Domination number in iterated line digraph of a complete bipartite digraph*

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Abstract: Let $K_{d,d}$ be a complete bipartite digraph. In this paper, we determine the exact value of the domination number in iterated line digraph of $K_{d,d}$.

Key words: Combinatorial problems; Domination number; Complete bipartite digraph; Line digraph

1 Introduction

An interconnection network can be modeled as a digraph, where each element can be represented as a vertex and the directed connection between two vertices is described by an arc. In this paper, we consider only finite strict directed graph G (digraph having no loops and no parallel arcs are allowed) with vertex set $V(G)$ and arc set $A(G)$. For a vertex $v \in V(G)$, the out-neighborhood of v is $N^+(v) = \{u|(v,u) \in A(G)\}$ and the in-neighborhood of v is $N^-(v) = \{u|(u,v) \in A(G)\}$. The closed out-neighborhood and closed in-neighborhood of v are $N^+[v] = N^+(v) \cup \{v\}$ and $N^-[v] = N^-(v) \cup \{v\}$, respectively. For a subset $F \subseteq V(G)$, the out-neighborhood of F is $N^+(F) = \bigcup_{v\in F} N^+(v)$ and the in-neighborhood of F

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is $N^-(F) = \bigcup_{v \in F} N^-(v)$.

In a digraph G, a vertex u is said to dominate itself and all of its out-neighbors, that is u dominates v if either $u = v$ or (u, v) is an arc of G. A dominating set of a digraph G is a subset $D \subseteq V(G)$ such that every vertex in $V(G)$ is dominated by at least one vertex in D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of dominating sets of G.

Let $G = (V(G), A(G))$ be a digraph, $|V(G)| = n$, $|A(G)| = m$, $V(G) = \{v_1, v_2, \cdots, v_n\}$. The line digraph of G, denoted by $L(G)$, is the digraph with vertex set $V(L(G)) = \{a_{ij} | a_{ij} = (v_i, v_j)$ is an arc in $G\}$ and a vertex a_{ij} is adjacent to a vertex a_{st} in $L(G)$ if and only if $v_j = v_s$ in G. For an integer n, the n-th iterated line digraph of G is recursively defined as $L^n(G) = L(L^{n-1}(G))$ with $L^0(G) = G$.

Let $K_{d,d}$ is a complete bipartite digraph and $d \geq 1$, the $(n-1)$th iterated line digraph $L^{n-1}(K_{d,d})$ is studied by Liu et al [7] as $LCBD(d, n)$. For convenience, we use the symbols $LCBD(d, n)$ below. For $d \geq 2$ and $n \geq 1$, the bipartite digraph $LCBD(d, n)$ is a d-regular digraph, has $2d^n$ vertices and $2d^{n+1}$ arcs. $LCBD(d, n)$ can be also defined as follows: let $V_0 = \{1, 2, \cdots, d\}, V_1 = \{1', 2', \cdots, d'\}$. The vertex set of $LCBD(d, n)$ is

$$\{x_1 x_2 \cdots x_n : x_i \in V_j \ \ and \ \ x_{i+1} \in V_{j+1}, j = 0, 1 (mod\ 2), i = 1, 2, \cdots, n-1\},$$

and the arc set of $LCBD(d, n)$ consists of all arcs from one vertex $x_1 x_2 \cdots x_n$ to $x_2 \cdots x_n \alpha$, where x_n and α are not in the same set (V_0 or V_1). Furthermore some properties of $LCBD(d, n)$ are given such as the connectivity, spectrum and so on in [7]. But the domination number of $LCBD(d, n)$ has not been determined yet.

It is one of major areas in theoretical and algorithmic observation to study domination and its related topics for digraph. Domination of digraphs come up more naturally in modeling real world problems, especially in modeling interconnection network. A survey on domination in digraph has been written by Ghoshal et al [4]. For a comprehensive treatment of domination and its variations, we refer the reader to [5, 6]. Domination in digraph have been studied extensively in recent years such as in De Brujin and Kautz digraphs [1, 2, 8] and so on.

For terminologies not given here we refer the reader to [3].

2 Main results

The next Lemma which is induced in [1] provides a construction of a dominating set of $L(G)$ from that of G.

Lemma 2.1. *Assume that D is a dominating set of G, let D_1 be the subset of vertices of $L(G)$ defined by $D_1 = \{(u,v)|u \in D\}$. Then D_1 is a dominating set of $L(G)$.*

Proof. Let (x,y) be an arbitrary vertex in $L(G)$. If $(x,y) \in D_1$, then it is obvious. Assume that $(x,y) \notin D_1$, then vertex x is not in D by the definition of D_1. Therefore, there exists a vertex x' in D such that $(x',x) \in D_1$ and (x,y) is dominated by (x',x). □

Lemma 2.2. *Assume that D_n is a dominating set of $LCBD(d,n)$, let D_{n+1} be the subset of vertices of $LCBD(d,n+1)$ defined by*

$$D_{n+1} = \{x_1x_2\cdots x_n\alpha | x_1x_2\cdots x_n \in D_n,$$
$$x_n \text{ and } \alpha \text{ are not in the same set } (V_0 \text{ or } V_1)\}.$$

Then D_{n+1} is a dominating set of $LCBD(d,n+1)$ and $D_{n+1} = d|D_n|$.

Proof. As mentioned earlier, $LCBD(d,n+1)$=$L(LCBD(d,n))$. It is easy to see that D_{n+1} is corresponding to the set of vertices $(x_1x_2\cdots x_n, x_2x_3\cdots x_n\alpha)$ in $LCBD(d,n+1)$ such that $x_1x_2\cdots x_n \in D_n$. Hence, by Lemma 2.1, D_{n+1} is a dominating set of $LCBD(d,n+1)$. The equation $D_{n+1} = d|D_n|$ clearly holds. □

Next we determine the exact value of the domination number of $LCBD(d,n)$.

Theorem 2.3. *For $d \geq 2, n \geq 1$, $\gamma(LCBD(d,n)) = \lceil \frac{2d^n}{d+1} \rceil$.*

Proof. First, we show that $\gamma(LCBD(d,n)) \geq \lceil \frac{2d^n}{d+1} \rceil$.

Assume that D is a dominating set of $LCBD(d,n)$, let $\bar{D}$ be the set of vertices not in D. A vertex in $\bar{D}$ is dominated by at least one vertex in D, there are at least $|\bar{D}|$ arcs from vertices in D to $\bar{D}$. On the other hand, there are at most $d|D|$ arcs from vertices in D to $\bar{D}$ because $LCBD(d,n)$ is d-regular. Hence an inequality $d|D| \geq |\bar{D}|$ must hold. Since $|D| + |\bar{D}| = 2d^n$, we obtain that $|D| \geq \lceil \frac{2d^n}{d+1} \rceil$. That is $\gamma(LCBD(d,n)) \geq \lceil \frac{2d^n}{d+1} \rceil$.

Next we will construct a dominating set of $LCBD(d,n)$ with size $\lceil \frac{2d^n}{d+1} \rceil$. We consider the following two cases.

Case 1. n is odd.

Let $f(d,n) = \lceil \frac{2d^n}{d+1} \rceil$, by simple calculation, we have

$$\begin{aligned} f(d,1) &= 2, \\ f(d,n) &= \lceil \frac{2(d^n+1-1)}{d+1} \rceil \\ &= \lceil \frac{2(d+1)(d^{n-1}-d^{n-2}+\cdots+d^2-d+1)-2}{d+1} \rceil \\ &= 2(d^{n-1}-d^{n-2}+\cdots+d^2-d+1). \end{aligned}$$

By recursive, we have

$$\begin{aligned} f(d,n) &= 2d^2(d^{n-3}-d^{n-4}+\cdots+d^2-d+1)-2(d-1) \\ &= d^2 f(d,n-2)-2(d-1), n \geq 3. \end{aligned}$$

Now we will construct and show that there is a dominating set D_n of $LCBD(d,n)$ such that $|D_n| = f(d,n)$ by induction on n.

For $n = 1$, it is easily proved that $\gamma(LCBD(d,1)) = \gamma(K_{d,d}) = 2 = f(d,1)$. Without loss of generality, the set $D_1 = \{1, 1'\}(1 \in V_0, 1' \in V_1)$ is a dominating set of $K_{d,d}$.

Assume that D_{n-2} is a dominating set of $LCBD(d,n-2)$ such that $|D_{n-2}| = f(d,n-2)$. Let

$$\begin{aligned} F_3 = \{x_1\alpha\beta | x_1 \in D_1, &\alpha \text{ and } x_1 \text{ are not in the same set } (V_0 \text{ or } V_1), \\ &\beta \text{ and } x_1 \text{ are in the same set } (V_0 \text{ or } V_1)\}, \end{aligned}$$

$$\begin{aligned} F_{31} &= \{11'\beta | \beta \neq 1, \beta \in V_0\} \subseteq F_3, \\ F_{32} &= \{1'1\beta | \beta \neq 1', \beta \in V_1\} \subseteq F_3, \\ D_3 &= F_3 \backslash (F_{31} \cup F_{32}). \end{aligned}$$

For n is odd and $n \geq 3$, we can recursively construct D_n as follows: Let

$$\begin{aligned} F_n = \{x_1x_2\cdots x_{n-2}\alpha\beta | x_1x_2\cdots x_{n-2} \in D_{n-2}, \alpha \text{ and } x_{n-2} \text{ are not} \\ \text{in the same set } (V_0 \text{ or } V_1), \beta \text{ and } x_{n-2} \text{ are in the same set } (V_0 \text{ or } V_1)\}, \end{aligned}$$

$$\begin{aligned} F_{n1} &= \{11'1\cdots 1'\beta | \beta \neq 1, \beta \in V_0\}, \\ F_{n2} &= \{1'11'\cdots 1\beta | \beta \neq 1', \beta \in V_1\}. \end{aligned}$$

Thus, F_n is a dominating set of $LCBD(d,n)$ by applying two times the definition in Lemma 2.2, $F_{n1} \subseteq F_n$, $F_{n2} \subseteq F_n$ and $F_{n1} \cap F_{n2} = \emptyset$. Let

$$D_n = F_n \backslash (F_{n1} \cup F_{n2}),$$

then $|D_n| = |F_n| - |F_{n1} \cup F_{n2}| = d^2|D_{n-2}| - 2(d-1) = d^2 f(d, n-2) - 2(d-1) = f(d, n)$.

Next we show that D_n is a dominating set of $LCBD(d, n)$. Let x be an arbitrary vertex of $LCBD(d, n)$, we consider the following two cases:

Subcase 1.1. $x \in F_n$.

If $x \in D_n$, it is obvious. If $x \in (F_{n1} \cup F_{n2})$, by the above argument, we know that $x_1' = 11'1 \cdots 1'1 \in D_n$ and $x_2' = 1'11' \cdots 11' \in D_n$. If $x \in F_{n1}$, then it is dominated by vertex x_2'. If $x \in F_{n2}$, then it is dominated by vertex x_1'.

Subcase 1.2. $x \notin F_n$.

Since F_n is a dominating set of $LCBD(d, n)$, x is dominated by at least one vertex in F_n. Furthermore, no vertices of $N^-[x]$ are in the set $F_{n1} \cup F_{n2}$ because $N^+(F_{n1} \cup F_{n2}) = \bigcup_{v \in (F_{n1} \cup F_{n2})} N^+(v) \subseteq D_n$. Thus x is dominated by at least one vertex in D_n.

Case 2. n is even.

Since n is even, $n-1$ is odd, by simple calculation, we have

$$\begin{aligned}
f(d, n) &= \lceil \frac{2d^n}{d+1} \rceil \\
&= \lceil \frac{2d(d^{n-1} + 1 - 1)}{d+1} \rceil \\
&= \lceil \frac{2d(d+1)(d^{n-2} - d^{n-3} + \cdots + d^2 - d + 1) - 2d}{d+1} \rceil \\
&= 2d(d^{n-2} - d^{n-3} + \cdots + d^2 - d + 1) - 1 \\
&= df(d, n-1) - 1, n \geq 2.
\end{aligned}$$

By the above argument in Case 1, $LCBD(d, n-1)$ has a dominating set D_{n-1} such that $|D_{n-1}| = f(d, n-1)$. Let

$$\begin{aligned}
H_n = \{&x_1 x_2 \cdots x_{n-1}\alpha | x_1 x_2 \cdots x_{n-1} \in D_{n-1}; \\
&\alpha \text{ and } x_{n-1} \text{ are not in the same set } (V_0 \text{ or } V_1)\}.
\end{aligned}$$

Then H_n is a dominating set of $LCBD(d, n)$ by Lemma 2.2 and $|H_n| = d|D_{n-1}| = df(d, n-1)$.

The vertex $v = 1'11' \cdots 1'1 \in H_n$ by the construction of D_{n-1} and H_n. Set $D_n = H_n \backslash H_{n1}$, where $H_{n1} = \{1'11' \cdots 1'1\}$, then $|D_n| = df(d, n-1) - 1$.

Next we show that D_n is a dominating set of $LCBD(d, n)$. Let x be an arbitrary vertex of $LCBD(d, n)$, we consider the following two cases:

Subcase 2.1. $x \in H_n$.

If $x \in D_n$, it is obvious. If $x \in H_{n1}$, by the above argument, we know that $x' = 11'1 \cdots 1' \in D_n$, $x = 1'11' \cdots 1'1$ is dominated by x'.

Subcase 2.2. $x \notin H_n$.

Since H_n is a dominating set of $LCBD(d, n)$, x is dominated by at least one vertex in H_n. Furthermore, $1'11' \cdots 1'1 \notin N^-[x]$, because its every out-neighbor is contained in D_n. Thus x is dominated by at least one vertex in D_n. □

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