The Diameter of Connected Domination Critical Graphs *

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Abstract. Let $\gamma_c(G)$ be the connected domination number of G. A graph is k-γ_c-critical if $\gamma_c(G) = k$ and $\gamma_c(G + uv) < \gamma_c(G)$ for any nonadjacent pair of vertices u and v in the graph G. In this paper, we show that the diameter of a k-γ_c-critical graph is at most k and this upper bound is sharp.

Keywords. diameter, connected domination number, connected domination critical.

1 Introduction

We only consider finite connected and undirected graphs without loops or multiple edges.

Let $G = (V(G), E(G))$. The *open neighborhood* and *closed neighborhood* of a vertex v in the graph G are denoted by $N(v) = \{u \in V(G) : vu \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$, respectively. For a vertex set $S \subseteq V(G)$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$. The graph induced by $S \subseteq V$ is denoted by $G[S]$.

*The research is supported by Chinese Natural Science Foundations (90818020).
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A set $S \subseteq V(G)$ is a *dominating set* if and only if $N[S] = V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. A graph is *k-γ-critical* if $\gamma(G) = k$ and $\gamma(G + uv) < \gamma(G)$ for any nonadjacent pair of vertices u and v in the graph G.

Favaron, Sumner and Wojcicka [3] researched the diameter of k-γ-critical graphs and showed some results as follows:

Theorem 1.1. The diameter of a k-γ-critical graph is at most $2k - 2$.

Sampathkumar and Walikar [4] defined a *connected dominating set S* to be a dominating set S whose induced subgraph $G[S]$ is connected. The minimum cardinality of a connected dominating set of G is the *connected domination number $\gamma_c(G)$*.

Chen et. al. [1, 2] defined that a graph is *k-γ_c-critical* if $\gamma_c(G) = k$ and $\gamma_c(G + uv) < \gamma_c(G)$ for any nonadjacent pair of vertices u and v in the graph G. They characterized 1-γ_c-critical and 2-γ_c-critical graphs, and showed some results as follows:

Observation 1.2. If $\gamma_c(G + uv) < \gamma_c(G)$ for a connected graph and any nonadjacent pair of vertices u and v in the graph G, then every minimum connected dominating set S of $G + uv$ contains at least one of u and v. Moreover, if without loss of generality, $u \in S$ and $v \notin S$, then u is the only neighbor of v in S.

Theorem 1.3. The diameter of a 3-γ_c-critical graph is at most 3.

In this paper, we study the connected domination critical graphs. In section 2, we show that the diameter of a k-γ_c-critical graph is at most k and this bound is sharp.

2 The diameter of a k-γ_c-critical graph

Let $d(x, y)$ denote the distance of an arbitrary pair of vertices x and y of the graph G. For a connected graph G, an edge $e \in E(G)$ is called a cut edge of G, if $G - e$ is not connected.

Theorem 2.1. The diameter of a k-γ_c-critical graph is at most k.

Proof. For an arbitrary nonadjacent pair of vertices u and v, $\gamma_c(G + uv) \leq k - 1$. Let S be a minimum connected dominating set of $G + uv$, then there is at least one vertex of $\{u, v\}$, say u, which is belong to S.

Case 1: $v \in S$. If uv is not a cut edge of $G[S]$, then S is a connected dominating set of G. Hence, $\gamma_c(G) \leq \gamma_c(G + uv)$, a contradiction. Hence uv is a cut edge of $G[S]$. Let G_1 and G_2 be two components of $G[S] - uv$, which contain u and v respectively. Let $S_1 = V(G_1)$ and $S_2 = V(G_2)$.

Case 1.1: There exists a vertex $w \in V(G) - S$ such that $w \in (N(S_1) \cap N(S_2))$. Then there exist $x_1 \in S_1$ and $x_2 \in S_2$ such that w is adjacent to both x_1 and x_2. Hence $d(u,v) \leq d(u,x_1) + d(x_1,w) + d(w,x_2) + d(x_2,v) \leq |S_1| - 1 + 1 + 1 + |S_2| - 1 = |S_1| + |S_2| = |S| \leq k - 1$.

Case 1.2: $w \notin (N(S_1) \cap N(S_2))$ for any $w \in V(G) - S$. Then since G is a connected graph, there exists an adjacent pair of vertices w_1 and w_2 such that $w_1 \in N(S_1)$, $w_2 \in N(S_2)$. Further more, there exist $x_1 \in S_1$ and $x_2 \in S_2$ such that $w_1x_1, w_2x_2 \in E(G)$. Hence $d(u,v) \leq d(u,x_1) + d(x_1,w_1) + d(w_1,w_2) + d(w_2,x_2) + d(x_2,v) \leq |S_1| - 1 + 1 + 1 + 1 + |S_2| - 1 = |S_1| + |S_2| + 1 = |S| + 1 \leq k$.

Case 2: $v \notin S$. By Observation 1.2, we have u is the only neighbor of v in S. Since G is connected, there exists a vertex $x \in V(G) - S$ which is adjacent to v. Since S is a minimum connected dominating set of $G + uv$, then there exists a vertex $y \in S$ which is adjacent to x. Hence $d(u,v) \leq d(u,y) + d(y,x) + d(x,v) = |S| - 1 + 1 + 1 = k$.

By Cases 1-2, we have $d(u,v) \leq k$ for an arbitrary nonadjacent pair of vertices u and v, hence the diameter of a k-γ_c-critical graph is at most k.□

Let $u \in V(G)$. If $G - u$ is not connected, then u is *cut-vertex* of G. By the definition of the connected dominating set, we have

Lemma 2.2. Let $S \subseteq V(G)$ be a connected dominating set of G, then S contains all cut-vertices of G.

We construct one class of graphs G_{k-2} (see Figure 1) with diameter k

as follows:

$$V(G_{k-2}) = \{a_i, b_j, c_j \ : \ 0 \le i \le k-2, 1 \le j \le 2\}$$

$$E(G_{k-2}) = \{a_i a_{i+1}, a_{k-2}b_1, a_{k-2}c_1, b_1 b_2, b_1 c_1, b_2 c_2, c_1 c_2 \ : \ 0 \le i \le k-3\}$$

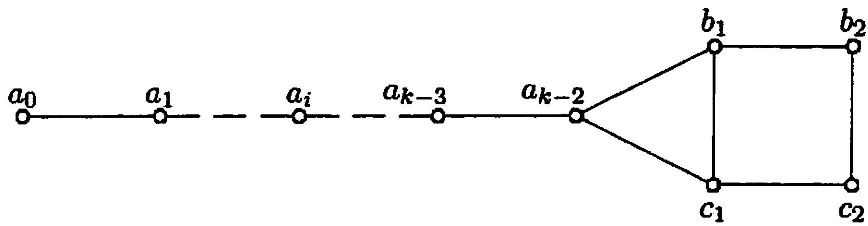

Figure 1: The graphs G_{k-2}

We prove that G_{k-2} is k-γ_c-critical graph for any $k \ge 3$.

Theorem 2.3. G_{k-2} is k-γ_c-critical graph for any $k \ge 3$.
Proof. Let $S = \{a_i, b_1, b_2 \ : \ 1 \le i \le k-2\}$, then S is a connected dominating set of G_{k-2} and $\gamma_c(G_{k-2}) \le k$.

Let S^* be an arbitrary connected dominating set of G_{k-2}. Then by Lemma 2.2, we have $a_i \in S^*$ for all $i \in \{1, 2, ..., k-2\}$. Since $\{b_2, c_2\}$ is dominated by S^*, one vertex of $\{b_1, c_1\}$, say b_1, has to belong to S^*. To dominate the vertex c_2, one vertex of $\{b_2, c_1\}$ has to belong to S^*. Hence $|S^*| \ge k-2+2 = k$, i.e. $\gamma_c(G_{k-2}) \ge k$. So $\gamma_c(G_{k-2}) = k$.

For an arbitrary pair of nonadjacent vertices u and v, we prove that there exists a connected dominating set S' of $G_{k-2}+uv$ such that $|S'| < k$. We consider five cases as follows:

Case 1: $u = a_i$ $(0 \le i \le k-3)$ and $v \in \{a_j : i+2 \le j \le k-2\} \cup \{b_1, b_2\}$, let $S' = \{a_s, b_1, b_2 \ : \ 1 \le s \le k-2, s \ne i+1\}$.

Case 2: $u = a_i$ $(0 \le i \le k-3)$ and $v \in \{c_1, c_2\}$, let $S' = \{a_s, c_1, c_2 \ : \ 1 \le s \le k-2, s \ne i+1\}$.

Case 3: $u = a_{k-2}$ and $v \in \{b_2, c_2\}$, let $S' = \{a_i, v \ : \ 1 \le i \le k-2\}$.

Case 4: $u = b_1$ and $v = c_2$, let $S' = \{a_i, b_1 \ : \ 1 \leq i \leq k-2\}$.

Case 5: $u = b_2$ and $v = c_1$, let $S' = \{a_i, c_1 \ : \ 1 \leq i \leq k-2\}$.

Hence G_{k-2} is k-γ_c-critical graph for any $k \geq 3$. □

By Theorem 2.3, we see that the upper bound of diameter in Theorem 2.1 is the best possible.

3 Open Problem

We constructed some k-γ_c-critical graphs with diameter k, and found a interesting thing: the k-γ_c-critical graphs include a subgraph H which is isomorphic to the graph G_{k-2}. So we pose the following open problem to end the paper.

Open Problem: Is there a subgraph H which is isomorphic to the graph G_{k-2} in all k-γ_c-critical graphs with diameter k?

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