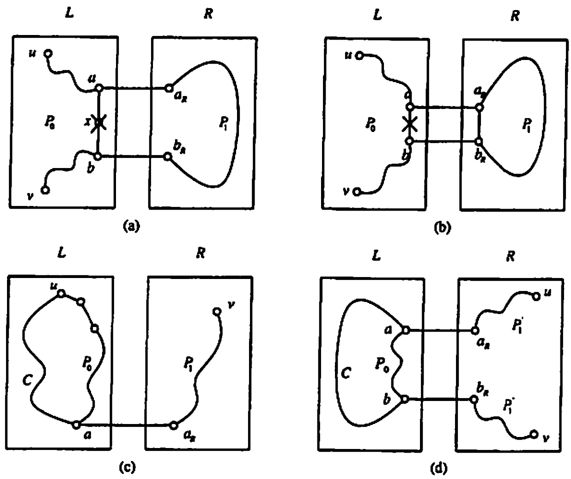


Figure 4 Illustration for Case 2 of Theorem1
A straight line represents an edge and a curve line represents a path between two vertices

exists a fault-free path P_0 of length l_0 between u and v in the amended L. If the path P_0 passes the faulty edge (x, y), we denote P_0 as $P_0 = \langle u, P_0(u,a), a, b, P_0(b,v), v\rangle$ with $a = x, b = y$; otherwise, we denote P_0 as $P_0 = \langle u, P_0(u,a), a, b, P_0(b,v), v\rangle$. By Lemma 2, there exists a path P_1 of length l_1 between a_R and b_R in R. Therefore, there exists a uv-path $P = \langle u, P_0(u,a), a, a_R, P_1, b_R, b, P_0(b,v), v\rangle$ of length l in $LTQ_n - F$ (See Figure 4(b)).

Subcase 2.2 $u \in V(L), v \in V(R)$.

Suppose that $2^{n-1} - 1 \le l \le 2^n - f_v - 1$. We can write $l = l_0 + l_1 + 1$, where $2^{n-2} - 1 \le l_0 \le 2^{n-1} - f_v - 1$ and $l_0 \ne \frac{2^{n-1}-f_v}{2}$, $\lceil \frac{n+3}{2} \rceil + 2 \le 2^{n-2} - 1 \le l_1 \le 2^{n-1} - 1$. By Lemma 1, there is a cycle C of length $2^{n-1} - f_v$ in $L - F$. There are two different paths of length l_0 starting from u in cycle C. Then, there exists a ua-path P_0 of length l_0 in C such that a is not incident to v. By Lemma 2, there exists a path P_1 of length l_1 between a_R and v in R. Therefore, $P = \langle u, P_0(u,a), a, a_R, P_1, v\rangle$ is a uv-path of length l in $LTQ_n - F$ (See Figure 4(c)).

Subcase 2.3 $u, v \in V(R)$. By Lemma 2, there exists a uv-path of length l with $2^{n-1} - 2 \le l \le 2^{n-1} - 1$ in R. Hence, we only need to consider the case for $2^{n-1} \le l \le 2^n - f_v - 1$.

Suppose that $l = 2^{n-1}$. By Lemma 8, there exist at least $2^{(n-3)}$ 2-dimensional edges of R that lie in paths of length $l_1 = 2^{n-1} - 2$ between u and v in R. Since $2^{(n-3)} > n - 3 = |F|$ for $n \ge 4$, there exists an edge $(a, b) \in E_2$ in a path P_1 of length $2^{n-1} - 2$ in R such that $(a_L, b_L) \notin F$. Then $P = \langle u, P_1(u,a), a, a_L, b_L, b, P_1(b,v), v\rangle$ is a uv-path of length l in $LTQ_n - F$.

Suppose that $2^{n-1}+1 \leq l \leq 2^n - f_v - 1$. We can write $l = l_0 + l_1 + 2$, where $1 \leq l_0 \leq 2^{n-1} - f_v - 1$ and $l_1 = 2^{n-1} - 2$. By Lemma 1, there is a cycle C of length $2^{n-1} - f_v$ in $L - F$. There exist two different vertices a and b in C such that the length of ab-path P_0 in C is l_0 and $a_R, b_R \notin \{u, v\}$. By Lemma 7, there exist two vertex-disjoint paths $P_1^{'}$ and $P_1^{''}$ in R such that: $P_1^{'}$ connects a_R to u, $P_1^{''}$ connects b_R to v and $V(P_1^{'}) \cup V(P_1^{''}) = V(R)$. Let $l_1 = |P_1^{'}| + |P_1^{''}|$. Therefore, $P = \langle u, P_1^{'}(u, a_R), a_R, a, P_0(a, b), b, b_R, P_1^{''}(b_R, v), v \rangle$ is a uv-path of length l in $LTQ_n - F$ (See Figure 4(d)).

This completes the proof of the theorem. ∎

Remarks The conditions in Theorem are tight in the following senses:
(1) For $n \geq 3$, if $l \leq 2^{n-1} - 2$, then the theorem is not necessary true. For example, there is no path of length 2 between any two adjacent vertices in LTQ_3.
(2) For $n \geq 3$, if $|F| \geq n - 2$, then the theorem is not necessary true. For example, let $F = \{000\}$, then there are no paths of lengths 3 and 5 between two vertices 011 and 010 in $LTQ_3 - F$.

5 Acknowledgements

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