On the adjacent vertex distinguishing total chromatic number of outer plane graph*

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Abstract

A total coloring of a simple graph G is called adjacent vertex distinguishing if for any two adjacent and distinct vertices u and v in G, the set of colors assigned to the vertices and the edges incident to u differs from the set of colors assigned to the vertices and the edges incident to v. In this paper we shall prove the adjacent vertex distinguishing total chromatic number of outer plane graph with $\Delta \leq 5$ is $\Delta+2$ if G have two adjacent maximum degree vertices, otherwise is $\Delta+1$.

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1. Introduction

In this paper we consider only simple, finite and undirected graphs. Let G be a graph. We denote its vertex set, edge set, maximum degree and minimum degree by $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$, respectively. Let $d(u)$ and $N(u)$ denote the degree of vertex u and the set of vertices adjacent to u in a graph G, respectively. Let $V_\Delta = \{v \mid v \in V(G) \text{ and } d(v) = \Delta(G)\}$, $G[V_\Delta]$ is the subset graph induced by V_Δ. A total coloring of graph G is a mapping from $V(G) \cup E(G)$ to set C which satisfy no two adjacent vertices or edges of G have the same color and the color of each vertex of G is distinct from the colors of its incident edges. A total coloring of a simple graph G is called adjacent vertex distinguishing if for any two adjacent and distinct vertices u and v in G, the set of colors assigned to the vertices and the edges incident to u differs from the set of colors assigned to the vertices and the edges incident to v., where the set of colors assigned to the vertices and the edges incident to u is denoted by $f[u] = \{f(u)\} \cup \{f(uv)|uv \in E(G)\}$. The minimal number of colors required for a adjacent vertex distinguishing total coloring of G is denoted by $\chi_{at}(G)$.

In 2002, Zhang Zhongfu [1] introduced the notion of adjacent vertex distinguishing (edge) coloring. A similar concept was discussed in [3]. Other articles involving such colorings appear in [2,3,4,7,9].

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In 2004, Zhang Zhongfu [5] first introduced the concept of adjacent vertex distinguishing total coloring. Adjacent vertex distinguishing total coloring conjecture given by Zhang Zhongfu [5] is as follows.

Conjecture For any simple graph G, then $\chi_{at}(G) \leq \Delta(G) + 3$.

The conjecture appears to be difficult even when the graph G are some special graphs such as paths, cycles, complete graphs and so on, they were proven by Zhang Zhongfu in [5]. In [8], Wang et al. prove that series paralled graph with maximum degree 3 satisfy adjacent vertex distinguishing total coloring conjecture. It is clear, for any simple graph G, $\chi_{at}(G) \geq \Delta(G) + 1$. If graph G has two adjacent vertices of maximal degree, then $\chi_{at}(G) \geq \Delta(G) + 2$.

Let G be a plane graph, if all vertices of G are on the boundary of one face f_0, then G is called outer plane graph, and the face f_0 is called the outer face (the others interior face). In this paper we shall prove the adjacent vertex distinguishing total chromatic number of outer plane graph with $\Delta \leq 5$ is $\Delta + 2$ if G has two adjacent maximum degree vertices, otherwise is $\Delta + 1$.

Definitions not given here may be found in [10,11,12,13].

2. Main result

Theorem 2.1 Let $G(V, E)$ be a 2-connected outer plane graph with $\Delta(G) = 2$, then $\chi_{at}(G) \leq 5$.

Proof. Because $G(V, E)$ be a 2-connected outer plane graph with $\Delta(G) = 2$, hence G is cycle C_n, by [5], we know $\chi_{at}(C_n) = 5, when\ n = 3$; $\chi_{at}(C_n) = 4, when\ n \geq 4$. ■

Lemma 2.2[6] Let $G(V, E)$ be a 2-connected outer plane graph with $\Delta(G) = 3$, then at least one of the following 2 items is true.

1. $\exists u, v \in V(G)$, s.t. $d(u) = d(v) = 2, uv \in E(G)$;
2. $\exists u, v, w \in V(G)$, s.t. $uv, uw, vw \in E(G)$, $d(u) = 2, d(v) = d(w) = 3$. ■

Theorem 2.3 Let $G(V, E)$ be a 2-connected outer plane graph with $\Delta(G) = 3$, then $\chi_{at}(G) = 5$.

Proof. Because $G(V, E)$ is a 2-connected outer plane graph with $\Delta(G) = 3$, then $\chi_{at}(G) \geq 5$. We now prove $\chi_{at}(G) \leq 5$ by using induction method on $p = \mid V(G) \mid$. Let $C = \{1, 2, 3, 4, 5\}$ be a color set.

If $\mid V(G) \mid = 4$, then $G(V, E)$ is a graph formed by deleting an edge from the complete graph K_4. By enumeration, the conclusion is true. Assume that the conclusion is true when $\mid V(G) \mid < p$. We prove the conclusion is true for $\mid V(G) \mid = p$. By lemma 2.2, we may distinguishing the following two cases.

Case 1. $\exists u, v \in V(G)$, s.t. $d(u) = d(v) = 2, uv \in E(G)$. Suppose $N(u) = \{u_0, v\}, N(v) = \{v_0, u\}$. We may assume that $d(u_0) \neq 2$ (Otherwise u, v are replaced by u_0, u, and so on). Let $G^* = G - u + u_0v$. Then G^* is also a 2-connected outer plane graph with $\Delta(G^*) = 3$ and $\mid V(G^*) \mid = \mid V(G) \mid -1 < p$. By induction hypothesis, G^* has a 5-adjacent vertex distinguishing total coloring f^*. Now we extend f^* of G^* to a 5-adjacent vertex distinguishing total coloring f of G.

1. Let $f(u_0u) = f^*(u_0v)$.
2. If $d(v_0) \neq 2$ or $f^*(v) \notin f^*[v_0]$, then let $f(uv) \in C - \{f(u_0u), f^*(v_0v), f^*(v)\}$; If $d(v_0) = 2$ and $f^*(v) \in f^*[v_0]$, then let $f(uv) \in C - f^*[v_0] - \{f(u_0u)\}$;
3. Let $f(u) \in C - \{f^*(u_0), f^*(v), f(u_0u), f(uv)\}$. The coloring of other elements is the same to f^*.

Case 2. If 1 of lemma 2.2 is not appear in G, then $\exists u, v, w \in V(G)$, s.t. $uv, uw, vw \in E(G)$, $d(u) = 2, d(v) = d(w) = 3$. Let $N(v) = \{u, v_1, w\}, N(w) = \{u, v, w_1\}$. we denote a new graph $G^* = G - u$. Then G^* is also a 2-connected outer plane graph with $\Delta(G^*) = 2$ *or* $\Delta(G^*) = 3$ and $|V(G^*)| = |V(G)| - 1 < p$. By theorem 2.1 or induction hypothesis, G^* has a 5-adjacent vertex distinguishing total coloring f^*. Now we extend f^* of G^* to a 5-adjacent vertex distinguishing total coloring f of G.

Subcase 2.1. If $d(w_1) \neq 3$ and $d(v_1) \neq 3$, then

(1) Let $f(uw) \in C - \{f^*(ww_1), f^*(wv), f^*(w)\}$;

(2) Let $f[w] = \{f(uw)\} \cup f^*[w]$,

if $f^*[v] \subseteq f[w]$, then let $f(uv) \in C - f[w]$; otherwise let $f(uv) \in C - \{f(uw), f^*(wv), f^*(v), f^*(vv_1)\}$;

(3) Let $f(u) \in C - \{f(uw), f(uv), f^*(w), f^*(v))\}$; The coloring of other elements is the same to f^*.

Subcase 2.2. If $d(w_1) = 3$ or $d(v_1) = 3$, without lose of generality, we may assume that $d(w_1) = 3$ and $d(v_1) \neq 3$ then

(1) If $f^*[w] \subseteq f^*[w_1]$, then let $f(uw) \in C - f^*[w_1]$; Otherwise let $f(uw) \in C - f^*[w]$;

(2) Let $f[w] = \{f(uw)\} \cup f^*[w]$, if $f^*[v] \subseteq f[w]$, then let $f(uv) \in C - f[w]$; Otherwise let $f(uv) \in C - f^*[v] - \{f(uw)\}$;

(3) Let $f(u) \in C - \{f(uw), f(uv), f^*(w), f^*(v)\}$. The coloring of other elements is the same to f^*.

Subcase 2.3. If $d(w_1) = 3$ and $d(v_1) = 3$, then,

2.3.1 If $f^*[w] \not\subseteq f^*[w_1]$ and $f^*[v] \not\subseteq f^*[v_1]$, then the discussion is the same as case 2.1.

2.3.2 If $f^*[w] \subseteq f^*[w_1]$ or $f^*[v] \subseteq f^*[v_1]$, without lose of generality, we may assume that $f^*[w] \not\subseteq f^*[w_1]$ and $f^*[v] \subseteq f^*[v_1]$, then

(1) Let $f(uv) \in C - f^*[v_1]$;

(2) let $f[v] = \{f(uv)\} \cup f^*[v]$, if $f^*[w] \subseteq f[v]$, then let $f(uw) \in C - f[v]$; Otherwise let $f(uw) \in C - f^*[w] - \{f(uv)\}$;

(3) Let $f(u) \in C - \{f(uw), f(uv), f^*(w), f^*(v)\}$. The coloring of other elements is the same to f^*.

2.3.3 If $f^*[w] \subseteq f^*[w_1]$ and $f^*[v] \subseteq f^*[v_1]$, then

(1) Let $f(uw) \in C - f^*[w_1]$;

(2) Let $f[w] = \{f(uw)\} \cup f^*[w]$,

if $f[w] = f^*[v_1]$, then let $f(uv) \in C - f^*[v_1]$;

if $f[w] \neq f^*[v_1]$, then let $\alpha \in f[w] - f^*[v_1], \beta \in f^*[v_1] - f[w]$,

— if $\alpha \notin f^*[v], \beta \in f^*[v]$ and $f(uw) \neq \alpha$, then let $f(uv) = \alpha$;

— if $\alpha \notin f^*[v], \beta \in f^*[v]$ and $f(uw) = \alpha$, then exchange the colors of uw and vw, let $f(uv) \in C - \{f(uw), f(wv), f^*(v), f^*(vv_1)\}$;

— if $\alpha, \beta \notin f^*[v]$, then there are the following 3 cases to be considered.

a) If $f(w) \neq \alpha$, then let $f(v) = \alpha, f(uv) = \beta$;

b) If $f(w) = \alpha, f(v_1) \neq \beta$, then let $f(v) = \beta, f(uv) = \alpha$;

c) If $f(w) = \alpha, f(v_1) = \beta$, then exchange the colors of uw and w, and recolor vertex v, let $f(v) = \alpha$, $f(uv) = \beta$.

(3) Let $f(u)$ be the color in C s.t. $f(u)$ is not the colors of uw, uv, w, v. The coloring of other elements is the same to f^*.

With all cases considered , f is a 5- adjacent vertex distinguishing total coloring of G. ■

Lemma 2.4[6] Let G be a 2-connected outer plane graph with $\Delta(G) = 4$. Then at least one of the following 5 items is true:

(1) $\exists u, v \in V(G)$, s.t. $d(u) = d(v) = 2$ and $uv \in E(G)$;

(2) $\exists u, v, w, u_1, v_1 \in V(G)$, s.t. $d(u) = d(v) = 2, d(u_1) = d(v_1) = d(w) = 4$ and $wu, wv, wu_1, wv_1, uu_1, vv_1, u_1v_1 \in E(G)$;

(3) $\exists u, u_1, v, v_1, v_2, w, w_1 \in V(G)$, s.t. $d(u) = d(v) = d(w) = 2, d(v_1) = d(v_2) = d(u_1) = d(w_1) = 4$ and $uu_1, uv_1, u_1v_1, vv_1, vv_2, v_1v_2, v_2w, v_2w_1, ww_1 \in E(G)$;

(4) $\exists u, v, w \in V(G)$, s.t. $uv, uw, vw \in E(G), d(u) = 2, d(v) = d(w) = 3$;

(5) $\exists u, v, w \in V(G)$, s.t. $uv, uw, vw \in E(G), d(u) = 2, d(v) = 3, d(w) = 4$.

Theorem 2.5 Let $G(V, E)$ be a 2-connected outer plane graph with $\Delta(G) = 4$, if $E(G[V_\Delta]) = \emptyset$, then $\chi_{at}(G) = 5$.

Proof. Because $E(G[V_\Delta]) = \emptyset$, then $\chi_{at}(G) \geq 5$. We now prove $\chi_{at}(G) \leq 5$ by using induction method on $p =\mid V(G) \mid$. Let $C = \{1, 2, 3, 4, 5\}$ be a color set.

If $\mid V(G) \mid= 5$, then $G(V, E)$ is a fan with order 5. By enumeration, the conclusion is true. Assume that the conclusion is true when $\mid V(G) \mid< p$. We prove the conclusion is true for $\mid V(G) \mid= p$. Since $E(G[V_\Delta]) = \emptyset$ and lemma 2.4, we know that item 2 and 3 do not hold, hence we may distinguish the following three cases:

Case 1. If $\exists u, v \in V(G)$, s.t. $d(u) = d(v) = 2$ and $uv \in E(G)$. In this time, the discussion is the same as case 1 of theorem 2.3.

Case 2. If item (1) of lemma 2.4 does not hold, but item (4) of lemma 2.4 holds. i.e. $\exists u, v, w \in V(G)$, s.t. $uv, uw, vw \in E(G), d(u) = 2, d(v) = d(w) = 3$. The proof is the same as case 2 of theorem 2.3.

Case 3. If both (1) and (4) of lemma 2,4 do not hold, then (5) of lemma 2,4 must hold. i.e. $\exists u, v, w \in V(G)$, s.t. $uv, uw, vw \in E(G), d(u) = 2, d(v) = 3, d(w) = 4$.

Assume that $N(v) = \{v_1, u, w\}, N(w) = \{u, v, w_1, w_2\}$ and $w_1 \neq w_2$. We define a new graph $G^* = G - u$. Then G^* is also a 2-connected outer plane graph with $\Delta(G^*) = 3$ *or* $\Delta(G^*) = 4$ and $\mid V(G^*) \mid=\mid V(G) \mid -1 < p$. By theorem 2.3 or induction hypothesis, G^* has a 5-adjacent vertex distinguishing total coloring f^*. Now we extend f^* of G^* to a 5-adjacent vertex distinguishing total coloring f of G.

(1) Let $f(uw) \in C - f^*[w]$;

(2) If $f^*[v] \nsubseteq f^*[v_1]$ or $d(v) \neq d(v_1)$, then let $f(uv) \in C - f^*[v] - \{f(uw)\}$;

If $f^*[v] \subseteq f^*[v_1]$, $d(v) = d(v_1)$ and $f(uw) \in f^*[v_1]$, then let $f(uv) \in C - f^*[v_1]$;

If $f^*[v] \subseteq f^*[v_1]$, $d(v) = d(v_1)$ and $f(uw) \notin f^*[v_1]$, then redefine the color of v as follows: $f(v) = f(uw)$, let $f(uv) \in C - \{f(uw), f^*(wv), f^*(vv_1)\}$

(3) Let $f(u) \in C - \{f(uw), f(uv), f^*(w), f(v)\}$. The coloring of other elements is the same to f^*.

From what discussed above, the conclusion is true. ■

Theorem 2.6 Let $G(V, E)$ be a 2-connected outer plane graph with $\Delta(G) = 4$, if $E(G[V_\Delta]) \neq \emptyset$, then $\chi_{at}(G) = 6$.

Proof. Because $E(G[V_\Delta]) \neq \emptyset$, then $\chi_{at}(G) \geq 6$. We now prove $\chi_{at}(G) \leq 6$ by using induction on $\mid V(G) \mid= p$. Let $C = \{1, 2, 3, 4, 5, 6\}$ be a color set.

By enumeration, the conclusion is true for the outer plane graph with order $\mid V(G) \mid= 6$ and $E(G[V_\Delta]) \neq \emptyset$. Assume that the conclusion is true when $\mid V(G) \mid< p$. We prove the conclusion is true for $\mid V(G) \mid= p$. By lemma 2.4, we may distinguishing the following five cases.

Case 1. If $\exists u, v \in V(G)$, s.t. $d(u) = d(v) = 2$ and $uv \in E(G)$. In this time, the discussion is the same as case 1 of theorem 2.3.

Case 2. If item (1) of lemma 2.4 does not hold, but item (2) of lemma 2.4 holds. i.e. $\exists u, v, w, u_1, v_1 \in V(G)$, s.t. $d(u) = d(v) = 2, d(u_1) = d(v_1) = d(w) = 4$ and $wu, wv, wu_1, wv_1, uu_1, vv_1, u_1v_1 \in E(G)$. We define a new graph $G^* = G - u$. Then G^*

is also a 2-connected outer plane graph with $\Delta(G^*) = 4$ and $|V(G^*)| = |V(G)| - 1 < p$. By induction hypothesis or theorem 2.5, G^* has a 6-adjacent vertex distinguishing total coloring f^*. Now we extend f^* of G^* to a 6-adjacent vertex distinguishing total coloring f of G.

Suppose $u_2 \in N(u_1) - \{u, w, v_1\}$.

Subcase 2.1 If $d(u_2) = 4$ and $f^*[u_2] = f^*[v_1]$.

(1) Suppose $l \in C - f^*[u_2] = C - f^*[v_1]$. If $l \in f^*[u_1]$, then let $f(uu_1) \in C - f^*[u_1]$; If $l \notin f^*[u_1]$, then let $f(uu_1) = l$.

(2) Suppose $l' \in f^*[v_1] - (f^*[u_1] \cup \{f(uu_1)\})$.

a) If $l, l' \in f^*[w]$, then let $f(uw) \in C - f^*[w] - \{f(uu_1)\}$;

b) If $l \in f^*[w], l' \notin f^*[w]$, then let $f(uw) = l'$;

c) If $l \notin f^*[w], l' \in f^*[w]$ and $f(uu_1) \neq l$, then let $f(uw) = l$;

d) If $l \notin f^*[w], l' \in f^*[w]$ and $f(uu_1) = l$, suppose $f^*(u_1w) = \alpha$, and exchanges the colors of uu_1 and u_1w, i.e. $f(u_1u) = \alpha, f(u_1w) = l$, and then let $f(uw) \in C - \{l, \alpha, f^*(w), f^*(wv_1), f^*(wv)\}$;

e) If $l, l' \notin f^*[w]$, firstly, we redefine the colors of wv and v as follows: $f(wv) = l, f(v) \in C - \{f^*(vv_1), f(wv), f^*(w), f^*(v_1)\}$, then let $f(uw) = l'$.

(3) Let $f(u)$ be the color in C s.t. $f(u)$ is not the color of uu_1, uw, u_1, w. The coloring of other elements is the same to f^*.

Subcase 2.2 If $d(u_2) = 4$ and $f^*[u_2] \neq f^*[v_1]$.

(1) Suppose $c_1 \in f^*[v_1] - f^*[u_2], c_2 \in f^*[u_2] - f^*[v_1]$.

- If $c_1, c_2 \in f^*[u_1]$, then let $f(uu_1) \in C - f^*[u_1]$;
- If $c_1 \notin f^*[u_1], c_2 \in f^*[u_1]$, then let $f(uu_1) = c_1$;
- If $c_1 \in f^*[u_1], c_2 \notin f^*[u_1]$, then let $f(uu_1) = c_2$;
- If $c_1, c_2 \notin f^*[u_1]$, then let $f(uu_1) = c_1$, meanwhile, modify the colors of u_1w, w, wv as follows (Note: new colors are still denoted by $f^*(u_1w), f^*(w), f^*(wv)$ hereinafter): $f^*(u_1w) = c_2, f^*(w) \in C - \{c_2, f^*(wv_1), f^*(v), f^*(v_1), f^*(u_1)\}, f^*(wv) \in C - \{c_2, f^*(wv_1), f^*(vv_1), f^*(w), f^*(v)\}$.

(2) Suppose $C' = \{f^*(u_1w), f^*(wv_1), f^*(wv), f^*(w)\}$, $f[u_1] = \{f^*(u_1), f^*(u_1u_2), f^*(u_1v_1), f^*(u_1w), f(uu_1)\}$ and $l \in f^*[v_1] - f[u_1]$.

1) If $c_2, l \in C'$, then let $f(uw) \in C - C' - \{f(uu_1)\}$;

2) If $c_2 \in C', l \notin C'$, then let $f(uw) = l$;

3) If $c_2 \notin C', l \in C'$ and $f(uu_1) \neq c_2$, then let $f(uw) = c_2$;

4) If $c_2 \notin C', l \in C'$ and $f(uu_1) = c_2$, firstly, suppose $f^*(u_1w) = c_3$ and exchanges the colors of uu_1 and u_1w i.e. $f(u_1u) = c_3, f(u_1w) = c_2$, then let $f(uw) \in C - \{c_2, c_3, f^*(w), f^*(wv_1), f^*(wv)\}$;

5) If $c_2, l \notin C'$, firstly, modify the color of wv and v as follows: $f(wv) = c_2, f(v) \in C - \{f^*(vv_1), f(wv), f^*(w), f^*(v_1)\}$, then let $f(uw) = l$.

(3) Let $f(u)$ be the color in C s.t. $f(u)$ is not the color of uu_1, uw, u_1, w. The coloring of other elements is the same to f^*.

Subcase 2.3 If $d(u_2) \neq 4$.

(1) If $f^*[u_1] \nsubseteq f^*[v_1]$, then let $f(uu_1) \in C - f^*[u_1]$; Otherwise let $f(uu_1) \in C - f^*[v_1]$.

(2) Suppose $f[u_1] = f^*[u_1] \cup \{f(uu_1)\}$, obviously, $f[u_1] \neq f^*[v_1]$, let $c_1 \in f[u_1] - f^*[v_1], c_2 \in f^*[v_1] - f[u_1]$ and $C' = \{f^*(u_1w), f^*(wv_1), f^*(wv), f^*(w)\}$.

- If $c_1, c_2 \in C'$, then let $f(uw) \in C - C' - \{f(uu_1)\}$;
- If $c_1 \in C',, c_2 \notin C'$, then let $f(uw) = c_2$;
- If $c_1 \notin C',, c_2 \in C'$ and $f(uu_1) \neq c_1$, then let $f(uw) = c_1$;
- If $c_1 \notin C',, c_2 \in C'$ and $f(uu_1) = c_1$, then after exchange the color of uu_1 and u_1w, let $f(uw) \in C - \{f(uu_1), f(u_1w), f^*(wv_1), f^*(wv), f^*(w)\}$
- If $c_1, c_2 \notin C'$, firstly, modify the color of wv and v as follows: $f(wv) = c_1, f(v) \in C - \{c_1, f^*(vv_1), f^*(w), f^*(v_1)\}$, then let $f(uw) = c_2$.

(3) Let $f(u) \in C - \{f(uu_1), f(uw), f^*(u_1), f^*(w)\}$. The coloring of other elements is the same to f^*.

Case 3. If item (1) and (2) of lemma 2.4 do not hold, but item (3) of lemma 2.4 holds. i.e. $\exists u, u_1, v, v_1, v_2, w, w_1 \in V(G)$, s.t. $d(u) = d(v) = d(w) = 2, d(v_1) = d(v_2) = d(u_1) = d(w_1) = 4$ and $uu_1, uv_1, u_1v_1, vv_1, vv_2, v_1v_2, v_2w, v_2w_1, ww_1 \in E(G)$; We define a new graph $G^* = G - v$. Then G^* is also a 2-connected outer plane graph with $\Delta(G^*) = 4$ and $|V(G^*)| = |V(G)| - 1 < p$. By induction hypothesis or theorem 2.5, G^* has a 6-adjacent vertex distinguishing total coloring f^*. Now we extend f^* of G^* to a 6-adjacent vertex distinguishing total coloring f of G.

(1) If $f^*[v_1] \subseteq f^*[u_1]$, then let $f(vv_1) \in C - f^*[u_1]$; Otherwise, let $f(vv_1) \in C - f^*[v_1]$.

(2) Suppose $f[v_1] = f^*[v_1] \cup \{f(vv_1)\}$.

- If $f[v_1] = f^*[w_1]$, then let $f(vv_2) \in C - f^*[w_1]$ when $f^*[v_2] \subseteq f^*[w_1]$; $f(vv_2) \in C - f^*[v_2] - \{f(vv_1)\}$ when $f^*[v_2] \nsubseteq f^*[w_1]$;
- If $f[v_1] \neq f^*[w_1]$, the proof is same as the proof of subcase 2.3.

(3) Let $f(v) \in C - \{f(vv_1), f(vv_2), f^*(v_1), f^*(v_2)\}$. The coloring of other elements is the same to f^*.

Case 4. If item (1), (2) and (3) of lemma 2.4 do not hold, but item (4) of lemma 2.4 holds. The discussion is the same as case 2 of the proof of theorem 2.3.

Case 5. If item (1), (2), (3) and (4) of lemma 2.4 do not hold, but item (5) of lemma 2.4 holds. i.e. $\exists u, v, w \in V(G)$, s.t. $uv, uw, vw \in E(G), d(u) = 2, d(v) = 3, d(w) = 4$. We define a new graph $G^* = G - u$. Then G^* is also a 2-connected outer plane graph with $\Delta(G^*) = 4$ and $|V(G^*)| = |V(G)| - 1 < p$. By induction hypothesis or theorem 2.5, G^* has a 6-adjacent vertex distinguishing total coloring f^*. Now we extend f^* of G^* to a 6-adjacent vertex distinguishing total coloring f of G.

Suppose $v_1 \in N(v) - \{u, w\}; w_1, w_2 \in N(w) - \{u, v\}$.

Subcase 5.1 $d(w_1) \neq 4$ and $d(w_2) \neq 4$. The discussion is similar to the case 3 of theorem 2.5.

Subcase 5.2 $d(w_1) = 4$ or $d(w_2) = 4$. Without loss of generality, we assume that $d(w_1) = 4$ and $d(w_2) \neq 4$.

(1) Let $f(uw) \in C - f^*[w_1]$ when $f^*[w] \subseteq f^*[w_1]$; $f(uw) \in C - f^*[w]$ when $f^*[w] \nsubseteq f^*[w_1]$;

(2) If $f^*[v] \subseteq f^*[v_1]$ and $d(v) = d(v_1)$, then let $f(uv) \in C - f^*[v_1] - \{f(uw)\}$; Otherwise let $f(uv) \in C - f^*[v] - \{f(uw)\}$;

(3) Let $f(u) \in C - \{f(uw), f(uv), f^*(w), f^*(v)\}$. The coloring of other elements is the same to f^*.

Subcase 5.3 $d(w_1) = 4$ and $d(w_2) = 4$.

(1) • If $f^*[w_1] = f^*[w_2]$, then when $f^*[w] \nsubseteq f^*[w_1]$ let $f(uw) \in C - f^*[w]$. Otherwise let $f(uw) \in C - f^*[w_1]$;

• If $f^*[w_1] \neq f^*[w_2]$, suppose $c_1 \in f^*[w_1] - f^*[w_2], c_2 \in f^*[w_2] - f^*[w_1]$. This time we will recolor the edge wv and the vertex v, then let $f(uw) = c_1, f(wv) = c_2$ and $f(v) \in \{f(wv), f^*(vv_1), f^*(w), f^*(v_1)\}$ or $f(uw) = c_2, f(wv) = c_1$ and $f(v) \in \{f(wv), f^*(vv_1), f^*(w), f^*(v_1)\}$ so that $f(wv) \neq f^*(vv_1)$.

(2) Suppose $f^*[v] = \{f(wv), f(v), f^*(vv_1)\}$, then if $f^*[v] \subseteq f^*[v_1]$ *and* $d(v) = d(v_1)$, then let $f(uv) \in C - f^*[v_1] - \{f(uw)\}$; otherwise let $f(uv) \in C - f^*[v] - \{f(uw)\}$.

(3) Let $f(u) \in C - \{f(uw), f(uv), f(w), f^*(v)\}$. The coloring of other elements is the same to f^*.

From what stated above, the proof is completed. ■

Lemma 2.7 Let $G(V, E)$ be a 2-connected outer plane graph with $\Delta(G) \geq 5$. Then at least one of the following statements holds in G.

1. There exist two adjacent vertices u and v of degree 2.
2. There exist two vertices u and v of degree 2 adjacent to one vertex w of degree 4.
3. There exists one vertex u of degree 2 adjacent to one vertex v of degree 3.

Lemma 2.7 has been proved in [6]. ■

Theorem 2.8 Let $G(V, E)$ be a 2-connected outer plane graph with $\Delta(G) = 5$, if $E(G[V_\Delta]) = \emptyset$, then $\chi_{at}(G) = 6$.

Proof. Because $E(G[V_\Delta]) = \emptyset$, then $\chi_{at}(G) \geq 6$. We now prove $\chi_{at}(G) \leq 6$. by using induction method on $p =| V(G) |$. Let $C = \{1, 2, 3, 4, 5, 6\}$ be a color set.

If $| V(G) |= 6$, then $G(V, E)$ is a fan with order 6. By enumeration, the conclusion is true. Assume that the conclusion is true when $| V(G) |< p$. We prove the conclusion is true for $| V(G) |= p$.

Case 1. Assume that the statement 1 of Lemma 2.7 holds, and $d(u) = d(v) = 2, uv \in E(G)$, u_0, v_0 are the another adjacent vertex of u and v respectively, obviously, $u_0 \neq v_0$. We define a new graph as $G^* = G - u + u_0v$. It is clear that G^* is also a 2-connected outer plane graph, where $| V(G^*) |=| V(G) | -1 < p$, $\Delta(G^*) = 5$ and $E(G^*[V_\Delta]) = \emptyset$. By the induction hypothesis, there is a 6-adjacent vertex distinguishing total coloring f^* of G^*. Now we extend f^* of G^* to a 6-adjacent vertex distinguishing total coloring f of G.

Subcase 1.1 If $d(u_0) \geq 3$ and $d(v_0) \geq 3$, then let $f(uu_0) = f^*(u_0v)$, $f(uv) \in C - f(uu_0) - f^*(vv_0) - f^*(v)$, $f(u) \in C - f^*(u_0) - f^*(v) - f(u_0u) - f(uv)$.

Subcase 1.2 If $d(u_0) = d(v_0) = 2$, then let $f(uu_0) = f^*(u_0v)$, $f(uv) \in C - f^*[v_0] - f^*(v) - f(u_0u)$, $f(u) \in C - f^*[u_0] - f^*(v) - f(uv)$.

Subcase 1.3 If $d(u_0) = 2, d(v_0) = 3$ or $d(u_0) = 3, d(v_0) = 2$. The proof methods of two cases are same, so we only consider the case $d(u_0) = 2, d(v_0) = 3$, let $f(uu_0) = f^*(u_0v)$, $f(uv) \in C - f(uu_0) - f^*(vv_0) - f^*(v)$, $f(u) \in C - f^*[u_0] - f^*(v) - f(uv)$.

The coloring of other elements is the same to f^*.

Case 2. If statement 1 of Lemma 2.7 does not appear, but statement 2 of Lemma 2.7 holds. Assume that $d(u) = d(v) = 2$, $d(w) = 4$, $uw, vw \in E(G), u_1(\neq w)$ *and* $v_1(\neq w)$ are another one adjacent vertex of u and v, respectively, and $w_1, w_2 \notin \{u, v\}$ are another two adjacent vertices of w. Thus it follows from the assumption that $d(u_1) \geq 3, d(v_1) \geq 3$.

Subcase 2.1 If $u_1 \notin \{w_1, w_2\}$ *and* $v_1 \notin \{w_1, w_2\}$ $(u_1 \neq v_1)$,, we define a new graph as $G^* = G - u - v + wu_1 + wv_1$. Thus G^* is also a 2-connected outer plane graph, where $| V(G^*) |< p$, $\Delta(G^*) = 5$ and $E(G^*[V_\Delta]) = \emptyset$. By the induction hypothesis, there is a 6-adjacent vertex distinguishing total coloring f^* of G^*. Now we extend f^* of G^* to a 6-adjacent vertex distinguishing total coloring f of G.

Let $f(uu_1) = f^*(wu_1), f(vv_1)) = f^*(wv_1), f(uw) = f^*(wv_1), f(vw) = f^*(wu_1)$, $f(u) \in C - f(uu_1) - f(uw) - f^*(w) - f^*(u_1), f(v) \in C - f(wv) - f(vv_1) - f^*(w) - f^*(v_1)$. The coloring of other elements is the same to f^*.

Subcase 2.2. If $u_1 \notin \{w_1, w_2\}$ *and* $v_1 \in \{w_1, w_2\}$, without loss of generality we may assume that $v_1 = w_2$. We define a new graph as $G^* = G - u + u_1w$. Thus G^* is also a 2-connected outer plane graph, where $| V(G^*) |< p$, $\Delta(G^*) = 5$ and $E(G^*[V_\Delta]) = \emptyset$.

By the induction hypothesis, there is a 6-adjacent vertex distinguishing total coloring f^* of G^*. Now we extend f^* of G^* to a 6-adjacent vertex distinguishing total coloring f of G. First, we color edge uu_1, let $f(uu_1) = f^*(wu_1)$.

——— If $f^*(wu_1) \neq f^*(vw_2)$, then let $f(uw) = f^*(wv), f(wv) = f^*(wu_1), f(u) = C - f^*(w) - f^*(u_1) - f(uw) - f(uu_1)$.

——— If $f^*(wu_1) = f^*(vw_2)$, then let $f(ww_2) = f^*(wu_1), f(w_2v) = f^*(ww_2), f(uw) = f^*(ww_2), f(u) = C - f^*(w) - f^*(u_1) - f(uw) - f(uu_1)$.

The coloring of other elements is the same to f^*.

Subcase 2.3. If $u_1 \in \{w_1, w_2\}$ *and* $v_1 \in \{w_1, w_2\}$, without loss of generality, we may assume that $u_1 = w_1, v_1 = w_2$.

2.3.1 If w_1 and w_2 are not adjacent, because statement 1 of Lemma 2.7 does not appear, then $d(w_1) \geq 3, d(w_2) \geq 3$, we will distinguish 6 cases in proof.

1. When $d(w_1) = d(w_2) = 3$, we define a new graph as $G^* = G - u - v + w_1w_2$. Thus G^* is also a 2-connected outer plane graph, where $| V(G^*) | < p$, $\Delta(G^*) = 5$ and $E(G^*[V_\Delta]) = \emptyset$. By the induction hypothesis, there is a 6-adjacent vertex distinguishing total coloring f^* of G^*. Now we extend f^* of G^* to a 6-adjacent vertex distinguishing total coloring f of G.

Let $f(uw_1) = f(vw_2) = f^*(w_1w_2)$, $f(uw) \in C - f(w_1u) - f^*(w_1w) - f^*(w_2w) - f^*(w)$, $f(vw) \in C - f(uw) - f^*(w_1w) - f^*(w_2w) - f(w_2v) - f^*(w)$, $f(u) \in C - f^*(w_1) - f^*(w) - f(w_1u) - f(wu)$, $f(v) \in C - f^*(w_2) - f^*(w) - f(wv) - f(w_2v)$.

2. When $d(w_1) = 3, d(w_2) = 4$, we define a new graph as $G^* = G - u - v + w_1w_2$. Thus G^* is also a 2-connected outer plane graph, where $| V(G^*) | < p$, $\Delta(G^*) = 5$ and $E(G^*[V_\Delta]) = \emptyset$. By the induction hypothesis, there is a 6-adjacent vertex distinguishing total coloring f^* of G^*. Now we extend f^* of G^* to a 6-adjacent vertex distinguishing total coloring f of G.

First, we color edges uw_1 and vw_2, let $f(uw_1) = f(vw_2) = f^*(w_1w_2)$, the coloring of other elements is the same to f^*, then we obtain a partial 6-adjacent vertex distinguishing total coloring f' of G. Suppose $C - f'[w_2] = \alpha$

If $f'(w_1w) = \alpha$ or $f'(w) = \alpha$, then let $f(uw) \in C - f(w_1u) - f'(w_1w) - f'(w_2w) - f'(w)$, $f(vw) \in C - f(uw) - f'(w_1w) - f'(w_2w) - f(w_2v) - f'(w)$, $f(u) \in C - f'(w_1) - f'(w) - f(w_1u) - f(wu)$, $f(v) \in C - f'(w_2) - f'(w) - f(wv) - f(w_2v)$.

If $f'(w_1w) \neq \alpha$ and $f'(w) \neq \alpha$, then let $f(wv) = \alpha$, $f(uw) \in C - f(w_1u) - f'(w_1w) - f'(w_2w) - f'(w) - f(wv)$, $f(u) \in C - f'(w_1) - f'(w) - f(w_1u) - f(wu)$, $f(v) \in C - f'(w_2) - f'(w) - f(wv) - f(w_2v)$.

3. When $d(w_1) = 3, d(w_2) = 5$, we define a new graph as $G^* = G - u - v + w_1w_2$. Thus G^* is also a 2-connected outer plane graph, where $| V(G^*) | < p$, $\Delta(G^*) = 5$ and $E(G^*[V_\Delta]) = \emptyset$. By the induction hypothesis, there is a 6-adjacent vertex distinguishing total coloring f^* of G^*. Now we extend f^* of G^* to a 6-adjacent vertex distinguishing total coloring f of G.

Let $f(uw_1) = f(vw_2) = f^*(w_1w_2)$, $f(uw) \in C - f(w_1u) - f^*(w_1w) - f^*(w_2w) - f^*(w)$, $f(vw) \in C - f(uw) - f^*(w_1w) - f^*(w_2w) - f(w_2v) - f^*(w)$, $f(u) \in C - f^*(w_1) - f^*(w) - f(w_1u) - f(wu)$, $f(v) \in C - f^*(w_2) - f^*(w) - f(wv) - f(w_2v)$.

4. When $d(w_1) = d(w_2) = 4$, we define a new graph as $G^* = G - u - v + w_1w_2$. Thus G^* is also a 2-connected outer plane graph, where $| V(G^*) | < p$, $\Delta(G^*) = 5$ and $E(G^*[V_\Delta]) = \emptyset$. By the induction hypothesis, there is a 6-adjacent vertex distinguishing

total coloring f^* of G^*. Now we extend f^* of G^* to a 6-adjacent vertex distinguishing total coloring f of G.

First, we color edges uw_1 and vw_2, let $f(uw_1) = f(vw_2) = f^*(w_1w_2)$, the coloring of other elements is the same to f^*, then we obtain a partial 6-adjacent vertex distinguishing total coloring f' of G.

By f', we know that $| f'[w_1] \cap f'[w_2] |= 4$, then existing two colors $\alpha, \beta \in C$ such that $\alpha \in f'[w_1]$, *but* $\alpha \notin f'[w_2]$; $\beta \in f'[w_2]$, *but* $\beta \notin f'[w_1]$.

— If $f'(w_1w) = \alpha, f'(w_2w) = \beta$, then let $f(uw) \in C - f(w_1u) - f'(w_1w) - f'(w_2w) - f'(w)$, $f(vw) \in C - f(uw) - f'(w_1w) - f'(w_2w) - f(w_2v) - f'(w)$, $f(u) \in C - f'(w_1) - f'(w) - f(w_1u) - f(wu)$, $f(v) \in C - f'(w_2) - f'(w) - f(wv) - f(w_2v)$.

— If $f'(w_1w) = \alpha, f'(w_2w) \neq \beta$, then

If $f'(w) = \beta$, let $f(uw) \in C - f(w_1u) - f'(w_1w) - f'(w_2w) - f'(w)$, $f(vw) \in C - f(uw) - f'(w_1w) - f'(w_2w) - f(w_2v) - f'(w)$, $f(u) \in C - f'(w_1) - f'(w) - f(w_1u) - f(wu)$, $f(v) \in C - f'(w_2) - f'(w) - f(wv) - f(w_2v)$.

If $f'(w) \neq \beta$, let $f(uw) = \beta$, $f(vw) \in C - f(uw) - f'(w_1w) - f'(w_2w) - f(w_2v) - f'(w)$, $f(u) \in C - f'(w_1) - f'(w) - f(w_1u) - f(wu)$, $f(v) \in C - f'(w_2) - f'(w) - f(wv) - f(w_2v)$.

— If $f'(w_1w) \neq \alpha, f'(w_2w) \neq \beta$, then

If $f'(w) \notin \{\alpha, \beta\}$, let $f(uw) = \beta$, $f(vw) = \alpha$, $f(u) \in C - f'(w_1) - f'(w) - f(w_1u) - f(wu)$, $f(v) \in C - f'(w_2) - f'(w) - f(wv) - f(w_2v)$.

If $f'(w) \in \{\alpha, \beta\}$, without loss of generality, we may assume $f'(w) = \alpha$, let $f(uw) = \beta$, $f(vw) \in C - f(uw) - f'(w_1w) - f'(w_2w) - f(w_2v) - f'(w)$, $f(u) \in C - f'(w_1) - f'(w) - f(w_1u) - f(wu)$, $f(v) \in C - f'(w_2) - f'(w) - f(wv) - f(w_2v)$.

Thus f is a 6-adjacent vertex distinguishing total coloring of G.

5. When $d(w_1) = 4$ *and* $d(w_2) = 5$, suppose $N(w_2) = \{x, y, z, w, v\}$. We define a new graph as $G^* = G - v$. Thus G^* is also a 2-connected outer plane graph, where $| V(G^*) |< p$, $\Delta(G^*) = 4$ *or* $\Delta(G^*) = 5$ and $E(G^*[V_\Delta]) = \emptyset$. By the theorem 2.5 or the induction hypothesis, there is a 6-adjacent vertex distinguishing total coloring f^* of G^*. Now we extend f^* of G^* to a 6-adjacent vertex distinguishing total coloring f of G.

First, we color edge w_2v, let $f(w_2v) = C - f^*(w_2x) - f^*(w_2y) - f^*(w_2z) - f^*(w_2w) - f^*(w_2)$, suppose $C - f^*(w_1) = \alpha$,

If $f^*(w_2w) = \alpha$ or $f^*(w) = \alpha$ or $f^*(uw) = \alpha$, then let $f(vw) \in C - f^*(uw) - f^*(w_1w) - f^*(w_2w) - f(w_2v) - f^*(w)$, $f(v) \in C - f^*(w_2) - f^*(w) - f(wv) - f(w_2v)$.

If $f^*(w_2w) \neq \alpha$, $f^*(w) \neq \alpha$, $f^*(uw) \neq \alpha$ and $f(w_2v) = \alpha$, then let $f(ww_2) = \alpha, f(w_2v) = f^*(w_2w)$ (i.e. to exchange the colors of edges ww_2 and vw_2), $f(vw) \in C - f^*(uw) - f^*(w_1w) - f^*(w_2w) - f(w_2v) - f^*(w)$, $f(v) \in C - f^*(w_2) - f^*(w) - f(wv) - f(w_2v)$.

If $f^*(w_2w) \neq \alpha$, $f^*(w) \neq \alpha$, $f^*(uw) \neq \alpha$ and $f(w_2v) \neq \alpha$, then let $f(wv) = \alpha, f(v) \in C - f^*(w_2) - f^*(w) - f(wv) - f(w_2v)$.

6. When $d(w_1) = d(w_2) = 5$, We define a new graph as $G^* = G - v$. Thus G^* is also a 2-connected outer plane graph, where $| V(G^*) |< p$, $\Delta(G^*) = 5$ and $E(G^*[V_\Delta]) = \emptyset$. By the induction hypothesis, there is a 6-adjacent vertex distinguishing total coloring f^* of G^*. Now we extend f^* of G^* to a 6-adjacent vertex distinguishing total coloring f of G.

let $f(w_2v) = C - f^*(w_2x) - f^*(w_2y) - f^*(w_2z) - f^*(w_2w) - f^*(w_2)$, $f(vw) \in C - f^*(uw) - f^*(w_1w) - f^*(w_2w) - f(w_2v) - f^*(w)$, $f(v) \in C - f^*(w_2) - f^*(w) - f(wv) - f(w_2v)$.

2.3.2 If w_1 and w_2 are adjacent, because statement 1 of Lemma 2.7 does not appear, G is a 2-connected graphs and $E(G^*[V_\Delta]) = \emptyset$, then $d(w_1) \geq 4, d(w_2) \geq 4$, and $d(w_1) = d(w_2) = 5$ is impossible, we will distinguish 2 cases in proof.

1. When $d(w_1) = d(w_2) = 4$, We define a new graph as $G^* = G - u - v + x + xw_1 + xw_2$, $x \notin V(G)$. Obviously, G^* is also a 2-connected outer plane graph, where $| V(G^*) | < p$, $\Delta(G^*) = 5$ and $E(G^*[V_\Delta]) = \emptyset$. By the induction hypothesis, there is a 6-adjacent vertex distinguishing total coloring f^* of G^*. Now we extend f^* of G^* to a 6-adjacent vertex distinguishing total coloring f of G.

First, we color edges uw_1 and vw_2, let $f(uw_1) = f^*(xw_1), f(vw_2) = f^*(xw_2)$, the coloring of other elements is the same to f^*, then we obtain a partial 6-adjacent vertex distinguishing total coloring f' of G.

By f', we know that $| f'[w_1] \cap f'[w_2] |= 4$, then existing two colors $\alpha, \beta \in C$ such that $\alpha \in f'[w_1]$, *but* $\alpha \notin f'[w_2]$; $\beta \in f'[w_2]$, *but* $\beta \notin f'[w_1]$.

— If $f'(w_1w) = \alpha, f'(w_2w) = \beta$, then let $f(uw) \in C - f(w_1u) - f'(w_1w) - f'(w_2w) - f'(w)$, $f(vw) \in C - f(uw) - f'(w_1w) - f'(w_2w) - f(w_2v) - f'(w)$, $f(u) \in C - f'(w_1) - f'(w) - f(w_1u) - f(wu)$, $f(v) \in C - f'(w_2) - f'(w) - f(wv) - f(w_2v)$.

— If $f'(w_1w) = \alpha, f'(w_2w) \neq \beta$, then

If $f'(w) = \beta$, let $f(uw) \in C - f(w_1u) - f'(w_1w) - f'(w_2w) - f'(w)$, $f(vw) \in C - f(uw) - f'(w_1w) - f'(w_2w) - f(w_2v) - f'(w)$, $f(u) \in C - f'(w_1) - f'(w) - f(w_1u) - f(wu)$, $f(v) \in C - f'(w_2) - f'(w) - f(wv) - f(w_2v)$.

If $f'(w) \neq \beta$, let $f(uw) = \beta$, $f(vw) \in C - f(uw) - f'(w_1w) - f'(w_2w) - f(w_2v) - f'(w)$, $f(u) \in C - f'(w_1) - f'(w) - f(w_1u) - f(wu)$, $f(v) \in C - f'(w_2) - f'(w) - f(wv) - f(w_2v)$.

— If $f'(w_1w) \neq \alpha, f'(w_2w) \neq \beta$, then

If $f'(w) \notin \{\alpha, \beta\}$, let $f(uw) = \beta$, $f(vw) = \alpha$, $f(u) \in C - f'(w_1) - f'(w) - f(w_1u) - f(wu)$, $f(v) \in C - f'(w_2) - f'(w) - f(wv) - f(w_2v)$.

If $f'(w) \in \{\alpha, \beta\}$, without loss of generality, we may assume $f'(w) = \alpha$, let $f(uw) = \beta$, $f(vw) \in C - f(uw) - f'(w_1w) - f'(w_2w) - f(w_2v) - f'(w)$, $f(u) \in C - f'(w_1) - f'(w) - f(w_1u) - f(wu)$, $f(v) \in C - f'(w_2) - f'(w) - f(wv) - f(w_2v)$.

Thus f is a 6-adjacent vertex distinguishing total coloring of G.

2. When $d(w_1) = 4$ *and* $d(w_2) = 5$, the proof is similar to 5 of 2.3.1.

Case 3. If both statements 1 and 2 of Lemma 2.7 do not hold, then statement 3 of Lemma 2.7 must hold. Suppose $d(u) = 2$ and $d(v) = 3$, $N(u) = \{w, v\}, N(v) = \{u, v_1, v_2\}$, obviously, $uv \in E(G), w \neq v$ *and* $d(w) \geq 3$.

For this case, it is easy to prove that there is a group of vertices u, v, w such that $w \in \{v_1, v_2\}$. Otherwise, we define a new graph G^* by deleting all such vertex u from G and adding edge wv into G, then G^* is also an outer plane graph, by the assumption of case 3, all statements 1, 2 and 3 of Lemma 2.7 do not occur in G^*, it is a contradiction.

So we may suppose that all such u, v, w satisfy $w \in \{v_1, v_2\}$. Without loss of generality, we may assume that $w = v_2$.

First, we define a new graph as $G^* = G - \{u\}$. Thus G^* is also a 2-connected outer plane graph, where $| V(G^*) | < p$, $\Delta(G^*) = 5$ and $E(G^*[V_\Delta]) = \emptyset$. By the induction hypothesis, there is a 6-adjacent vertex distinguishing total coloring f^* of G^*. Now we extend f^* of G^* to a 6-adjacent vertex distinguishing total coloring f of G.

Subcase 3.1 When $d(w) = 3$, Let $N(w) = \{u, v, w_1\}$, then,

1 When $d(w_1) = 3$, then,

▲▲ If $f^*(w) \notin f^*[w_1]$ or $f^*(wv) \notin f^*[w_1]$, then,

Firstly, we color edge wu,

Let $f(uw) \in C - f^*(ww_1) - f^*(w) - f^*(wv) - f^*(v) - f^*(vv_1)$.

Next, we color edge uv,

—— If $d(v_1) \neq 3$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$.

—— If $d(v_1) = 3$, then,

(1) If $f^*(v) \notin f^*[v_1]$ or $f^*(wv) \notin f^*[v_1]$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$

(2) If $f^*(v) \in f^*[v_1]$ and $f^*(wv) \in f^*[v_1]$, then, let $f(uv) \in C - f^*[v_1] - f(uw)$.

Finally, we color vertex u, let $f(u) \in C - f^*(w) - f^*(v) - f(uw) - f(uv)$

▲▲ If $f^*(w) \in f^*[w_1]$ and $f^*(wv) \in f^*[w_1]$, then,

▲ If $f^*(v) \in f^*[w_1], f^*(vv_1) \notin f^*[w_1]$ or $f^*(v) \notin f^*[w_1], f^*(vv_1) \in f^*[w_1]$, the proof of two cases is same, so we only considered the case $f^*(v) \in f^*[w_1], f^*(vv_1) \notin f^*[w_1]$
Firstly, we color edge wu,

let $f(uw) \in C - f^*[w_1] - f^*(vv_1)$.

Next, we color edge uv,

—— If $d(v_1) \neq 3$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$.

—— If $d(v_1) = 3$, then,

(1) If $f^*(v) \notin f^*[v_1]$ or $f^*(wv) \notin f^*[v_1]$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$

(2) If $f^*(v) \in f^*[v_1]$ and $f^*(wv) \in f^*[v_1]$, then, let $f(uv) \in C - f^*[v_1] - f(uw)$.

Finally, we color vertex u, let $f(u) \in C - f^*(w) - f^*(v) - f(uw) - f(uv)$

▲ If $f^*(v) \in f^*[w_1], f^*(vv_1) \in f^*[w_1]$, then,

Firstly, we color edge wu, let $f(uw) \in C - f^*[w_1]$.

Next, we color edge uv,

—— If $d(v_1) \neq 3$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$.

—— If $d(v_1) = 3$, then,

(1) If $f^*(v) \notin f^*[v_1]$ or $f^*(wv) \notin f^*[v_1]$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$

(2) If $f^*(v) \in f^*[v_1]$ and $f^*(wv) \in f^*[v_1]$, then, let $f(uv) \in C - f^*[v_1] - f(uw)$.

Finally, we color vertex u, let $f(u) \in C - f^*(w) - f^*(v) - f(uw) - f(uv)$

▲ If $f^*(v) \notin f^*[w_1], f^*(vv_1) \notin f^*[w_1]$, then, Firstly, we color edge wu,

let $f(uw) \in C - f^*[w_1]$, then we know $f(uw) \in \{f^*(v), f^*(vv_1)\}$

Next, we color edge uv,

—— If $d(v_1) \neq 3$, then, let $f(uv) \in C - f^*(wv) - f^*(v) - f^*(vv_1) - f^*(ww_1)$.

—— If $d(v_1) = 3$, then,

(1) If $f^*(v) \notin f^*[v_1]$ or $f^*(wv) \notin f^*[v_1]$, then, let $f(uv) \in C - f^*(wv) - f^*(v) - f^*(vv_1) - f^*(ww_1)$

(2) If $f^*(v) \in f^*[v_1]$ and $f^*(wv) \in f^*[v_1]$, then, let $f(uv) \in C - f^*[v_1] - f^*(ww_1)$.

Finally, we color vertex u, let $f(u) \in C - f^*(w) - f^*(v) - f(uw) - f(uv)$

2 When $d(w_1) \neq 3$, then,

Firstly, we color edge wu,

Let $f(uw) \in C - f^*(ww_1) - f^*(w) - f^*(wv) - f^*(v) - f^*(vv_1)$.

Next, we color edge uv,

—— If $d(v_1) \neq 3$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$.

—— If $d(v_1) = 3$, then,

(1) If $f^*(v) \notin f^*[v_1]$ or $f^*(wv) \notin f^*[v_1]$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$

(2) If $f^*(v) \in f^*[v_1]$ and $f^*(wv) \in f^*[v_1]$, then, let $f(uv) \in C - f^*[v_1] - f(uw)$.

Finally, we color vertex u, let $f(u) \in C - f^*(w) - f^*(v) - f(uw) - f(uv)$

Subcase 3.2 When $d(w) = 4$, Let $N(w) = \{u, v, w_1, w_2\}$, then,

• **3.2.1** • . If $d(w_1) \neq 4$ *and* $d(w_2) \neq 4$,

Firstly, we color edge uw, then, let $f(uw) \in C - f^*(ww_1) - f^*(ww_2) - f^*(w) - f^*(wv)$.

Next, we color edge uv,

—— If $d(v_1) \neq 3$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$.

—— If $d(v_1) = 3$, then,

(1) If $f^*(v) \notin f^*[v_1]$ or $f^*(wv) \notin f^*[v_1]$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$

(2) If $f^*(v) \in f^*[v_1]$ and $f^*(wv) \in f^*[v_1]$, then, let $f(uv) \in C - f^*[v_1] - f(uw)$.

Finally, we color vertex u, let $f(u) \in C - f^*(w) - f^*(v) - f(uw) - f(uv)$

• **3.2.2** • . If $d(w_1) = d(w_2) = 4$, then $4 \leq | f^*(w_1) \cap f^*(w_2) | \leq 5$,

▲ If $| f^*[w_1] \cap f^*[w_2] | = 4$, $\Rightarrow$ existing 2 colors $\alpha, \beta \in C$ such that $\alpha \in f^*[w_1]$, *but* $\alpha \notin f^*[w_2]$, $\beta \in f^*[w_2]$, *but* $\beta \notin f^*[w_1] \Longrightarrow \alpha \notin f^*[w_1] \cap f^*[w_2]$ *and* $\beta \notin f^*[w_1] \cap f^*[w_2]$ $(\alpha \neq \beta)$

•• If $f^*[w_1] \cap f^*[w_2] = \{f^*(ww_1), f^*(ww_2), f^*(w), f^*(wv)\}$.

• If $f^*(v) \notin \{\alpha, \beta\}$ and $f^*(vv_1) \notin \{\alpha, \beta\}$, then, Firstly, we color edge uw, let $f(uw) \in \{\alpha, \beta\}$, without loss of generality, we may assume $f(uw) = \alpha$,

Next, we color edge uv, first we recolor edge wv, let $f(wv) = \beta$,

If $d(v_1) \neq 3$, then, let $f(uv) \in C - f(uw) - f(wv) - f^*(v) - f^*(vv_1)$.

If $d(v_1) = 3$, then,

(1) If $f^*(v) \notin f^*[v_1]$ or $f^*(wv) \notin f^*[v_1]$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$

(2) If $f^*(v) \in f^*[v_1]$ and $f^*(wv) \in f^*[v_1]$, then, let $f(uv) \in C - f^*[v_1] - f(uw)$.

Finally, we color vertex u, let $f(u) \in C - f^*(w) - f^*(v) - f(uw) - f(uv)$.

• If $f^*(v) \in \{\alpha, \beta\}$ and $f^*(vv_1) \notin \{\alpha, \beta\}$ or $f^*(v) \notin \{\alpha, \beta\}$ and $f^*(vv_1) \in \{\alpha, \beta\}$. The proof methods of two cases are same, so we only consider the case $f^*(v) \in \{\alpha, \beta\}$ and $f^*(vv_1) \notin \{\alpha, \beta\}$ without loss of generality, we may assume $f^*(v) = \alpha$

Firstly, we color edge uw, let $f(uw) = \alpha$,

Next, we color edge uv, first we recolor edge wv, let $f(wv) = \beta$,

If $d(v_1) \neq 3$, then, let $f(uv) \in C - f(uw) - f(wv) - f^*(v) - f^*(vv_1)$.

If $d(v_1) = 3$, then,

(1) If $f^*(v) \notin f^*[v_1]$ or $f^*(wv) \notin f^*[v_1]$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$

(2) If $f^*(v) \in f^*[v_1]$ and $f^*(wv) \in f^*[v_1]$, then, let $f(uv) \in C - f^*[v_1] - f(uw)$.

Finally, we color vertex u, let $f(u) \in C - f^*(w) - f^*(v) - f(uw) - f(uv)$.

• If $f^*(v) \in \{\alpha, \beta\}$ and $f^*(vv_1) \in \{\alpha, \beta\}$, without loss of generality, we may assume $f^*(v) = \beta, f^*(vv_1) = \alpha$.
Firstly, we color edge uw, let $f(uw) = \alpha$,

Next, we color edge uv, first, we recolor edge wv, let $f(wv) = \beta$, second, we recolor vertex v, let $f(v) \in C - f^*(w) - f^*(v_1) - f(wv) - f^*(vv_1)$.

If $d(v_1) \neq 3$, then, let $f(uv) \in C - f(uw) - f(wv) - f^*(v) - f^*(vv_1)$.

If $d(v_1) = 3$, then,

(1) If $f^*(v) \notin f^*[v_1]$ or $f^*(wv) \notin f^*[v_1]$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$

(2) If $f^*(v) \in f^*[v_1]$ and $f^*(wv) \in f^*[v_1]$, then, let $f(uv) \in C - f^*[v_1] - f(uw)$.

Finally, we color vertex u, let $f(u) \in C - f^*(w) - f^*(v) - f(uw) - f(uv)$.

•• If $f^*[w_1] \cap f^*[w_2] \neq \{f^*(ww_1), f^*(ww_2), f^*(w), f^*(wv)\}$, then,

• First, we color edge uw,

If α or $\beta \in \{f^*(ww_1), f^*(ww_2), f^*(w), f^*(wv)\}$, without loss of generality, we may assume $\alpha \in \{f^*(ww_1), f^*(ww_2), f^*(w), f^*(wv)\}$ and $\beta \notin \{f^*(ww_1), f^*(ww_2), f^*(w), f^*(wv)\}$, then, let $f(uw) = \beta$,

If α and $\beta \in \{f^*(ww_1), f^*(ww_2), f^*(w), f^*(wv)\}$.

Let $f(uw) \in C - f^*(ww_1) - f^*(ww_2) - f^*(w) - f^*(wv)$

• Second, we color edge uv,

If $d(v_1) \neq 3$, then, let $f(uv) \in C - f(uw) - f(wv) - f^*(v) - f^*(vv_1)$.

If $d(v_1) = 3$, then,

(1) If $f^*(v) \notin f^*[v_1]$ or $f^*(wv) \notin f^*[v_1]$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$

(2) If $f^*(v) \in f^*[v_1]$ and $f^*(wv) \in f^*[v_1]$, then, let $f(uv) \in C - f^*[v_1] - f(uw)$.

• Finally, we color vertex u, let $f(u) \in C - f^*(w) - f^*(v) - f(uw) - f(uv)$.

▲ If $| f^*[w_1] \cap f^*[w_2] |= 5$ (w_1, w_2 are not adjacent), $\Rightarrow f^*[w_1] = f^*[w_2]$.

Firstly, we color edge uw,

If $f^*(w) \notin f^*[w_1](= f^*[w_2])$ or $f^*(wv) \notin f^*[w_1](= f^*[w_2])$, let $f(uw) \in C - f^*(ww_1) - f^*(ww_2) - f^*(w) - f^*(wv)$.

If $f^*(w) \in f^*[w_1](= f^*[w_2])$ and $f^*(wv) \in f^*[w_1](= f^*[w_2])$, let $f(uw) \in C - f^*[w_1]$.

Next, we color edge uv,

If $d(v_1) \neq 3$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$.

If $d(v_1) = 3$, then,

(1) If $f^*(v) \notin f^*[v_1]$ or $f^*(wv) \notin f^*[v_1]$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$

(2) If $f^*(v) \in f^*[v_1]$ and $f^*(wv) \in f^*[v_1]$, then, let $f(uv) \in C - f^*[v_1] - f(uw)$.

Finally, we color vertex u, let $f(u) \in C - f^*(w) - f^*(v) - f(uw) - f(uv)$.

• 3.2.3 • . If $d(w_1) = 4, d(w_2) \neq 4$ or $d(w_1) \neq 4, d(w_2) = 4$. The proof methods of two cases are same, so we only consider the case $d(w_1) = 4, d(w_2) \neq 4$.

Firstly, we color edge uw,

If $f^*(ww_2) \notin f^*[w_1]$ or $f^*(wv) \notin f^*[w_1]$ or $f^*(w) \notin f^*[w_1]$, Let $f(uw) \in C - f^*(ww_1) - f^*(ww_2) - f^*(w) - f^*(wv)$;

If $f^*(ww_2) \in f^*[w_1]$, $f^*(wv) \in f^*[w_1]$ and $f^*(w) \in f^*[w_1]$, let $f(uw) \in C - f^*[w_1]$;

Next, we color edge uv,

If $d(v_1) \neq 3$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$.

If $d(v_1) = 3$, then,

(1) If $f^*(v) \notin f^*[v_1]$ or $f^*(wv) \notin f^*[v_1]$, then, let $f(uv) \in C - f(uw) - f^*(wv) - f^*(v) - f^*(vv_1)$

(2) If $f^*(v) \in f^*[v_1]$ and $f^*(wv) \in f^*[v_1]$, then, let $f(uv) \in C - f^*[v_1] - f(uw)$.

Finally, we color vertex u, let $f(u) \in C - f^*(w) - f^*(v) - f(uw) - f(uv)$.

Subcase 3.3 When $d(w) = 5$, because $E(G[V_\Delta]) = \emptyset$, then the proof is easy, and omitted here.

With all cases considered , f is a 6- adjacent vertex distinguishing total coloring of G. ■

Theorem 2.10 Let $G(V, E)$ be a 2-connected outer plane graph with $\Delta(G) = 5$, if $E(G[V_\Delta]) \neq \emptyset$, then $\chi_{at}(G) = 7$.

Proof. Because $E(G[V_\Delta]) \neq \emptyset$, then $\chi_{at}(G) \geq 7$. We now prove $\chi_{at}(G) \leq 7$ by using induction on $| V(G) |= p$. By enumeration, the conclusion is true for the outer plane graph with order $| V(G) |= 8$ and $E(G[V_\Delta]) \neq \emptyset$. Assume that the conclusion is true when $| V(G) |< p$. We prove the conclusion is true for $| V(G) |= p$.

The proof of theorem 2.10 are the same as that of theorem 2.9 except 2.3.2 of subcase 2.3 and subcase 3.3 of theorem 2.9.

Hence we only prove the 2.3.2 of subcase 2.3 and subcase 3.3. Let $C = \{1, 2, 3, 4, 5, 6, 7\}$ be denote a color set and the same notations as the 2.3.2 of subcase 2.3 and subcase 3.3.

2.3.2 (1) If $d(w_1) = d(w_2) = 5$, first we define a new graph $G^* = G - u - v + x + xw_1 + xw_2$, where $x \notin V(G)$. Obviously, G^* is also a 2-connected outer plane graph, where $| V(G^*) |< p$, $\Delta(G^*) = 5$ and $E(G^*[V_\Delta]) \neq \emptyset$. By the induction hypothesis, there is a 7-adjacent vertex distinguishing total coloring f^* of G^*. Now we extend f^* of G^* to a 7-adjacent vertex distinguishing total coloring f of G.

First, we color edges uw_1 and vw_2, let $f(uw_1) = f^*(xw_1), f(vw_2) = f^*(xw_2)$.

Next, we color edges uw, vw, let $f(uw) \in C - f(uw_1) - f^*(ww_1) - f^*(ww_2) - f^*(w)$; $f(vw) \in C - f(uw) - f^*(ww_1) - f^*(ww_2) - f^*(w) - f(w_2v)$.

Finally, we color vertices u, v, let $f(u) \in C - f^*(w_1) - f^*(w) - f(uw_1) - f(uw)$; $f(v) \in C - f^*(w) - f^*(w_2) - f(vw) - f(vw_2)$.

The coloring of other elements is the same to f^*, thus f is a 7-adjacent vertex distinguishing total coloring of G.

(2) The proofs of other cases are same as that of theorem 2.2.

Subcase 3.3 When $d(w) = 5$, first we define a new graph $G^* = G - v + wv_1$, obviously, G^* is also a 2-connected outer plane graph, where $| V(G^*) |< p$, $\Delta(G^*) = 5$ and $E(G^*[V_\Delta]) \neq \emptyset$. By the induction hypothesis, there is a 7-adjacent vertex distinguishing total coloring f^* of G^*. Now we extend f^* of G^* to a 7-adjacent vertex distinguishing total coloring f of G.

First, we color edge wv, let $f(wv) = f^*(wv_1)$;

Next, we color edges uv, vv_1,

If $d(v_1) = 3$, suppose $N(v_1) = \{v, x, y\}$, then let $f(uv) \in C - f^*(u) - f^*(wu) - f(wv) - f^*(v_1) - f^*(v_1x) - f^*(v_1y)$, $f(vv_1) \in C - f(uv) - f(wv) - f^*(v_1) - f^*(v_1x) - f^*(v_2y)$;

Finally, we color vertex v, let $f(v) \in C - f^*(u) - f^*(w) - f^*(v_1) - f(uv) - f(wv) - f(vv_1)$.

The coloring of other elements is the same to f^*, thus f is a 7-adjacent vertex distinguishing total coloring of G.

With above cases considered , theorem 2.10 is then proven. ■

This method of proof and this result are helpful for proving the adjacent vertex distinguishing total chromatic number of outer plane graph with $\Delta \geq 6$ is $\Delta + 2$ if G have two adjacent maximum degree vertices, otherwise is $\Delta + 1$, since when $\Delta(G) \geq 5$, they have same property lemma 2.7, but analysis of many cases are different, therefore we will find more effectual method to prove the general result in future works.

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