The Graham's pebbling conjecture for $K_{r,s}^{-k_1} \times K_{m,n}^{-k_2}$*

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Abstract: The pebbling number $f(G)$ of a graph G is the smallest number n such that, however n pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. Graham conjectured that for any connected graphs G and H, $f(G\times H) \le f(G)f(H)$, where $G\times H$ represents the Cartesian product of G and H. In this paper, we prove that $f(G\times H) \le f(G)f(H)$ when G has the two-pebbling property and $H = K_{r,s}^{-k}$, a graph obtained from the $r\times s$ complete bipartite graph $K_{r,s}$ by deleting k edges which form a matching. We also show that Graham's conjecture holds for $K_{r,s}^{-k_1} \times K_{m,n}^{-k_2}$.
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1. Introduction

Pebbling in graph was first considered by Chung [1]. Consider a graph G with a fixed number of pebbles placed on its vertices (i.e., a pebbling of the graph G). A pebbling move consists of the removal of two pebbles from a vertex, and the placement of one of those pebbles on an adjacent vertex. The pebbling number of a vertex v in a graph G is the smallest number $f(G,v)$ such that it is possible to move a pebble to v by a sequence of pebbling moves for every placement of $f(G,v)$ pebbles on G. The pebbling

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number of a graph G, denoted by $f(G)$, is the maximum of $f(G,v)$ over all vertices v. It is clear that $f(G) \geq |V(G)|$ and $f(G') \geq f(G)$, where G' is a spanning subgraph of G.

We say a graph G satisfies the two-pebbling property (respectively, the odd-two-pebbling property) if two pebbles can be moved to any specified target vertex when the total starting number of pebbles is $2f(G) - q + 1$ (respectively, $2f(G) - r + 1$), where q is the number of vertices with at least one pebble (respectively, r is the number of vertices with an odd number of pebbles). Evidently, graphs which are two-pebbling are also odd-two-pebbling. Given a pebbling of G, a transmitting subgraph of G is a path $x_0 x_1 \cdots x_k$ such that there are at least two pebbles on x_0 and at least one pebble on each of the other vertices in this path, except possibly x_k. In this case, we can transmit a pebble from x_0 to x_k.

Graham's Conjecture: [1] For any connected graphs G and H,

$$f(G \times H) \leq f(G)f(H).$$

There are a few results supporting Graham's conjecture. Chung [1] proved this conjecture when G is a complete graph K_m on m vertices and H has the two-pebbling property. Moews [6] confirmed this conjecture for trees. Snevily and Foster [7] further proved it when G is a tree or an even cycle of at least 10 vertices and H has the two-pebbling property. Herscovici and Higgins [5] proved it when $G = H = C_5$. Herscovici [4] also proved it when G is a cycle for a variety of graphs H, including all cycles. Wang [8] verified the case where G is a complete multi-partite graph and H has the two-pebbling property. Recently, Feng and Ju [2,3] proved Graham's conjecture when G and H are fan graphs or wheel graphs, and also showed it when G is a complete bipartite graph and H has the two-pebbling property. In this paper, we prove that $f(G \times H) \leq f(G)f(H)$ when G has the two-pebbling property and $H = K_{r,s}^{-k}$, where $s \geq r \geq 4$ and $k \leq r$. We also show that Graham's conjecture holds for $K_{r,s}^{-k_1} \times K_{m,n}^{-k_2}$, where $s \geq r \geq 4$, $n \geq m \geq 4$, $k_1 \leq r$ and $k_2 \leq m$. Our proof technique closely follows that of [3].

Throughout this paper G will denote a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex v of G, $p(v)$ will refer to the number of pebbles on v.

2. The pebbling number and the two-pebbling property for $K_{r,s}^{-k}$

In this paper, without loss of generality, we always assume that $\{u_1, \ldots, u_r\}$ and $\{v_1, \ldots, v_s\}$ is the bipartite partition of the vertex set of $K_{r,s}$, and $K_{r,s}^{-k}$

is the graph obtained from $K_{r,s}$ by deleting edges $u_1v_1, u_2v_2, \dots, u_kv_k$.

Theorem 2.1. Let $s \geq r \geq 4$. Then $f(K_{r,s}^{-r}) = r + s$.

Proof. Let $r+s$ pebbles be placed on $K_{r,s}^{-r}$ and $p(x)$ denote the number of pebbles on the vertex x. We first consider the target vertex u_i, where $1 \leq i \leq r$. Without loss of generality, we assume that the target vertex is u_1 and $p(u_1) = 0$. If there is a vertex in $\{v_2, v_3, \dots, v_s\}$ with at least two pebbles or $p(v_1) \geq 2^3$, then we can put one pebble on u_1. If there is a vertex in $\{u_2, u_3, \dots, u_r\}$ with at least four pebbles or there are at least two vertices in $\{u_2, u_3, \dots, u_r\}$ with at least two pebbles each, then we can put one pebble on u_1. Assume that $p(v_i) \leq 1$ for $i = 2, 3, \dots, s$ and $p(v_1) \leq 2^3 - 1$. We consider the following cases.

Case 1. $6 \leq p(v_1) \leq 7$. There exists a vertex $x \in \{u_2, \dots, u_r, v_2, \dots, v_s\}$ such that $p(x) \geq 1$. If $p(u_i) \geq 1$ $(i \neq 1)$, then we can move three pebbles from v_1 to u_i, and hence a pebble can be moved to u_1 from u_i. If $p(v_j) \geq 1$ $(j \neq 1)$, then we can move two pebbles from v_1 to u_k, where $2 \leq k \leq r$ and $k \neq j$, and hence $\{u_k, v_j, u_1\}$ forms a transmitting subgraph.

Case 2. $4 \leq p(v_1) \leq 5$. If $p(v_{j_0}) = 1$ for some j_0, then two pebbles can be moved to u_{i_0} from v_1, where $i_0 \neq j_0$. Therefore, $\{u_{i_0}, v_{j_0}, u_1\}$ forms a transmitting subgraph. Suppose $p(v_i) = 0$ for $i = 2, 3, \dots, s$. There must be at least $(r + s - 5)$ pebbles on $\{u_2, u_3, \dots, u_r\}$. If $p(u_{i_0}) \geq 2$ for some i_0, then we can move two pebbles from v_1 to u_{i_0}. By using four pebbles on u_{i_0}, we can put a pebble on u_1 from u_{i_0}. Assume $p(u_i) \leq 1$ for $i = 2, 3, \dots, r$. Then there are at least two vertices in $\{u_2, u_3, \dots, u_r\}$ with one pebble each. Let $p(u_{i_0}) = 1$ and $p(u_{j_0}) = 1$. We can move a pebble from v_1 to u_{i_0} and a pebble from v_1 to u_{j_0}. Therefore, one pebble can be moved to u_1.

Case 3. $2 \leq p(v_1) \leq 3$. If $p(u_{i_0}) \geq 3$ for some i_0, then we can move a pebble from v_1 to u_{i_0}. The total number of pebbles on u_{i_0} is at least four, and hence one pebble can be moved to u_1 from u_{i_0}. Suppose $p(u_i) \leq 2$ for $2 \leq i \leq r$. If there are two vertices u_{i_0} and u_{j_0} with $p(u_{i_0}) = p(u_{j_0}) = 2$, then one pebble can be moved to u_1. If there are two vertices u_{i_0} and u_{j_0} with $p(u_{i_0}) = 2$ and $p(u_{j_0}) = 1$. We can move one pebble from v_1 to u_{j_0}. Hence we can put one pebble on u_1. Assume that $p(u_i) \leq 1$ for $i = 2, 3, \dots, r$. Since $p(v_i) \leq 1$ for $i = 2, \dots, s$, we can choice two vertices u_i, v_j such that $p(u_i) = p(v_j) = 1$ and $i \neq j$. Therefore, $\{v_1, u_i, v_j, u_1\}$ forms a transmitting subgraph.

Case 4. $p(v_1) \leq 1$. It is impossible that $p(u_i) \leq 1$ for $i = 2, 3, \dots, r$. We may assume that there is only one vertex $u_{i_0} \in \{u_2, u_3, \dots, u_r\}$ with $2 \leq p(u_{i_0}) \leq 3$. We can choice one vertex v_j such that $j \neq 1, i_0$ and $p(v_j) = 1$. Therefore, $\{u_{i_0}, v_j, u_1\}$ forms a transmitting subgraph.

Similarly, we can prove that one pebble can be moved to v_i for every placement of $r + s$ pebbles on $K_{r,s}^{-r}$ for the case of the target vertex v_i, where $1 \leq i \leq s$. So $f(K_{r,s}^{-r}) = r + s$. □

Since $K_{r,s}^{-r}$ is a spanning subgraph of $K_{r,s}^{-k}$, $K_{r,s}^{-k}$ is a spanning subgraph of $K_{r,s}$ and $f(K_{r,s}) = r + s$ (see [3]), we have the following

Corollary 2.1. Let $s \geq r \geq 4$ and $k \leq r$. Then $f(K_{r,s}^{-k}) = r + s$.

Theorem 2.2. Let $s \geq r \geq 4$. Then $K_{r,s}^{-r}$ satisfies the two-pebbling property.

Proof. Let p be the number of pebbles on $K_{r,s}^{-r}$, q be the number of vertices with at least one pebble and $p + q = 2(r + s) + 1$. Firstly, we consider the target vertex u_i, where $1 \leq i \leq r$. Without loss of generality, we assume that the target vertex is u_1. If $p(u_1) = 1$, then the number of pebbles on all the vertices except u_1 is $2(r + s) + 1 - q - 1 \geq r + s$. Since $f(K_{r,s}^{-r}) = r + s$, we can put one additional pebble on u_1 by using $2(r + s) + 1 - q - 1$ pebbles. If $p(u_1) = 0$, then we consider the following two cases.

(1) Suppose that $p(v_i) = 0$ for some v_i, where $2 \leq i \leq s$. It is obvious that $q \leq r + s - 2$. If there is a vertex in $\{v_2, v_3, \dots, v_s\}$ with at least two pebbles or there exists a transmitting subgraph $\{u_j, v_k, u_1\}$, then by using no more than three pebbles, we can put one pebble on u_1. Clearly, the remaining $2(r + s) + 1 - q - 3$ pebbles will be sufficient to put one more pebble on u_1. Otherwise, we consider the following subcases.

(1.1) Suppose that there are at least two vertices in $\{v_2, v_3, \dots, v_s\}$ with one pebble each. Let $p(v_{i_0}) = p(v_{j_0}) = 1$, where $i_0, j_0 \geq 2$. We may assume that $p(u_k) \leq 1$ for $k \in \{2, 3, \dots, r\}$. If there is no more than one vertex in $\{u_2, u_3, \dots, u_r\}$ with one pebble, then $q \leq s + 1$, $2(r+s)+1-q \geq 2r + s$ pebbles on $K_{r,s}^{-r}$ and $p(v_1) \geq 2r \geq 8$. Then it is possible to move 4 pebbles total from v_1 to any two vertices u' and u'' adjacent to v_1 and form transmitting graphs $\{u', v_{i_0}, u_1\}$ and $\{u'', v_{j_0}, u_1\}$. If there are at least two vertices in $\{u_2, u_3, \dots, u_r\}$ with one pebble each. Let $p(u_{k_0}) = p(u_{\ell_0}) = 1$. Since $q \leq r+s-2$, $2(r+s)+1-q \geq r+s+3$ pebbles on $K_{r,s}^{-r}$ and $p(v_1) \geq 4$, we can move a pebble to u_{k_0} and a pebble to u_{ℓ_0}. Therefore, $\{u_{k_0}, v_{i_0}, u_1\}$ and $\{u_{\ell_0}, v_{j_0}, u_1\}$ (or $\{u_{\ell_0}, v_{i_0}, u_1\}$ and $\{u_{k_0}, v_{j_0}, u_1\}$) form two transmitting subgraphs.

(1.2) Suppose that there is only one vertex in $\{v_2, v_3, \dots, v_s\}$ with one pebble. Clearly, $q \leq r + 1$. Let $p(v_{i_0}) = 1$, where $i_0 \geq 2$. We may assume that $p(u_k) \leq 1$ for $u_k \in N(v_{i_0})$. If there is an integer $j_0 \in \{2, 3, \dots, r\}$ such that $p(u_{j_0}) = 1$ and $j_0 \neq i_0$, then $\{v_1, u_{j_0}, v_{i_0}, u_1\}$ forms a transmitting subgraph. By using no more than four pebbles, we can move one pebble from the transmitting subgraph to u_1. Clearly, the remaining $2(r + s) + 1 - q - 4 \geq 2(r + s) + 1 - (r + 1) - 4$ pebbles will be sufficient to put one more pebble on u_1. Let $p(u_i) = 0$ for each $i \neq i_0$. Then $q \leq 3$. So $2(r + s) + 1 - q \geq 2(r + s) - 2$ pebbles on $K_{r,s}^{-r}$. There are at least 13 pebbles on $\{v_1, u_{i_0}\}$. If $p(u_{i_0}) \geq 3$, then $\left\lceil \frac{p(v_1)}{2} \right\rceil$ pebbles can be moved to

u_{i_0} from v_1. The total number of pebbles on u_{i_0} is $p(u_{i_0})+\left\lceil\frac{13-p(u_{i_0})}{2}\right\rceil \geq 8$. So we can move 2 pebbles to u_1. If $1 \leq p(u_{i_0}) \leq 2$, then $p(v_1) \geq 11$. So we can move 3 pebbles to u_{i_0} and move 2 pebbles to u_{j_0}, where $j_0 \neq i_0$. Thus $\{u_{j_0}, v_{i_0}, u_1\}$ forms a transmitting subgraph and there are at least four pebbles on u_{i_0}. Therefore, two pebbles can be moved to u_1. If $p(u_{i_0}) = 0$, then $p(v_1) \geq 13$. We can move 6 pebbles to u_{j_0}, where $j_0 \neq i_0$. Thus, 3 pebbles can be put on v_{i_0}. The total number of pebbles on v_{i_0} is four. Therefore, two pebbles can be moved to u_1 from v_{i_0}.

(1.3) Suppose that $p(v_i) = 0$ for $2 \leq i \leq s$. It is clear that $q \leq r$ and $p \geq 2s + r + 1$. If there is one vertex in $\{u_2, u_3, \ldots, u_r\}$ with at least four pebbles or there are at least two vertices in $\{u_2, u_3, \ldots, u_r\}$ with at least two pebbles each, then we can always move one pebble to u_1 by using four pebbles. So there are still $2s + r + 1 - 4 \geq (r + s) + (s - 4) + 1$ pebbles on $\{u_2, u_3, \ldots, u_r, v_1\}$. Hence we can put one more pebble on u_1. If there is only one vertex in $\{u_2, u_3, \ldots, u_r\}$ with three pebbles (and the other vertices with at most one pebble), then we can move one pebble from v_1 to this vertex . So by using four pebbles on this vertex, one pebble can be moved to u_1. There are still $2s + r + 1 - 5 \geq (r + s) + (s - 4)$ pebbles on $\{u_2, u_3, \ldots, u_r, v_1\}$, and we can put one more pebble on u_1. We now assume that $p(u_2) + p(u_3) + \ldots + p(u_r) \leq 2 + (r - 2) = r$ and $p(u_i) \leq 2$ for $2 \leq i \leq r$. Then $p(v_1) \geq 2s + r + 1 - r = 2s + 1$.

If $q \geq 5$, then there exist four vertices u_{i_0},u_{i_1},u_{i_2} and u_{i_3} with $p(u_{i_k}) \geq 1$ for $0 \leq k \leq 3$. So we can put a pebble on each of the four vertices from v_1. Thus each has two pebbles. Therefore two pebbles can be moved to u_1 from these vertices.

If $q = 4$, then there exist three vertices u_{j_1}, u_{j_2} and u_{j_3} with $p(u_{j_1})$, $p(u_{j_2})$, $p(u_{j_3}) \geq 1$. Obviously, $p = 2(r + s) + 1 - 4$. If $p(u_{j_1}) = p(u_{j_2}) = p(u_{j_3}) = 1$, then $p(v_1) \geq 10$. So we can move one pebble to each of u_{j_1} and u_{j_2}, and move three pebbles to u_{j_3}. We can move two pebbles from those vertices to u_1 . If $p(u_{j_1}) + p(u_{j_2}) + p(u_{j_3}) = 4$, then $p(v_1) \geq 9$. So two pebbles can be moved to the vertex which has two pebbles. In addition, we can move a pebble to each of the other two vertices. Therefore two pebbles can be moved to u_1 from these vertices.

If $q = 3$, then we can move two pebbles to u_1 similar to the case $q = 4$.

If $q = 2$, then there exists only one vertex u_j with $2 \geq p(u_j) \geq 1$. Obviously, we can move $\left\lceil\frac{p(v_1)}{2}\right\rceil = \left\lceil\frac{2(r+s)+1-2-p(u_j)}{2}\right\rceil$ pebbles to u_j. Since $p(u_j) + \left\lceil\frac{2(r+s)+1-2-p(u_j)}{2}\right\rceil \geq r + s \geq 8$, two pebbles can be moved to u_1 from u_j.

If $q = 1$, then $p(v_1) \geq 2(r + s) \geq 16$. We can move two pebbles from v_1 to u_1 .

(2) Suppose that $p(v_i) = 1$ for every v_i, where $2 \leq i \leq s$. If there is a

vertex in $\{u_2, u_3, \ldots, u_r\}$ with at least four pebbles or there are at least two vertices in $\{u_2, u_3, \ldots, u_r\}$ with at least two pebbles each, then it is obvious that two pebbles can be moved to u_1. If there are two vertices u_{i_0} and u_{j_0} with $3 \geq p(u_{i_0}) \geq 2$ and $p(u_{j_0}) = 1$, then $p = 2(r+s)+1-q \geq r+s+2$ and $p(v_1) \geq r+s+2-(s-1)-(3+r-2) = 2$. So we can move one pebble from v_1 to u_{j_0}. Thus $\{u_{i_0}, v_{j_0}, u_1\}$ and $\{u_{j_0}, v_{i_0}, u_1\}$ form two transmitting graphs. If $p(u_i) \leq 1$ for $2 \leq i \leq r$ and there are two vertices u_{i_0} and u_{j_0} with $p(u_{i_0}) = p(u_{j_0}) = 1$, then $p(v_1) \geq 4$, and we can move one pebble to u_{i_0} by using two pebbles on v_1. So $\{v_1, u_{j_0}, v_{i_0}, u_1\}$ and $\{u_{i_0}, v_{j_0}, u_1\}$ form two transmitting graphs. If there is only one vertex $u_{i_0} \in \{u_2, u_3, \ldots, u_r\}$ with $0 \leq p(u_{i_0}) \leq 3$ (and the other vertices having no pebbles). It is clear that $q \leq s+1$, $p \geq 2(r+s)+1-q = 2r+s$ and $p(v_1) \geq 2r+s-(s-1)-p(u_{i_0})$. So $\left\lceil \frac{2r+s-(s-1)-p(u_{i_0})}{2} \right\rceil$ pebbles can be put on u_{i_0}, and the total number of pebbles on u_{i_0} is $p(u_{i_0}) + \left\lceil \frac{2r+s-(s-1)-p(u_{i_0})}{2} \right\rceil \geq 4$. Therefore, two pebbles can be moved to u_1.

This proves that if the target vertex is u_1, then we can move two pebbles to u_1 for every placement of $2(r+s)+1-q$ pebbles on $K_{r,s}^{-r}$. If the target vertex is v_i for $1 \leq i \leq s$, then we can use a similar way to draw the conclusion. □

Since $K_{r,s}^{-r}$ is a spanning subgraph of $K_{r,s}^{-k}$ and $f(K_{r,s}^{-k}) = f(K_{r,s}^{-r})$, we have the following

Corollary 2.2. Let $s \geq r \geq 4$ and $k \leq r$. Then $K_{r,s}^{-k}$ also satisfies the two-pebbling property.

3. Main Results

In order to prove the main results, we need the following preliminaries. Let G and H be two graphs. The Cartesian products of G and H, denote by $G \times H$, is the graph whose vertex set is Cartesian product

$$V(G \times H) = V(G) \times V(H) = \{(x,y) | x \in V(G), y \in V(H)\},$$

and two vertices (x,y) and (x',y') are adjacent if and only if $x = x'$ and $yy' \in E(H)$, or $xx' \in E(G)$ and $y = y'$. We can depict $G \times H$ by drawing a copy of H at every vertex of G and connecting each vertex in one copy of H to the corresponding vertex in an adjacent copy of H. We write $\{x\} \times H$ (respectively, $G \times \{y\}$) for the subgraph of vertices whose projection onto $V(G)$ is the vertex x (respectively, whose projection onto $V(H)$ is y). If the vertices of G are labelled x_i, then for any distribution of pebbles on $G \times H$, we write p_i for the number of pebbles on $\{x_i\} \times H$, q_i for the number of occupied vertices of $\{x_i\} \times H$, r_i for the number of vertices of $\{x_i\} \times H$ with an odd number of pebbles.

Theorem 3.1. [1,3] Let G satisfy the two-pebbling property. Then

$$f(K_m \times G) \le f(K_m)f(G) = mf(G),$$
$$f(K_{m,n} \times G) \le f(K_{m,n})f(G) = (m+n)f(G).$$

The following Lemma 3.1 describes how many pebbles we can transfer from one copy of H to an adjacent copy of H in $G \times H$. The proof is straightforward.

Lemma 3.1. [5] Let $x_ix_j \in E(G)$. Suppose that in $G \times H$, we have p_i pebbles occupying q_i vertices of $\{x_i\} \times H$. If $q_i - 1 \le k \le p_i$, and if k and p_i have the same parity, then k pebbles can be retained on $\{x_i\} \times H$, while moving $(p_i - k)/2$ pebbles onto $\{x_j\} \times H$. If k and p_i have opposite parity, we must leave $k+1$ pebbles on $\{x_i\} \times H$, so we can only move $(p_i - k - 1)/2$ pebbles onto $\{x_j\} \times H$. In particular, we can always move at least $(p_i - q_i)/2$ pebbles onto $\{x_j\} \times H$.

Lemma 3.2. [3] Suppose that G satisfies the odd-two-pebbling property and p pebbles are placed on $K_{1,n} \times G$ in such a way that there are r vertices with odd number of pebbles. Let v_0 be the vertex of $K_{1,n}$ with degree n. If $p+r > 2(n+1)f(G)$, then two pebbles can be moved to (v_0, y) by a sequence of pebbling moves.

Lemma 3.3. [3] Suppose that G satisfies the odd-two-pebbling property, p pebbles are assigned to vertices of $K_2 \times G$ and r is the number of vertices with an odd number of pebbles. If $p+r > 4f(G)$, then two pebbles can be moved to any specified vertex of $K_2 \times G$ by a sequence of pebbling moves.

Lemma 3.4. Suppose that G satisfies the odd-two-pebbling property, and p pebbles are assigned to vertices of $K_{2,2} \times G$ and r is the number of vertices with an odd number of pebbles. If $p+r > 8f(G)$, then two pebbles can be moved to any specified vertex of $K_{2,2} \times G$ by a sequence of pebbling moves.

Proof. Without loss of generality, we assume that the target vertex is (u_1, y) for some $y \in G$. $K_{2,2} \times G$ can be partitioned into two subgraphs, say, M_1 and M_2, as follows. M_1 is $A \times G$ and M_2 is $B \times G$, where A is the induced subgraph of $\{u_1, v_2\}$ in $K_{2,2}$ and B is the induced subgraph of $\{u_2, v_1\}$ in $K_{2,2}$. Clearly, $A = B = K_2$. For $i = 1$ and 2, suppose that M_i contains p_i pebbles with r_i vertices having an odd number of pebbles. It is evident that $p = p_1 + p_2$, $r = r_1 + r_2$ and $p_i + r_i$ is even for $i = 1$ and 2. If $p_1 + r_1 > 4f(G)$, then two pebbles can be moved to (u_1, y) by Lemma 3.3. Now we assume that $p_1 + r_1 \le 4f(G)$ and $p_2 + r_2 > 4f(G)$. Let q_2 be the number of occupied vertices of M_2. Clearly, $r_2 \le q_2 \le |V(K_2 \times G)| \le f(K_2 \times G) \le 2f(G)$ (see Theorem 3.1). Hence, $p_2 \ge 4f(G) - r_2 + 2 \ge q_2 + 2$, and p_2 and $4f(G) - r_2 + 2$ have the same parity. Similar to Lemma 3.1, we can move $\frac{(p_2+r_2)-(4f(G)+2)}{2}$ pebbles to M_1 while keeping $4f(G)-r_2+2$ pebbles on M_2. Since $p_2+r_2>8f(G)-p_1-r_1$, we have $p_1+\frac{(p_2+r_2)-(4f(G)+2)}{2}>p_1+\frac{8f(G)-p_1-r_1-4f(G)-2}{2}=2f(G)+\frac{p_1-r_1}{2}-1$, in other words, we have $p_1+\frac{(p_2+r_2)-(4f(G)+2)}{2}\geq 2f(G)$. By Theorem 3.1, we can move one pebble from $2f(G)$ pebbles on M_1 to (u_1,y). Additionally, there are still $4f(G)-r_2+2$ pebbles on M_2. By Lemma 3.3, two pebbles can be moved to (v_1,y) on M_2. Since (u_1,y) and (v_1,y) are adjacent, one pebble can be moved to (u_1,y) from two pebbles on (v_1,y). □

Lemma 3.5. Let u be a vertex of $K_{3,3}^{-1}$ with degree 3, $y\in G$ and G satisfy the two-pebbling property. Then $f(K_{3,3}^{-1}\times G,(u,y))\leq 6f(G)$.

Proof. Without loss of generality, we assume that the target vertex is (u_3,y) for some $y\in G$ and $6f(G)$ pebbles are placed on $K_{3,3}^{-1}\times G$. $K_{3,3}^{-1}\times G$ can be partitioned into two subgraphs, say, M_1 and M_2, as follows. M_1 is $A\times G$ and M_2 is $B\times G$, where A is the induced subgraph of $\{u_2,u_3,v_1,v_3\}$ in $K_{3,3}^{-1}$ and B is the induced subgraph of $\{u_1,v_2\}$ in $K_{3,3}^{-1}$. Clearly, $A=K_{2,2}$ and $B=K_2$. For $i=1$ and 2, suppose that M_i contains p_i pebbles with r_i vertices having an odd number of pebbles. If $p_1\geq 4f(G)$, then one pebble can be moved to (u_3,y) by Theorem 3.1. Now we assume that $p_1<4f(G)$. Let $p_1=4f(G)-t$ and $p_2=2f(G)+t$ for some positive integer t. If $t\leq 2f(G)-r_2$, then we can place at least $(p_2-r_2)/2$ pebbles on vertices of M_1 from M_2 by a sequence of pebbling moves, and hence, in M_1, we have altogether

$$\begin{aligned}p_1+\frac{p_2-r_2}{2}&=4f(G)-t+\frac{6f(G)-(4f(G)-t)-r_2}{2}\\&=\frac{6f(G)+(4f(G)-t)-r_2}{2}\\&\geq 4f(G)\end{aligned}$$

pebbles. By Theorem 3.1, we can move one pebble to (u_3,y). If $t>2f(G)-r_2$, then $p_2+r_2=2f(G)+t+r_2>4f(G)$. Two pebbles can be moved to (v_2,y) by Lemma 3.3, and hence we can move one pebble to (u_3,y). □

Theorem 3.2. Let $s\geq 4$ and G satisfy the two-pebbling property. Then

$$f(K_{4,s}^{-4}\times G)\leq f(K_{4,s}^{-4})f(G)=(4+s)f(G).$$

Proof. Let $(4+s)f(G)$ pebbles be placed on $K_{4,s}^{-4}\times G$. Firstly, let the target vertex be (u_i,y) for some $y\in G$, where $1\leq i\leq 4$. Without loss of generality, we assume that the target vertex is (u_1,y) for some $y\in G$. $K_{4,s}^{-4}\times G$ can be partitioned into two subgraphs, say, M_1 and M_2, as follows. M_1 is $A\times G$ and M_2 is $B\times G$, where A is the induced subgraph of $\{u_1,u_2,v_3,v_4,\ldots,v_s\}$ in $K_{4,s}^{-4}$ and B is the induced subgraph of $\{u_3,u_4,v_1,v_2\}$ in $K_{4,s}^{-4}$. Clearly, $A=K_{2,s-2}$ and $B=K_{2,2}$. For $i=1$

and 2, suppose that M_i contains p_i pebbles with r_i vertices having an odd number of pebbles. If $p_1 \geq sf(G)$, then one pebble can be moved to (u_1, y) by Theorem 3.1. Now we assume that $p_1 < sf(G)$. Let $p_1 = sf(G) - t$ and $p_2 = 4f(G) + t$ for some positive integer t. If $t \leq 4f(G) - r_2$, then we can place at least $(p_2 - r_2)/2$ pebbles on vertices of M_1 from M_2 by a sequence of pebbling moves, and hence, in M_1, we have altogether

$$p_1 + \frac{p_2 - r_2}{2} = sf(G) - t + \frac{(4+s)f(G) - (sf(G) - t) - r_2}{2} \geq sf(G)$$

pebbles. By Theorem 3.1, we can move one pebble to (u_1, y). If $t > 4f(G) - r_2$, then $p_2 + r_2 = 4f(G) + t + r_2 > 8f(G)$. Two pebbles can be moved to (v_2, y) by Lemma 3.4, and hence one pebble can be moved to (u_1, y).

Next, let the target vertex be (v_i, y) for some $y \in G$, where $1 \leq i \leq 4$. Without loss of generally, we assume that the target vertex is (v_1, y). $K_{4,s}^{-4} \times G$ can be partitioned into two subgraphs, say, M_1 and M_2, as follows. M_1 is $A \times G$ and M_2 is $B \times G$, where A is the induced subgraph of $\{u_3, u_4, v_1, v_2, v_5, \ldots, v_s\}$ in $K_{4,s}^{-4}$ and B is the induced subgraph of $\{u_1, u_2, v_3, v_4\}$ in $K_{4,s}^{-4}$. Clearly, $A = K_{2,s-2}$ and $B = K_{2,2}$. Using the same way as the above, we can prove that one pebble can be moved to (v_1, y).

Finally, if $s > 4$, then let the target vertex be (v_j, y) for some $y \in G$, where $5 \leq j \leq s$. $K_{4,s}^{-4} \times G$ can be partitioned into two subgraphs, say, M_1 and M_2, as follows. M_1 is $A \times G$ and M_2 is $B \times G$, where A is the induced subgraph of $\{u_1, u_2, u_3, v_2, v_4, v_j\}$ in $K_{4,s}^{-4}$ and B is the induced subgraph of $\{u_4, v_1, v_3, v_5, \ldots, v_{j-1}, v_{j+1}, \ldots, v_s\}$ in $K_{4,s}^{-4}$. Clearly, $A = K_{3,3}^{-1}$ and $B = K_{1,s-3}$. For $i = 1$ and 2, suppose that M_i contains p_i pebbles with r_i vertices having an odd number of pebbles. If $p_1 \geq 6f(G)$, then one pebble can be moved to (v_j, y) by Lemma 3.5. Now we assume that $p_1 < 6f(G)$. Let $p_1 = 6f(G) - t$ and $p_2 = (s-2)f(G) + t$ for some positive integer t. If $t \leq (s-2)f(G) - r_2$, then we can place at least $(p_2 - r_2)/2$ pebbles on vertices of M_1 from M_2 by a sequence of pebbling moves, and hence, in M_1, we have altogether

$$p_1 + \frac{p_2 - r_2}{2} = 6f(G) - t + \frac{(4+s)f(G) - (6f(G) - t) - r_2}{2} \geq 6f(G)$$

pebbles. By Lemma 3.5, we can move one pebble to (v_j, y). If $t > (s-2)f(G) - r_2$, then $p_2 + r_2 > (s-2)f(G) + (s-2)f(G) = 2(s-2)f(G)$. Two pebbles can be moved to (u_4, y) by Lemma 3.2, and hence one pebble can be moved to (v_j, y). □

Theorem 3.3. Let $s \geq r \geq 4$ and G satisfy the two-pebbling property. Then

$$f(K_{r,s}^{-r} \times G) \leq f(K_{r,s}^{-r})f(G) = (r+s)f(G).$$

Proof. Use induction on r. It is known from Theorem 3.2 that the result holds for $r = 4$. Now suppose that Theorem 3.3 holds for $r-1 (r \geq 5)$. We will prove that Theorem 3.3 holds for r. Let $(r+s)f(G)$ pebbles be placed on $K_{r,s}^{-r} \times G$. Firstly, let the target vertex be (u_i, y) for some $y \in G$, where $1 \leq i \leq r$. Without loss of generality, we assume that the target vertex is (u_1, y) for some $y \in G$. $K_{r,s}^{-r} \times G$ can be partitioned into two subgraphs, say, M_1 and M_2, as follows. M_1 is $A \times G$ and M_2 is $B \times G$, where A is the induced subgraph of $\{u_1, u_2, \ldots, u_{r-1}, v_1, v_2, \ldots, v_{r-2}, v_r, v_{r+1}, \ldots, v_s\}$ in $K_{r,s}^{-r}$ and B is the induced subgraph of $\{u_r, v_{r-1}\}$ in $K_{r,s}^{-r}$. Clearly, $A = K_{r-1,s-1}^{-(r-2)}$ and $B = K_2$. For $i = 1$ and 2, suppose that M_i contains p_i pebbles with r_i vertices having an odd number of pebbles. If $p_1 \geq (r+s-2)f(G)$, then by $K_{r-1,s-1}^{-(r-1)}$ being a spanning subgraph of $K_{r-1,s-1}^{-(r-2)}$ and the induction hypothesis, we have

$$f(K_{r-1,s-1}^{-(r-2)} \times G) \leq f(K_{r-1,s-1}^{-(r-1)} \times G) \leq f(K_{r-1,s-1}^{-(r-1)}) f(G) = (r+s-2)f(G),$$

and hence we can move one pebble to (u_1, y). Let $p_1 = (r+s-2)f(G) - t$ and $p_2 = 2f(G) + t$ for some positive integer t. If $t \leq 2f(G) - r_2$, then we can place at least $(p_2 - r_2)/2$ pebbles on vertices of M_1 from M_2 by a sequence of pebbling moves, and hence, in M_1, we have altogether

$$p_1 + \frac{p_2 - r_2}{2} = (r+s-2)f(G) - t + \frac{(r+s)f(G) - ((r+s-2)f(G) - t) - r_2}{2}$$
$$\geq (r+s-2)f(G)$$

pebbles. So we can move one pebble to (u_1, y). If $t > 2f(G) - r_2$, then $p_2 + r_2 > 2f(G) + 2f(G) - r_2 + r_2 = 4f(G)$. So two pebbles can be moved to (v_{r-1}, y) by Lemma 3.3, and hence one pebble can be moved to (u_1, y).

Secondly, let the target vertex is (v_i, y) for some $y \in G$, where $1 \leq i \leq r$. We assume that the target vertex is (v_1, y). $K_{r,s}^{-r} \times G$ can be partitioned into two subgraphs, say, M_1 and M_2, as follows. M_1 is $A \times G$ and M_2 is $B \times G$, where A is the induced subgraph of $\{u_1, u_2, \ldots, u_{r-1}, v_1, v_2, \ldots, v_{r-2}, v_r, v_{r+1}, \ldots, v_s\}$ in $K_{r,s}^{-r}$ and B is the induced subgraph of $\{u_r, v_{r-1}\}$ in $K_{r,s}^{-r}$. Clearly, $A = K_{r-1,s-1}^{-(r-2)}$ and $B = K_2$. Using the same way as the above, we can move one pebble to (v_1, y).

Finally, if $s > r$, then let the target vertex be (v_j, y) for some $y \in G$, where $r+1 \leq j \leq s$. $K_{r,s}^{-r} \times G$ can be partitioned into two subgraphs, say, M_1 and M_2, as follows. M_1 is $A \times G$ and M_2 is $B \times G$, where A is the induced subgraph of $\{u_1, u_2, \ldots, u_{r-1}, v_1, v_2, \ldots, v_{r-3}, v_r, v_j\}$ in $K_{r,s}^{-r}$ and B is the induced subgraph of $\{u_r, v_{r-2}, v_{r-1}, v_{r+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_s\}$ in $K_{r,s}^{-r}$. Clearly, $A = K_{r-1,r-1}^{-(r-3)}$ and $B = K_{1,s-r+1}$. For $i = 1$ and 2, suppose that M_i contains p_i pebbles with r_i vertices having an odd number of pebbles.

If $p_1 \geq (2r-2)f(G)$, then by $K_{r-1,r-1}^{-(r-1)}$ being a spanning subgraph of $K_{r-1,r-1}^{-(r-3)}$ and the induction hypothesis, we have

$$f(K_{r-1,r-1}^{-(r-3)} \times G) \leq f(K_{r-1,r-1}^{-(r-1)} \times G) \leq f(K_{r-1,r-1}^{-(r-1)})f(G) = (2r-2)f(G),$$

and hence we can move one pebble to (v_j, y). Now we assume that $p_1 < (2r-2)f(G)$. Let $p_1 = (2r-2)f(G) - t$ and $p_2 = (s-r+2)f(G) + t$ for some positive integer t. If $t \leq (s-r+2)f(G) - r_2$, then we can place at least $(p_2 - r_2)/2$ pebbles on vertices of M_1 from M_2 by a sequence of pebbling moves, and hence, in M_1, we have altogether

$$\begin{aligned} p_1 + \frac{p_2 - r_2}{2} &= (2r-2)f(G) - t + \frac{(r+s)f(G) - ((2r-2)f(G) - t) - r_2}{2} \\ &\geq (2r-2)f(G) \end{aligned}$$

pebbles. So we can move one pebble to (v_j, y). If $t > (s-r+2)f(G) - r_2$, then $p_2 + r_2 > (s-r+2)f(G) + (s-r+2)f(G) = 2(s-r+2)f(G)$. Two pebbles can be moved to (u_r, y) by Lemma 3.2, and hence one pebble can be moved to (v_j, y). □

Since $K_{r,s}^{-r} \times G$ is a spanning subgraph of $K_{r,s}^{-k} \times G$, the following corollaries are obvious by Theorem 3.3.

Corollary 3.1. Let $s \geq r \geq 4$, $k \leq r$ and G satisfy the two-pebbling property. Then

$$f(K_{r,s}^{-k} \times G) \leq f(K_{r,s}^{-k})f(G).$$

Corollary 3.2. Let $s \geq r \geq 4$, $n \geq m \geq 4$, $k_1 \leq r$ and $k_2 \leq m$. Then

$$f(K_{r,s}^{-k_1} \times K_{m,n}^{-k_2}) \leq f(K_{r,s}^{-k_1})f(K_{m,n}^{-k_2}).$$

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