A construction of a Franklin magic square of order 16

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1 Introduction

These days, sudokus are very popular, but in fact those sudokus are popularized magic squares. Of course there are a few differences. In a sudoku of order-9, the one that is best known, we just fill in the numbers from 1 to 9. Finally when our sudoku is finished the sum of every row, every column and every order-3 square will equal 45. In case of an order-n magic square, we have to fill in the numbers from 1 to n^2, just once. The sum of each row, column and both the diagonals will then be the same. About those magic squares there has lately been a real hype.
On Friday, 23 March 2007, an article appeared in several newspapers, announcing that 3 pupils from the Netherlands have found a magic square of order 12. Moreover that square was not only normal magic (def. see 2.2), it was also almost complete Franklin magic (def. see 2.3). The only missing characteristic was that each half row was not equal to 435, which is half of the magic sum of 870. But instead of this missing characteristic, they found a lot of other geometric figures with as result the magic sum.
That article [4] took my interest and I decided to try constructing a magic square of order 16 by myself. Here I must mention that squares of this order are already known. In 1544 the German mathematician M. Stifel published a normal magic square of order 16 and in 1767 Ferguson published a Franklin magic square of order 16 that was constructed by Franklin himself a few years earlier. Other examples have been found by the mathematical followers of Benjamin Franklin (see Chapters 6 and 9 of the book Benjamin Franklin's Numbers, Princeton University Press, 2007). Finally in 2001 they found another letter from Franklin, which contained a square of order 16 that was not only Franklin magic, but also normal and pandiagonal magic (def. see 2.4)! That square is considered in [2]. The square

that we will discuss is not the same. Before we explain the construction, we will first give some definitions to make everything clear.

2 Some definitions

2.1 The magic sum

The *magic sum* of an order-n magic square equals the sum of the smallest and the biggest number in the square, multiplied by half of the order. This is also the sum of all the numbers divided by n. So we get:

$$\frac{1}{n}\sum_{i=1}^{n^2} i = \frac{n^2(n^2+1)}{2n} = \frac{n(n^2+1)}{2}.$$

2.2 Normal magic squares

A *normal order-n magic square* is a square in which every row, column and both the diagonals have the same sum. It is not very difficult to construct such a square, more information can be found in [1].

2.3 Franklin magic squares

A magic square is a *Franklin magic square* if it has the following 3 characteristics. (see also Chapter 10 of [3])

1. The sum of all the numbers in every half column or half row beginning on the side of the square equals half of the magic sum. Notice that the order has to be even, otherwise we could not divide the rows or columns in two!
2. The sum of all the numbers on every *bent row* and every parallel bent row equals the magic sum. A bent row starts, as we can see in figure 1, in an entry on the side of the square. Then it goes diagonal to an axis, from where it follows its symmetric image.
3. The sum of all the numbers in every order-2 square it contains, equals the magic sum multiplied with $\frac{4}{n}$, in which n is the order. This is $\frac{n(n^2+1)}{2}\frac{4}{n}$ and consequently the sum of all the numbers in every order-2 square it contains, equals $2(n^2+1)$. For an order-8 square this sum is half the magic sum, and for an order-16 square this sum is the magic sum divided by 4.

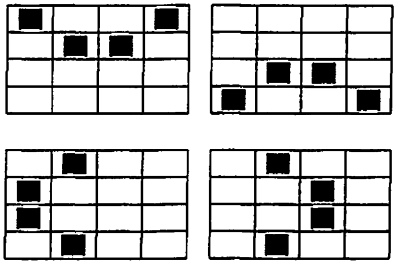

Figure 1: Some bent rows in an order-4 square.

Comment: In a Franklin magic square the diagonals do not have to be equal to the magic sum!

2.4 Pandiagonal magic squares

An order-n square is *pandiagonal magic* if the sum of all the numbers on every broken diagonal equals the magic sum. A *broken diagonal* starts, as we can see in figure 2, in an entry in the first row in a certain column, let us say in column k. From there it goes diagonal to the left or the right side of the square. In the first case the broken diagonal goes further in the entry in the last column on row $k+1$. In the other case it goes further in the entry in the first column, on row $n-k+2$.

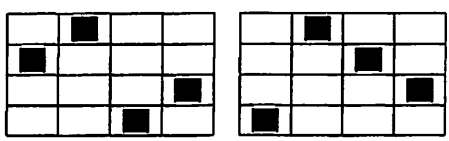

Figure 2: Examples of broken diagonals.

3 Construction

Now we have given the definitions, we can start with the real work, the construction of the order-16 magic square. To be clear, we will give every entry in the square a notation such as for matrices. We will indicate every entry with A$[i,j]$ in which i is the row, running from 1 to 16, and j the column, also running from 1 to 16. We start on the upper side of the square at the left, in entry A[1,1], where we place number 1, then we place number 16 in A[1,3]. In this way we fill in every number from 1 to 16 on the first and the last row in such a way that the sum of the entries A[1,j] and A[1,$j+2$] with $j \in \{1,5,9,13\}$ and the sum of the entries A[16,j] and A[16,$j+2$] with $j \in \{2,6,10,14\}$ equals 17. To make our construction unique, we also define 7 other entries. In A[1,7] we place number 2, in A[1,9] number 5, in A[1,15] number 6, in A[16,2] number 4, in A[16,8] number 7, in A[16,10] number 8 and finally in A[16,16] we place number 3. Hereafter we can fill in the other entries on these 2 rows when we notice the next requirement (*).

(*) The sum of the entries A[k,j] and A[$17-k,j$] with j and $k \in \{1,...,16\}$ equals 257. (In general for an order-n square this sum is $1+n^2$.)

To continue we fill in the third and the 14^{th} row. In this case the sum of the entries A[3,2k] and A[16,2k] with $k \in \{1,...,8\}$, and the sum of the entries A[1,2$k-1$] and A[14,2$k-1$] with $k \in \{1,...,8\}$ equals 33. In this way we fill in all the numbers from 17 to 32. Then we can fill in the other entries on these 2 rows if we use our requirement (*). Now we can fill in the second, the 4^{th}, the 13^{th} and the 15^{th} row. In this case, the sum of the entries A[1,2$k-1$] and A[4,2$k-1$] with $k \in \{1,...,8\}$, the sum of the entries A[13,2k] and A[16,2k] with $k \in \{1,...,8\}$, the sum of the entries A[2,2k] and A[3,2k] with $k \in \{1,...,8\}$ and the sum of the entries A[14,2$k-1$] and A[15,2$k-1$] with $k \in \{1,...,8\}$ equals 65. Now we can fill in the other entries on these 4 rows by using our requirement (*).
In this way half the square has already been filled in. Meanwhile, the attentive reader can see that there is a relation between the j-th column and the ($j+4$)-th column. Let us look for example to the first and the 5^{th} column. On the first row we see there the numbers 1 and 15. Consequently we have that the number on the 5^{th} column equals the number on the first column plus 14. On the second row we find the numbers 224 and 210, here is the number of the 5^{th} column, the number of the first column minus 14. This pattern repeats itself in the next rows and we can see now that, if we want that every order-4 square it contains, equals the magic sum, we have to assure that the number on the 5^{th} row in the 5^{th} column equals the number on the 5^{th} row in the first column plus 14. Furthermore the number on the 12^{th} row in the 5^{th} column has to be equal to the number on the same row in the first column minus 14.
To obtain this we will make a connection between the 5^{th} and the last row and a connection between the 12^{th} and the first row. We do this by making the sum of the entries A[5,2k] and A[16,2k] with $k \in \{1,...,8\}$ and the sum

of the entries A[1,2k – 1] and A[12,2k – 1] with $k \in \{1, ..., 8\}$ equal to 81. With our requirement (*), we can fill in the rest of the 5^{th} and the 12^{th} row. Now we can fill in the 7^{th} and the 10^{th} row, by using that the sum of the entries A[7,2k – 1] and A[12,2k – 1] with $k \in \{1, ..., 8\}$ and the sum of the entries A[10,2k] and A[5,2k] with $k \in \{1, ..., 8\}$ equals 161. Afterwards we can fill in the rest of the 7^{th} and the 10^{th} row with our requirement (*). Finally only the 6^{th}, the 8^{th}, the 9^{th} and the 11^{th} row remain. We can fill in these rows by making the sum of the entries A[5,2k] and A[8,2k] with $k \in \{1, ..., 8\}$, the sum of the entries A[6,2k – 1] and A[7,2k – 1] with $k \in \{1, ..., 8\}$, the sum of the entries A[9,2k – 1] and A[12,2k – 1] with $k \in \{1, ..., 8\}$ and the sum of the entries A[10,2k] and A[11,2k] with $k \in \{1, ..., 8\}$ equal to 193.
So at last, we will have filled in all the numbers from 1 to 256 in our square.

1	253	16	244	15	247	2	250	5	249	12	248	11	243	6	254
224	36	209	45	210	42	223	39	220	40	213	41	214	46	219	35
225	29	240	20	239	23	226	26	229	25	236	24	235	19	230	30
64	196	49	205	50	202	63	199	60	200	53	201	54	206	59	195
177	77	192	68	191	71	178	74	181	73	188	72	187	67	182	78
112	148	97	157	98	154	111	151	108	152	101	153	102	158	107	147
81	173	96	164	95	167	82	170	85	169	92	168	91	163	86	174
144	116	129	125	130	122	143	119	140	120	133	121	134	126	139	115
113	141	128	132	127	135	114	138	117	137	124	136	123	131	118	142
176	84	161	93	162	90	175	87	172	88	165	89	166	94	171	83
145	109	160	100	159	103	146	106	149	105	156	104	155	99	150	110
80	180	65	189	66	186	79	183	76	184	69	185	70	190	75	179
193	61	208	52	207	55	194	58	197	57	204	56	203	51	198	62
32	228	17	237	18	234	31	231	28	232	21	233	22	238	27	227
33	221	48	212	47	215	34	218	37	217	44	216	43	211	38	222
256	4	241	13	242	10	255	7	252	8	245	9	246	14	251	3

Figure 3: The order-16 magic square.

4 Characteristics

If we check this order-16 square, we will see that it is normal magic, Franklin magic and pandiagonal magic. Because of these 3 characteristics we can find a lot of geometric figures with as result the magic sum. Examples are: every circle of 16 entries (see figure 3), every cross made of two diagonals with each 8 entries (see figure 3), every hourglass (see figure 3)...
To be clear we will give below a list of all the characteristics that can

be found in the order-16 magic square, that we have constructed in section 3. Notice that some characteristics are a consequence of some other characteristics and that many geometric figures are almost trivial equal to 2056, because we can make them by removing some entries, which have the magic sum or a part of the magic sum as result, out of a big square with a well chosen order. For example, we can deduce characteristic (14) from characteristic (13).

4.1 Enumeration of the characteristics

1. The sum of all the numbers in every row equals 2056.
2. The sum of all the numbers in every column equals 2056.
3. The sum of all the numbers in both diagonals equals 2056.
4. The sum of all the numbers in every broken diagonal equals 2056.
5. The sum of all the numbers in every bent row (there are 64 bent rows) equals 2056.
6. The sum of all the numbers in every order-4 square equals 2056.
7. The sum of all the numbers in every circle of 16 entries (see figure 3) equals 2056.
8. The sum of all the numbers in every cross made of two diagonals with each 8 entries (see figure 3) equals 2056.
9. The sum of all the numbers in every hourglass (upright or laid down: see figure 3) equals 2056.
10. The sum of all the numbers in every half column or half row beginning on the side of the square or from the horizontal or the vertical axis equals 1028. (2056/2)
11. The sum of all the numbers in every fourth of a column or row beginning from column or row 1 (mod 4) equals 514. (2056/4)
12. If we cut both diagonals in two, we receive 4 pieces of 8 entries with as result 1028.
13. The sum of all the numbers in every cross made of two diagonals with each 4 entries (analogous as in (8)) equals 1028.

14. The sum of all the numbers in every circle of 8 entries (that we can create by removing the cross as mentioned in (13) in an order-4 square) equals 1028.

15. The sum of all the numbers in every order-2 square equals 514. (a fourth of the magic sum)

16. Finally there is almost a symmetric partition of the prime numbers: there are above and under the horizontal axis 27 prime numbers; and left and right from the vertical axis respectively 26 and 28 prime numbers

5 Observations

As you can see in characteristic (16) we suddenly mention prime numbers. But we have a reason for this. The partition in characteristic (16) can also be found in the order-12 square that has been constructed by the 3 Dutch pupils. In that square we find 17 prime numbers left and right from the vertical axis and above and under the horizontal axis we find respectively 18 and 16 prime numbers. This is consequently analogous as in (16) because you can rotate the order-16 square in figure 3 over 90^o, 180^o or 270^o without losing any characteristic.
Columns become rows and vice versa, diagonals stay diagonals, rotated broken and bent diagonals become other broken and bent diagonals, and because they all have the magic sum 2056, characteristics (4) and (5) stay valid. In this way we can go further: rotated circles, squares and crosses become other circles, squares and crosses... and so we will see that, indeed, every characteristic is conserved.
In general we can say this. If we have p prime numbers in a magic square of order 12 or order 16, we can find for both orders a magic square where there are $\frac{p}{2}$ prime numbers at both sides of one axis, and $\frac{p}{2}-1$ and $\frac{p}{2}+1$ prime numbers at the two sides of the other axis. Notice that the number of prime numbers in an order-n magic square, so all the prime numbers smaller than n^2, has to be even because we have to divide that number in two.
We can now ask ourselves if we can find this partition in squares with a different order. The answer is yes! The order-8 square below is normal magic, Franklin magic and pandiagonal magic and it has the prime number partition that we have discussed above. This square can also be find on website [5].

1	46	51	32	35	62	17	16
60	23	10	37	26	7	44	53
14	33	64	19	48	49	30	3
55	28	5	42	21	12	39	58
9	38	59	24	43	54	25	8
63	20	13	34	29	4	47	50
6	41	56	27	40	57	22	11
52	31	2	45	18	15	36	61

We can now ask ourselves if we can always find an order-$8k$ square, with k a natural number different from zero, where this partition is valid if p is even and if the square is normal magic, Franklin magic and pandiagonal magic. Few people will be surprised when I claim that it will be difficult to find an answer to this question. Especially if k becomes bigger. In the future someone may find an answer, but no one has yet found it. And what about for squares of order $8k + 4$ with k a natural number? Can we find for these squares a magic square with our prime number partition? It is easy to see that we will have the same problem. It will again be difficult to check it, if k becomes bigger. In addition we will have to ask ourselves how magic those squares of order $8k + 4$ must be. Up to now an order-12 square that is complete Franklin magic has not been found. The most interesting question that we can ask ourselves now is, whether we can find such an order-12 square, in which the sum of the numbers of every half row equals the half magic sum. It is probable that nowadays many people are searching for an order-12 Franklin magic square, and so it could be only a question of time until someone find such a square. But who can prove that this square really exists?
More information about magic squares can be found in [3]. And if you want to discuss about the discoveries around magic squares you can always go to website [6].

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[5] http://www.region.com.ar/amela/franklinsquares/

[6] www.wiskundemeisjes.nl/20070322/magisch-vierkant/

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