SOME TRANSFORMATIONS FOR THE WELL-POISED ${}_7F_6(1)$-SERIES

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ABSTRACT. Several transformations about ${}_7F_6(1)$-series are established by applying the modified Abel lemma on summation by parts. As consequence, a reciprocal relation on balanced ${}_3F_2(1)$-series is derived, which may also be considered as a nonterminating extension of Saalschütz's theorem (1891).

1. Introduction and Motivation

For a complex x and an integer $n \in \mathbb{Z}$, define the rising shifted-factorial by

$$(x)_n = \Gamma(x+n)/\Gamma(x)$$

where the Γ-function is given by the Euler integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad \text{with} \quad \Re(t) > 0.$$

When $n \in \mathbb{N}_0$, it reduces the usual rising shifted-factorial

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N}.$$

For the sake of brevity, the quotients of shifted factorials and Γ-function will be abbreviated respectively as

$$\begin{aligned}\begin{bmatrix}\alpha, \beta, \cdots, \gamma \\ A, B, \cdots, C\end{bmatrix}_n &= \frac{(\alpha)_n(\beta)_n\cdots(\gamma)_n}{(A)_n(B)_n\cdots(C)_n}, \\ \Gamma\begin{bmatrix}\alpha, \beta, \cdots, \gamma \\ A, B, \cdots, C\end{bmatrix} &= \frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}.\end{aligned}$$

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Following Bailey [1] and Slater [7], the generalized hypergeometric series for an indeterminate z read as

$$ {}_{1+p}F_q\left[\begin{matrix}a_1, a_2, \cdots, a_p\\ b_1, \cdots, b_q\end{matrix}\middle| z\right] = \sum_{n=0}^{\infty}\left[\begin{matrix}a_0, a_1, \cdots, a_p\\ 1, b_1, \cdots, b_q\end{matrix}\right]_n z^n $$

where $\{a_i\}$ and $\{b_j\}$ are complex parameters such that no zero factors appear in the denominators of the summands on the right hand sides.

Throughout the paper, if Ω_n is used to denote partial sum of some series, then the corresponding letter Ω without subscript will stand for the limit of Ω_n (if it exists of course) when $n \to \infty$.

Recently, Chu and Wang [2, 3] have utilized Abel's lemma on summation by parts to deal with the terminating hypergeometric series. In the present paper, we shall use the same lemma with the " remainder term" to investigate the following partial sum

$$ \begin{aligned} &S_n(a, b, c, d, e, f, g)\\ &:= \sum_{k=0}^{n-1}(a+2k)\left[\begin{matrix}b, & c, & d, & e, & f, & g\\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g\end{matrix}\right]_k, \end{aligned} $$

where $1+3a = b+c+d+e+f$. Some transformations for both terminating and nonterminating ${}_7F_6(1)$-series will be established.

2. Nonterminating ${}_7F_6(1)$-Series Transformation

Theorem 1. *For five indeterminate a, b, c, d, e subject to $1+2a = b+c+d+e+f$, there holds the following transformation from very well poised ${}_7F_6(1)$-series to balanced ${}_3F_2(1)$-series and ${}_4F_3(1)$-series*

$$ \begin{aligned} &{}_7F_6\left[\begin{matrix}a, 1+a/2, & b, & c, & d, & e, & f\\ a/2, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f\end{matrix}\middle| 1\right]\\ &=\Gamma\left[\begin{matrix}1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d\\ 1+a, 1+a-b-c, 1+a-b-d, 1+a-c-d\end{matrix}\right] {}_3F_2\left[\begin{matrix}b, & c, & d\\ & 1+a-e, & 1+a-f\end{matrix}\middle| 1\right]\\ &-\Gamma\left[\begin{matrix}1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a-f, e+f-a\\ 1+a, b, c, d, 1+e, 1+f, 1+e+f-a\end{matrix}\right] {}_4F_3\left[\begin{matrix}1, 1+a-b-c, 1+a-b-d, 1+a-c-d\\ 1+e, 1+f, 1+e+f-a\end{matrix}\middle| 1\right]. \end{aligned} $$

When $f \to 0$, this theorem induces the following relation between two balanced ${}_3F_2(1)$-series.

Corollary 2 (Nonterminating transformation between two ${}_3F_2(1)$-series).

$$\begin{aligned}{}_3F_2\left[\begin{matrix}b, & c, & d\\ a, & 1+b+c+d-a\end{matrix}\middle|1\right] &= \Gamma\left[\begin{matrix}a, a-b-c, a-b-d, a-c-d\\ a-b, a-c, a-d, a-b-c-d\end{matrix}\right]\\ &+\Gamma\left[\begin{matrix}a, a-b-c, a-b-d, a-c-d, 1+b+c+d-a\\ b, \quad c, \quad d, \quad 1+a-b-c-d, \quad 2a-b-c-d\end{matrix}\right]\\ &\times {}_3F_2\left[\begin{matrix}a-b-c, a-b-d, a-c-d\\ 1+a-b-c-d, 2a-b-c-d\end{matrix}\middle|1\right].\end{aligned}$$

This differs essentially from the following Saalschützian transformation [6] (see also [5, Eq. 5.2] or [1, §3.8])

$$\begin{aligned}{}_3F_2\left[\begin{matrix}b, & c, & d\\ a, & 1+b+c+d-a\end{matrix}\middle|1\right] &= \Gamma\left[\begin{matrix}a, a-b-c, 1+b+c+d-a, 1+d-a\\ a-b, a-c, 1+b+d-a, 1+c+d-a\end{matrix}\right]\\ &+\frac{1}{b+c-a}\Gamma\left[\begin{matrix}a, 1+b+c+d-a\\ b, \quad c, \quad 1+d\end{matrix}\right]{}_3F_2\left[\begin{matrix}a-b, a-c, 1\\ 1+a-b-c, 1+d\end{matrix}\middle|1\right].\end{aligned}$$

In Corollary 2 or the last transformation due to Saalschütz, if b, c or d is a negative integer, the second term on the right vanishes and we obtain Saalschütz's theorem [1, §2.2]

$${}_3F_2\left[\begin{matrix}a, b, & -n\\ c, & 1+a+b-c-n\end{matrix}\middle|1\right] = \left[\begin{matrix}c-a, & c-b\\ c, & c-a-b\end{matrix}\right]_n.$$

In order to prove Theorem 1, we shall utilize the modified Abel lemma on summation by parts. For an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators ∇ and Δ, respectively, by

$$\nabla\tau_k = \tau_k - \tau_{k-1} \quad\text{and}\quad \Delta\tau_k = \tau_k - \tau_{k+1}.$$

It should be pointed out that Δ is adopted for convenience in the present paper, which differs from the usual operator Δ only in the minus sign. Then **Abel's lemma** on summation by parts with the " remainder term" may be modified as follows:

$$\sum_{k=0}^{n-1} B_k \nabla A_k = \{A_{n-1}B_n - A_{-1}B_0\} + \sum_{k=0}^{n-1} A_k\Delta B_k.$$

In fact, it is almost trivial to check the following expression

$$\sum_{k=0}^{n-1} B_k \nabla A_k = \sum_{k=0}^{n-1} B_k\{A_k - A_{k-1}\} = \sum_{k=0}^{n-1} A_kB_k - \sum_{k=0}^{n-1} A_{k-1}B_k.$$

Replacing k by $k+1$ for the last sum, we can reformulate the equation as follows

$$\begin{aligned}\sum_{k=0}^{n-1} B_k \nabla A_k &= A_{n-1}B_n - A_{-1}B_0 + \sum_{k=0}^{n-1} A_k\{B_k - B_{k+1}\}\\ &= A_{n-1}B_n - A_{-1}B_0 + \sum_{k=0}^{n-1} A_k \triangle B_k,\end{aligned}$$

which is exactly the equality stated in the modified Abel lemma. □

Proof of Theorem 1. For two sequences given by

$$\begin{aligned}A_k &= \begin{bmatrix} 1+b, & 1+c, & 1+d, & 1+2a-b-c-d \\ 1+a-b, & 1+a-c, & 1+a-d, & 1-a+b+c+d \end{bmatrix}_k,\\ B_k &= \begin{bmatrix} e, & f, & g, & 1-a+b+c+d \\ 1+a-e, & 1+a-f, & 1+a-g, & 2a-b-c-d \end{bmatrix}_k;\end{aligned}$$

it is almost trivial to compute the relations

$$\varpi := A_{-1}B_0 = \frac{(a-b)(a-c)(a-d)(b+c+d-a)}{b\,c\,d\,(2a-b-c-d)},$$

$$\mathfrak{R} := \frac{A_{n-1}B_n}{A_{-1}B_0} = \frac{b+c+d-a+n}{b+c+d-a}\begin{bmatrix} b, & c, & d, & e, & f, & g \\ a-b, & a-c, & a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_n;$$

and the following differences

$$\begin{aligned}\nabla A_k &= \frac{(a+2k)(a-b-c)(a-b-d)(a-c-d)}{b\,c\,d\,(2a-b-c-d)}\\ &\times \begin{bmatrix} b, & c, & d, & 2a-b-c-d \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+b+c+d-a \end{bmatrix}_k,\\ \triangle B_k &= \frac{(1+a+2k)(1+a-e-f)(1+a-e-g)(1+a-f-g)}{(1+a-e)(1+a-f)(1+a-g)(1+a-e-f-g)}\\ &\times \begin{bmatrix} e, & f, & g, & 1+b+c+d-a \\ 2+a-e, & 2+a-f, & 2+a-g, & 1+2a-b-c-d \end{bmatrix}_k.\end{aligned}$$

By means of the modified Abel lemma on summation by parts, we can manipulate the following S-series

$$\begin{aligned}&S_n(a,b,c,d,e,f,g) \times \frac{(a-b-c)(a-b-d)(a-c-d)}{b\,c\,d\,(2a-b-c-d)}\\ &= \sum_{k=0}^{n-1} B_k \nabla A_k = \varpi(\mathfrak{R}-1) + \sum_{k=0}^{n-1} A_k \triangle B_k.\end{aligned}$$

Writing explicitly the last partial sum

$$\sum_{k=0}^{n-1} A_k \triangle B_k = \frac{(1+a-e-f)(1+a-e-g)(1+a-f-g)}{(1+a-e)(1+a-f)(1+a-g)(1+a-e-f-g)}$$

$$\times \sum_{k=0}^{n-1}(1+a+2k)\begin{bmatrix} 1+b, & 1+c, & 1+d, & e, & f, & g \\ 1+a-b, & 1+a-c, & 1+a-d, & 2+a-e, & 2+a-f, & 2+a-g \end{bmatrix}_k,$$

we find the following recurrence relation

$$S_n(a,b,c,d,e,f,g) = S_n(1+a,1+b,1+c,1+d,e,f,g) \tag{1a}$$

$$\times \frac{b\,c\,d\,(1+a-e-f)(1+a-e-g)(1+a-f-g)}{(b+c-a)(b+d-a)(c+d-a)(1+a-e)(1+a-f)(1+a-g)} \tag{1b}$$

$$+\frac{(a-b)(a-c)(a-d)(b+c+d-a)}{(b+c-a)(b+d-a)(c+d-a)}\{1-\mathcal{R}(a,b,c,d,e,f,g)\}. \tag{1c}$$

Iterating the last relation m-times, we get the following transformation

$$S_n(a,b,c,d,e,f,g) = S_n(m+a,m+b,m+c,m+d,e,f,g)$$

$$\times \begin{bmatrix} b, & c, & d, & 1+a-e-f, & 1+a-e-g, & 1+a-f-g \\ b+c-a, & b+d-a, & c+d-a, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_m$$

$$+\frac{(a-b)(a-c)(a-d)}{(b+c-a)(b+d-a)(c+d-a)}$$

$$\times \sum_{k=0}^{m-1}(b+c+d-a+2k)\{1-\mathcal{R}(k+a,k+b,k+c,k+d,e,f,g)\}$$

$$\times \begin{bmatrix} b, & c, & d, & 1+a-e-f, & 1+a-e-g, & 1+a-f-g \\ 1+b+c-a, & 1+b+d-a, & 1+c+d-a, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_k.$$

Reformulating the $\mathcal{R}$-function by singling out k-factorials

$$\mathcal{R}(k+a,k+b,k+c,k+d,e,f,g) = \frac{b+c+d-a+n+2k}{b+c+d-a+2k}$$

$$\times \begin{bmatrix} k+b, & k+c, & k+d, & e, & f, & g \\ 1+k+a-e, & 1+k+a-f, & 1+k+a-g, & a-b, & a-c, & a-d \end{bmatrix}_n$$

$$= \frac{b+c+d-a+n+2k}{b+c+d-a+2k}\begin{bmatrix} b, & c, & d, & e, & f, & g \\ a-b, & a-c, & a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_n$$

$$\times \begin{bmatrix} n+b, & n+c, & n+d, & 1+a-e, & 1+a-f, & 1+a-g \\ b, & c, & d, & 1+n+a-e, & 1+n+a-f, & 1+n+a-g \end{bmatrix}_k$$

and then denoting by $\mathcal{S}_m$ the partial sum of another well-poised series

$$\mathcal{S}_m(a,b,c,d,e,f,g) := S_m(b+c+d-a,\, b,\, c,\, d,\, 1+a-f-g, 1+a-e-g, 1+a-e-f)$$

$$= \sum_{k=0}^{m-1}(b+c+d-a+2k)\begin{bmatrix} b, & c, & d, & 1+a-e-f, & 1+a-e-g, & 1+a-f-g \\ 1+b+c-a, & 1+b+d-a, & 1+c+d-a, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_k,$$

we establish the following transformation formula on well-poised series.

Proposition 3 (Reciprocal relation on well-poised series). *For seven indeterminate a, b, c, d, e, f, g subject to $1+3a = b+c+d+e+f+g$, there holds*

$$\begin{aligned} S_n(a,b,c,d,e,f,g) &= S_n(m+a; m+b, m+c, m+d, e, f, g) \\ &\times \begin{bmatrix} b,\ c,\ d,\ 1+a-e-f,\ 1+a-e-g,\ 1+a-f-g \\ b+c-a, b+d-a, c+d-a, 1+a-e, 1+a-f, 1+a-g \end{bmatrix}_m \\ &+ \frac{(a-b)(a-c)(a-d)}{(b+c-a)(b+d-a)(c+d-a)} \Big\{ \mathcal{S}_m(a,b,c,d,e,f,g) \\ &- \mathcal{S}_m(2n+a, n+b, n+c, n+d, n+e, n+f, n+g) \\ &\times \begin{bmatrix} b, & c, & d, & e, & f, & g \\ a-b, & a-c, & a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_n \Big\}. \end{aligned}$$

In this Proposition, letting $m = n$, $b \to a$, $c = 1-n$ and shifting n to $n+1$ in succession, we see that the last two lines are annihilated and the S_n-sum on the right hand side of the second line reduces to one. We recover Dougall's very well poised terminating series identity.

Corollary 4 (Dougall [4]).

$$\begin{aligned} &{}_7F_6 \left[\begin{matrix} a, & 1+a/2, & b, & c, & d, & e, & -n \\ & a/2, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+n \end{matrix} \middle| 1 \right] \\ &= \begin{bmatrix} 1+a, & 1+a-b-c, & 1+a-b-d, & 1+a-c-d \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-b-c-d \end{bmatrix}_n \end{aligned}$$

where $1+n+2a = b+c+d+e$.

The limiting case $m, n \to \infty$ of Proposition 3 leads to the following nonterminating transformation formula.

Proposition 3′ (Nonterminating well-poised series transformation)

$$\begin{aligned} &S(a,b,c,d,e,f,g) - \frac{(a-b)(a-c)(a-d)}{(b+c-a)(b+d-a)(c+d-a)} \mathcal{S}(a,b,c,d,e,f,g) \\ &= \Gamma \begin{bmatrix} b+c-a, b+d-a, c+d-a, 1+a-e, 1+a-f, 1+a-g \\ b, \ c, \ d, \ 1+a-e-f, 1+a-e-g, 1+a-f-g \end{bmatrix} {}_4F_3 \left[\begin{matrix} 1, & e, & f, & g \\ & 1+a-b, & 1+a-c, & 1+a-d \end{matrix} \middle| 1 \right] \\ &\frac{\Gamma \begin{bmatrix} 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a-f, 1+a-g \\ b, \ c, \ d, \ e, \ f, \ g \end{bmatrix}}{(b+c-a)(b+d-a)(c+d-a)} {}_4F_3 \left[\begin{matrix} 1, 1+a-e-f, 1+a-e-g, 1+a-f-g \\ 1+b+c-a, 1+b+d-a, 1+c+d-a \end{matrix} \middle| 1 \right], \end{aligned}$$

where $1+3a = b+c+d+e+f+g$.

Letting $b \to a$ further and then relabeling the parameter g by b, we get the three-term transformation formula displayed in Theorem 1. □

3. Further Relation Between Terminating ${}_7F_6(1)$-Series

Applying recurrence (1a-1c) to

$$S_n(1+a,e,f,g,1+b,1+c,1+d) = S_n(1+a,1+b,1+c,1+d,e,f,g),$$

we have

$$S_n(1+a,1+b,1+c,1+d,e,f,g) = S_n(2+a,1+b,1+c,1+d,1+e,1+f,1+g)$$
$$\times \frac{e\,f\,g\,(b+c-a)(b+d-a)(c+d-a)}{(1+a-e-f)(1+a-e-g)(1+a-f-g)(1+a-b)(1+a-c)(1+a-d)}$$
$$+ \frac{(1+a-e)(1+a-f)(1+a-g)(1+a-e-f-g)}{(1+a-e-f)(1+a-e-g)(1+a-f-g)}\{1-\mathfrak{R}(1+a,e,f,g,1+b,1+c,1+d)\}.$$

Substituting the last expression into (1a-1c) and then simplifying the result, we get the following relation

$$S_n(a,b,c,d,e,f,g) = S_n(2+a,1+b,1+c,1+d,1+e,1+f,1+g)$$
$$\times \frac{b\,c\,d\,e\,f\,g}{(1+a-b)(1+a-c)(1+a-d)(1+a-e)(1+a-f)(1+a-g)}$$
$$+ \frac{(a-b)(a-c)(a-d)(a-b-c-d)}{(a-b-c)(a-b-d)(a-c-d)}\{1-\mathfrak{R}(a,b,c,d,e,f,g)\}$$
$$+ \frac{b\,c\,d\,(2a-b-c-d)}{(a-b-c)(a-b-d)(a-c-d)}\{1-\mathfrak{R}(1+a,e,f,g,1+b,1+c,1+d)\}.$$

Iterating the last relation m-times, we get the following transformation

$$S_n(a,b,c,d,e,f,g) = S_n(2m+a,m+b,m+c,m+d,m+e,m+f,m+g)$$
$$\times \begin{bmatrix} b, & c, & d, & e, & f, & g \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_m$$
$$- \frac{(a-b)\,(a-c)\,(a-d)}{(a-b-c)(a-b-d)(a-c-d)}$$
$$\times \sum_{k=0}^{m-1} (b+c+d-a+k)\{1-\mathfrak{R}(2k+a,k+b,k+c,k+d,k+e,k+f,k+g)\}$$
$$\times \begin{bmatrix} b, & c, & d, & e, & f, & g \\ a-b, & a-c, & a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_k$$
$$+ \frac{b\,c\,d}{(a-b-c)(a-b-d)(a-c-d)}$$
$$\times \sum_{k=0}^{m-1} (2a-b-c-d+k)\{1-\mathfrak{R}(1+2k+a,k+e,k+f,k+g,1+k+b,1+k+c,1+k+d)\}$$
$$\times \begin{bmatrix} 1+b, & 1+c, & 1+d, & e, & f, & g \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_k.$$

Reformulating the both $\mathfrak{R}$-functions by singling out k-factorials

$$\mathfrak{R}(2k+a,k+b,k+c,k+d,k+e,k+f,k+g)$$
$$=\frac{b+c+d-a+n+k}{b+c+d-a+k}\begin{bmatrix} b, & c, & d, & e, & f, & g \\ a-b, & a-c, & a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_n$$
$$\times\begin{bmatrix} n+b,n+c,n+d,n+e,n+f,n+g,a-b,a-c,a-d,1+a-e,1+a-f,1+a-g \\ b,c,d,e,f,g,n+a-b,n+a-c,n+a-d,1+n+a-e,1+n+a-f,1+n+a-g \end{bmatrix}_k;$$

$$\mathfrak{R}(1+2k+a,k+e,k+f,k+g,1+k+b,1+k+c,1+k+d)$$
$$=\frac{2a-b-c-d+n+k}{2a-b-c-d+k}\begin{bmatrix} 1+b, & 1+c, & 1+d, & e, & f, & g \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_n$$
$$\times\begin{bmatrix} 1+n+b,1+n+c,1+n+d,n+e,n+f,n+g,1+a-b,1+a-c,1+a-d,1+a-e,1+a-f,1+a-g \\ 1+b,1+c,1+d,e,f,g,1+n+a-b,1+n+a-c,1+n+a-d,1+n+a-e,1+n+a-f,1+n+a-g \end{bmatrix}_k,$$

and then denoting by U_m, V_m the following two partial series

$$U_m(a,b,c,d,e,f,g)$$
$$:=\sum_{k=0}^{m-1}(b+c+d-a+k)\begin{bmatrix} b, & c, & d, & e, & f, & g \\ a-b, & a-c, & a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_k;$$

$$V_m(a,b,c,d,e,f,g)$$
$$:=\sum_{k=0}^{m-1}(2a-b-c-d+k)\begin{bmatrix} 1+b, & 1+c, & 1+d, & e, & f, & g \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_k,$$

we establish the following six-term transformation formula.

Proposition 5. *For seven indeterminate a,b,c,d,e,f,g subject to $1+3a = b+c+d+e+f+g$, there holds*

$$S_n(a,b,c,d,e,f,g)=S_n(2m+a,m+b,m+c,m+d,m+e,m+f,m+g)$$
$$\times\begin{bmatrix} b, & c, & d, & e, & f, & g \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_m$$
$$+\frac{(a-b)(a-c)(a-d)}{(b+c-a)(b+d-a)(c+d-a)}\Big\{U_m(a,b,c,d,e,f,g)$$
$$-U_m(2n+a,n+b,n+c,n+d,n+e,n+f,n+g)\begin{bmatrix} b, & c, & d, & e, & f, & g \\ a-b, & a-c, & a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_n\Big\}$$
$$-\frac{b\,c\,d}{(b+c-a)(b+d-a)(c+d-a)}\Big\{V_m(a,b,c,d,e,f,g)$$
$$-V_m(2n+a,n+b,n+c,n+d,n+e,n+f,n+g)\begin{bmatrix} 1+b, & 1+c, & 1+d, & e, & f, & g \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{bmatrix}_n\Big\}.$$

In this theorem, letting $m = n$, $f \to a$, $g = 1 - n$ and shifting n to $n+1$ in succession, then recalling Dougall's summation formula displayed in Corollary 4, we find the following interesting transformation on terminating ${}_7F_6(1)$-series.

Theorem 6 (Terminating transformation on ${}_7F_6(1)$-series).

$$
{}_7F_6\left[\begin{matrix} a, 1+b+c+d-a, & b, & c, & d, & e, & -n \\ b+c+d-a, & a-b, & a-c, & a-d, & 1+a-e, & 1+a+n \end{matrix}\middle| 1\right]
$$
$$
= \left[\begin{matrix} a, a-b-c, a-b-d, a-c-d \\ a-b, a-c, a-d, a-b-c-d \end{matrix}\right]_{n+1} - \frac{bcd\,(2a-b-c-d)}{(a-b)(a-c)(a-d)(a-b-c-d)}
$$
$$
\times {}_7F_6\left[\begin{matrix} a, & 1+2a-b-c-d, & 1+b, & 1+c, & 1+d, & e, & -n \\ & 2a-b-c-d, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+n \end{matrix}\middle| 1\right]
$$

where $1+n+2a = b+c+d+e$.

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