Hamiltonicity of transformation graph G^{+--}*

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Abstract

The transformation graph G^{+--} of a graph G is the graph with vertex set $V(G) \cup E(G)$, in which two vertices u and v are joined by an edge if one of the following conditions holds: (i) $u, v \in V(G)$ and they are adjacent in G, (ii) $u, v \in E(G)$ and they are not adjacent in G, (iii) one of u and v is in $V(G)$ while the other is in $E(G)$, and they are not incident in G. In this paper, for any graph G, we determine the independence number and the connectivity of G^{+--}. Furthermore, we show that for a graph G with no isolated vertices, G^{+--} is hamiltonian if and only if G is not a star and $G \notin \{2K_2, K_3\}$.

Key words: Transformation graph; Hamilton cycle

1 Introduction

All graphs considered here are finite, undirected and simple. We refer to [1] for unexplained terminology and notations. Let $G = (V(G), E(G))$ be a graph. $|V(G)|$ and $|E(G)|$ are called the *order* and the *size* of G, respectively. For two vertices u and v of G, if there is an edge e joining them, we say u and v are *adjacent*. In this case, both u and v are end vertices of e, and u (or v) and e are said to be *incident*. Two edges e and f are also called to be adjacent if they have an end vertex in common.

For a graph G, the symbols $\Delta(G), \delta(G), \kappa(G)$ and $\alpha(G)$ denote the maximum degree, the minimum degree, the connectivity and the independence number of G, respectively.

As usual, K_n and P_n denote the complete graph and path of order n, respectively. For two positive integers r and s, $K_{r,s}$ is the complete bipartite graph with two partite sets containing r and s vertices. In particular, $K_{1,s}$ is called a star. For $s \geqslant 2$, $K_{1,s} + e$ is the graph obtained from $K_{1,s}$ by adding a new edge which joins two vertices of degrees one. $K_{r,s} - e$ is the graph obtained from $K_{r,s}$ by deleting an edge. We say two graphs G and H

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are disjoint if they have no vertex in common, and denotes their union by $G + H$; such a graph is called the disjoint union of G and H. The disjoint union of k copies of G is written as kG. The join $G \vee H$ of G and H is the graph obtained from $G + H$ by joining each vertex of G to each vertex of H.

The *complement* of G, denoted by $\overline{G}$, is the graph with the same vertex set as G, but where two vertices are adjacent if and only if they are not adjacent in G. The total graph $T(G)$ of G is the graph whose vertex set is $V(G) \cup E(G)$, and in which two vertices are adjacent if and only if they are adjacent or incident in G.

Wu and Meng [9] introduced some new graphical transformations which generalize the concept of total graph. Let $G = (V(G), E(G))$ be a graph, and α, β be two elements of $V(G) \cup E(G)$. We define the associativity of α and β is $+$ if they are adjacent or incident, and $-$, otherwise. Let xyz be a 3-permutation of the set $\{+, -\}$. We say that α and β correspond to the first term x (resp. the second term y or the third term z) if both α and β are in $V(G)$ (resp. both α and β are in $E(G)$, or one of α and β is in $V(G)$ and the other is in $E(G)$). The transformation graph G^{xyz} of G is defined on the vertex set $V(G) \cup E(G)$. Two vertices α and β of G^{xyz} are joined by an edge if and only if their associativity in G is consistent with the corresponding term of xyz.

Therefore, one can obtain eight graphical transformations of graphs, since there are eight distinct 3-permutation of $\{+, -\}$. Note that G^{+++} is just the total graph $T(G)$ of G, and G^{---} is the complement of $T(G)$. Fleischner and Hobbs [6] showed that G^{+++} is hamiltonian if and only if G contains an EPS-subgraph, that is, a connected spanning subgraph S which is the edge-disjoint union of a (not necessarily connected) graph E, all of whose vertices have even degree, with a (possibly empty) forest P each of whose component is a path. Ma and Wu [8] showed that for a graph G of order $n \geq 3$, G^{---} is hamiltonian if and only if G is not isomorphic to any graph in $\{K_{1,n-1}, K_{1,n-1} + e, K_{1,n-2} + K_1\} \cup \{K_2 + 2K_1, K_3 + K_1, K_3 + 2K_1, K_4\}$. Wu, Zhang and Zhang [10] proved that for any graph G of order n, G^{-++} is hamiltonian if and only if $n \geq 3$. Recently, Xu and Wu [11] showed, for a graph G of order $n \geq 4$, G^{-+-} is hamiltonian if and only if G is not isomorphic to any graph in $\{K_{1,n-1}, K_{1,n-1} + e, K_{1,n-2} + K_1\} \cup \{2K_1 + K_2\}$. Yi and Wu [12] showed that for a graph of order p and size q, if $q \geq p - 1$, G^{++-} is hamiltonian. We refer to [2, 3, 4, 7, 13] for more results on G^{xyz}.

In this paper, we shall investigate the transformation graph G^{+--} of a graph G. G^{+--} is the graph with $V(G^{+--}) = V(G) \cup E(G)$, in which two vertices u and v are joined by an edge if one of the following conditions holds: (*i*) $u, v \in V(G)$ and they are adjacent in G, (*ii*) $u, v \in E(G)$ and they are not adjacent in G, (*iii*) one of u and v is in $V(G)$ while the other is in $E(G)$, and they are not incident in G.

For any graph G, we determine the independence number and the connectivity of G^{+--}. Furthermore, for a graph G with no isolated vertices, we obtain a necessary and sufficient condition for G^{+--} to be hamiltonian.

Theorem 1.1. *For a graph G with no isolated vertices, G^{+--} is hamiltonian if and only if G is not a star and $G \notin \{2K_2, K_3\}$.*

2 Independence number and connectivity of G^{+--}

We start with some simple observations. Let G be a graph of order p and size q. Then the order of G^{+--} is $p+q$, $d_{G^{+--}}(x) = d_G(x) + q - d_G(x) = q$ for $x \in V(G)$ and $d_{G^{+--}}(e) = p + q - d_G(u) - d_G(v) - 1$ for any $e = uv \in E(G)$. Let $\Delta'(G)$ be the maximum value of $d_G(u) + d_G(v)$, where u and v are taken over all adjacent vertices in G. So

$$\delta(G^{+--}) = min\{q, p + q - \Delta'(G) - 1\}.$$

Wu and Meng [9] proved that G^{+--} is connected if and only if G has at least two edges, and $diam(G^{+--}) \leq 4$ if G has at least two edges, and the equality holds if and only if $G \cong P_3$.

In proof of main theorem, we use the following classical theorem, due to Chvátal and Erdös [5].

Theorem 2.1. *Let G be a graph of order at least three. If $\alpha(G) \leq \kappa(G)$, then G is hamiltonian.*

In the subsequent two theorems, we shall determine the independence number and connectivity of G^{+--} for a graph G.

Theorem 2.2. *For any graph G, $\alpha(G^{+--}) = max\{\alpha(G), \Delta(G) + 1\}$.*

Proof. Since both an independent set and a vertex together with its incident edges of G are an independent set of G^{+--}, $\alpha(G^{+--}) \geq max\{\alpha(G), \Delta(G) + 1\}$. So, to complete the proof, it suffices to show that $\alpha(G^{+--}) \leq max\{\alpha(G), \Delta(G) + 1\}$. Let S be a maximum independent set of G^{+--} and $S = S_1 \cup S_2$, where $S_1 \subseteq V(G)$ and $S_2 \subseteq E(G)$. Let us consider three cases.

Case 1. $|S_1| = 0$.

Then $S = S_2$. Since all elements of S_2 are edges which are pairwise adjacent in G, $G[S] = G[S_2]$ is a star or a triangle. Therefore, $\alpha(G^{+--}) = |S| = |S_2| \leq \Delta(G) + 1$.

Case 2. $|S_1| = 1$.

Let $S_1 = \{u\}$. Then all elements of S_2 are incident with u in G, thus $|S_2| \leq d_G(u) \leq \Delta(G)$, and $|S| = |S_1| + |S_2| \leq \Delta(G) + 1$.

Case 3. $|S_1| \geq 2$.

We shall show that $S_2 = \emptyset$. Otherwise, we take $e \in S_2$. By the definition of G^{+--}, all elements of S_1 are end vertices of e in G. Since S_1 is also an independent set of G, it is impossible. So $|S_2| = 0$, thus $|S| = |S_1| \leq \alpha(G)$.

By cases 1, 2 and 3, $\alpha(G^{+--}) \leq max\{\alpha(G), \Delta(G)+1\}$. □

Theorem 2.3. *For a graph G of order p and size q, we have*

$$\kappa(G^{+--}) = \begin{cases} \delta(G^{+--}) - 1 & \textit{if } p > \Delta'(G) \textit{ and } G \textit{ has an isolated edge} \\ \delta(G^{+--}) & \textit{otherwise} \end{cases}$$

Proof. Suppose $\kappa(G^{+--}) < \delta(G^{+--})$. One can easily check that $\kappa(G^{+--}) = \delta(G^{+--})$ if $p < 3$, so assume $p \geq 3$ in sequel. We shall prove that $\kappa(G^{+--}) = \delta(G^{+--}) - 1$, and furthermore $p > \Delta'(G)$ and G has an isolated edge. Let S be a minimum cut of G^{+--} with $|S| < \delta(G^{+--})$, and $H_1, H_2, \cdots, H_k$ be all components of $G^{+--} - S$. Without loss of generality, suppose H_1 is a component of $G^{+--} - S$ with the maximum $|V(H_1) \cap E(G)|$, namely, $|V(H_1) \cap E(G)| \geq |V(H_i) \cap E(G)|$ for each $i = 2, \cdots, k$. By the choice of S, every component of $G^{+--} - S$ is nontrivial (or has at least two vertices).

Claim 1. $|V(H_1) \cap E(G)| = 1$.

Proof of Claim 1. Suppose $|V(H_1) \cap E(G)| \neq 1$. If $|V(H_1) \cap E(G)| = 0$ then $E(G) \subseteq S$ and thus $|S| \geq q$, which contradicts the assumption that $|S| < \delta(G^{+--}) \leq q$. Hence $|V(H_1) \cap E(G)| \geq 2$. Suppose that $\{e_1, e_2\} \subseteq V(H_1) \cap E(G)$. We consider two cases.

Case 1. e_1 and e_2 are adjacent in G.

Assume $e_i = uu_i$ for $i = 1, 2$ in G, and $e_3, \cdots, e_d$ be all the remaining edges which are incident with u in G. We claim that u_1 and u_2 must be adjacent in G. Otherwise, $V(H_2) \cup \cdots \cup V(H_k) \subseteq \{u, e_3, \cdots, e_d\}$. But, since $\{u, e_3, \cdots, e_d\}$ is an independent set of G^{+--}, $|V(H_i)| = 1$ for each $i = 2, \cdots, k$, which contradicts the fact that all components of $G^{+--} - S$ are nontrivial. Hence u_1 and u_2 are adjacent in G, and if let $e = u_1u_2$, then $e \in V(H_2) \cup \cdots \cup V(H_k)$. Furthermore $k = 2$ since $V(H_2) \cup \cdots \cup V(H_k) \subseteq \{u, e_3, \cdots, e_d, e\}$. Next by showing $V(H_1) = \{e_1, e_2\}$ we obtain a contradiction.

First of all, $e = u_1u_2 \in V(H_2)$ implies that $V(H_1) \cap V(G) \subseteq \{u_1, u_2\}$. On the other hand, since H_2 is nontrivial, $V(H_2)$ contains u or some e_i for some $i \geq 3$. But each of conditions $u \in V(H_2)$ and $e_i \in V(H_2)$ for some $i \geq 3$ implies that $V(H_1) \cap V(G) = \emptyset$ because both u_1 and u_2 are adjacent to u and e_j for any $j \geq 3$ in G^{+--}. By the same reasoning, we can obtain $V(H_1) \cap E(G) \subseteq \{e_1, e_2\}$. Since H_1 is nontrivial, $V(H_1) = \{e_1, e_2\}$. But, e_1 and e_2 are not adjacent in G^{+--}, which destroys that H_1 is a nontrivial component of $G^{+--} - S$.

Case 2. e_1 and e_2 are not adjacent in G.

Assume $e_i = u_i v_i$ for $i = 1, 2$ in G. Then $V(H_2) \cup \cdots \cup V(H_k) \subseteq E(G)$, and for $e \in V(H_2) \cup \cdots \cup V(H_k)$, its end vertices belong to $\{u_1, u_2, v_1, v_2\}$ in G by the definition of G^{+--}. Moreover, since all H_i are nontrivial, two situations might occur. Namely, $k = 2$ or $k = 3$. If $k = 2$, $V(H_2) \in \{\{u_1u_2, v_1v_2\}, \{u_1v_2, u_2v_1\}\}$; if $k = 3$, $\{V(H_2), V(H_3)\} = \{\{u_1u_2,\ v_1v_2\}, \{u_1v_2,\ u_2v_1\}\}$. Interchanging the role of two elements of H_2 with those of H_1, we obtain $V(H_1) = \{e_1, e_2\}$ in any cases of $k = 2$ and $k = 3$. Therefore, if $k = 2$ then $\Delta'(G) \geq 4$ and $|S| = |V(G)| + |E(G)| - 4 > p + q - \Delta'(G) - 1 \geq \delta(G^{+--})$, a contradiction; if $k = 3$ then $\Delta'(G) \geq 6$, and $|S| = |V(G)| + |E(G)| - 6 > p + q - \Delta'(G) - 1 \geq \delta(G^{+--})$. Again a contradiction.

This proves Claim 1. □

Suppose $V(H_1) \cap E(G) = \{e\}$ and let u and v be the end vertices of e in G. Then $(V(H_2) \cup \cdots \cup V(H_k)) \cap V(G) \subseteq \{u, v\}$. Since $|V(H_i) \cap E(G)| \leq 1$ by Claim 1, each component H_i with $i \geq 2$ must contain u or v. Moreover, since u and v are also adjacent in G^{+--}, $k = 2$.

Claim 2. $V(H_2) \cap E(G) = \emptyset$.

Proof of Claim 2. Otherwise, let $V(H_2) \cap E(G) = \{e'\}$. Then e and e' are adjacent in G, and without loss of generality, let u be their common end vertex in G. Let w be a neighbor of e in $V(H_1)$. Then $w \in V(G)$ by Claim 1, and $w \notin \{u, v\}$. By the definition of G^{+--}, w must be the other end vertex of e' in G. It follows that u and w are adjacent in G^{+--} and thus $u \notin V(H_2)$. Since $V(H_2) \cap V(G) \subseteq \{u, v\}$, $V(H_2) \cap V(G) = \{v\}$ and $u \in S$ since u and v are adjacent in G^{+--}. Hence $V(H_2) = \{e', v\}$. Interchanging the role of $V(H_1)$ and $V(H_2)$, one can obtain that $V(H_1) = \{e, w\}$. So $|S| = p + q - 4$. Combining this with $|S| < \delta(G^{+--}) \leq q$, $p = 3$. Since $e, e' \in E(G)$, $q \geq 2$, and thus $G \cong P_3$ or $G \cong K_3$. But, it is easy to check that $\kappa(G^{+--}) = 1 = \delta(G^{+--})$ for $G \cong P_3$ or $G \cong K_3$, a contradiction. □

By Claim 2, $V(H_2) \subseteq \{u, v\}$, and $V(H_2) = \{u, v\}$ since $|V(H_2)| \geq 2$. It follows that $(N_G(u) \cup N_G(v)) \setminus \{u, v\} \subseteq S$. Therefore

$$|S| \geq q - 1 + max\{d_G(u), d_G(v)\} - 1. \qquad (1)$$

Together with $|S| < \delta(G^{+--}) = min\{q, p+q-\Delta'(G)-1\}$, we have

$$q - 1 + max\{d_G(u), d_G(v)\} - 1 \leq q - 1 \qquad (2)$$

and

$$q - 1 + max\{d_G(u), d_G(v)\} - 1 \leq p + q - \Delta'(G) - 2. \quad (3)$$

It is easy to see that $d_G(u) = d_G(v) = 1$ from (2) and thus $|S| \geq q-1$ by (1) and $p > \Delta'(G)$ from (3). Recall that $|S| < \delta(G^{+--}) \leq q$, we have $|S| = q - 1 = \delta(G^{+--}) - 1$. This proves what we desired, i.e., $\kappa(G^{+--}) = \delta(G^{+--}) - 1$, and $p > \Delta'(G)$ and G has an isolated edge.

One the other hand, if $p > \Delta'(G)$ and G has an isolated edge then $\delta(G^{+--}) = min\{q, p+q-\Delta'(G)-1\} = q$, but $\kappa(G^{+--}) \leq q-1$ since $E(G) \setminus \{e\}$ is a vertex cut of G^{+--} such that $G[\{u,v\}]$ is a component of $G^{+--} - S$, where e is an isolated edge of G with $e = uv$.

The proof is complete. □

3 The Proof of Main Theorem

The following result is obvious, so its proof is omitted.

Lemma 3.1. *Let G be a graph of size q with no isolated vertices. Then $\alpha(G) \leq q$ and the equality holds if and only if G is disjoint union of stars.*

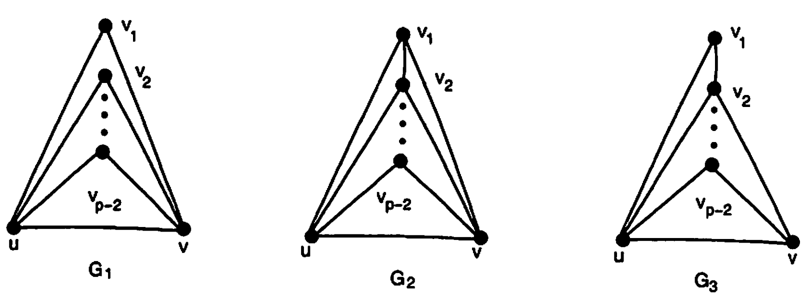

Figure 1. Several graphs with order p

Lemma 3.2. *For a graph G of order p, if $G \in \{K_{1,p-1} + e, G_1, G_2\}$ and $p \geq 4$, or $G \cong G_3$ and $p \geq 5$, where G_1, G_2 and G_3 are shown in Figure 1, then G^{+--} is hamiltonian.*

Proof. For $G \cong K_{1,p-1} + e$, let $V(K_{1,p-1} + e) = \{v_0, v_1, \cdots, v_{p-1}\}$ and $E(K_{1,p-1}+e) = \{e_1, e_2, \cdots, e_{p-1}, e_{12}\}$, where $e_i = v_0 v_i$ for $i = 1, 2, \cdots, p-1$, and $e_{12} = v_1 v_2$. Then we can find a Hamilton cycle of $(K_{1,p-1} + e)^{+--}$:

$$v_0 v_2 e_1 v_3 e_2 \cdots v_i e_{i-1} \cdots v_{p-1} e_{p-2} v_1 e_{p-1} e_{12} v_0.$$

Note that $G_1 \cong K_2 \vee \overline{K_{p-2}}$ and suppose $G \cong G_1$. Let $V(G) = \{u, v, v_1, v_2, \cdots, v_{p-2}\}$ and $E(G) = \{e, e_1^u, e_2^u, \cdots, e_{p-2}^u, e_1^v, e_2^v, \cdots, e_{p-2}^v\}$, where $e = uv$, $e_i^u = v_i u$, $e_i^v = v_i v$ for $i = 1, 2, \cdots, p-2$. Then the following is a Hamilton cycle of G^{+--} :

$$u e_1^v e_{p-2}^u v v_1 e v_2 e_1^u e_2^v v_3 e_2^u \cdots e_i^v v_{i+1} e_i^u \cdots e_{p-3}^v v_{p-2} e_{p-3}^u e_{p-2}^v u.$$

Observe that G_2 is obtained from G_1 by adding an edge which joins two vertices of degree two. If $G \cong G_2$, let $V(G) = \{u, v, v_1, v_2, \cdots, v_{p-2}\}$ and $E(G) = \{e, e', e_1^u, e_2^u, \cdots, e_{p-2}^u, e_1^v, e_2^v, \cdots, e_{p-2}^v\}$, where $e = uv, e' = v_1 v_2$, $e_i^u = v_i u$, $e_i^v = v_i v$ for $i = 1, 2, \cdots, p-2$. Then we can find a Hamilton cycle of G^{+--} for $p > 4$:

$$u e_1^v e_{p-2}^u v v_1 e v_2 e_1^u e_2^v v_3 e_2^u \cdots e_i^v v_{i+1} e_i^u \cdots e_{p-3}^v v_{p-2} e_{p-3}^u e_{p-2}^v e' u.$$

If $p = 4$, $G \cong K_4$ and $u e_1^v e_2^u v e' e v_2 e_1^u e_2^v v_1 u$ is a Hamilton cycle of G^{+--}.

Notice that G_3 is obtained from G_2 by deleting an edge as shown in Figure 1. If $G \cong G_3$, let $V(G) = \{u, v, v_1, v_2, \cdots, v_{p-2}\}$ and $E(G) = \{e, e', e_1^u, e_2^u, \cdots, e_{p-2}^u, e_2^v, e_3^v, \cdots, e_{p-2}^v\}$, where $e = uv$, $e' = v_1 v_2$, $e_i^u = v_i u$ for $i = 1, 2, \cdots, p-2$, and $e_i^v = v_i v$ for $i = 2, 3, \cdots, p-2$. Then we can find a Hamilton cycle of G^{+--} :

$$u e_2^v e_{p-2}^u v e_1^u v_2 v_1 e v_3 e_2^u e_3^v v_4 e_3^u \cdots e_i^v v_{i+1} e_i^u \cdots e_{p-3}^v v_{p-2} e_{p-3}^u e_{p-2}^v e' u.$$

Lemma 3.3. *Let G be a graph of order $p \geq 4$ and size q with no isolated vertices. If G is not a star and $max\{p, q\} \leq \Delta'(G)$ then G^{+--} is hamiltonian.*

Proof. By contradiction, suppose G is a counterexample with minimum order p. First note that $p > 4$. Otherwise, $G \in \{P_4, C_4, K_{1,3}+e, K_4-e, K_4\}$ by $max\{p, q\} \leq \Delta'(G)$. It is easy to check that G^{+--} is hamiltonian if $G \in \{P_4, C_4\}$ and we have seen that from Lemma 3.2, it is hamiltonian if $G \in \{K_{1,3} + e, K_4 - e, K_4\}$.

Let $e = uv$ be an edge of G such that $d_G(u) + d_G(v) = \Delta'(G)$. Without loss of generality, $d_G(u) \geq d_G(v)$. Observe that for any graph G, $q \geq \Delta'(G) - 1$. So, we consider two cases.

Case 1. $q = \Delta'(G) - 1$.

We claim that $\delta(G) = 1$. If $\delta(G) \geq 2$, by $q = \Delta'(G) - 1$, we have $N(u) \setminus \{v\} = N(v) \setminus \{u\}$ and thus $G \cong K_2 \vee \overline{K_{p-2}}$. By Lemma 3.2 G^{+--} is hamiltonian, a contradiction. The claim is true.

Since $d_G(u) \geq d_G(v)$, u has a neighbor, say w, with degree one in G. Let $e' = uw$ and $H = G - w$. Then H has the order $p - 1$ and size $q - 1$, and $\Delta'(H) = \Delta'(G) - 1$. By the choice of G and w, H^{+--} is hamiltonian. Let C be a Hamilton cycle of H^{+--}. Note that the order of C is $p + q - 2$. Recall that $d_{G^{+--}}(w) = q$ and $d_{G^{+--}}(e') = p + q - d_G(u) - d_G(w) - 1 = p + q - d_G(u) - 2$. So, $d_{G^{+--}}(e') \geq max\{p - 1, q - 1\}$ since $d_G(u) \leq p - 1$ and $q \geq \Delta(G) + 1 \geq d_G(u) + 1$. By inserting e' and w into C we shall obtain a Hamilton cycle of G^{+--}, which contradicts the choice of G.

First we insert e' into C. If $p \neq q$, since $d_{G^{+--}}(e') \geq max\{p - 1, q - 1\}$ and the length of C is $p + q - 2$, we can insert e' into C, and obtain a cycle of length $p + q - 1$. For the case $p = q$, $G \not\cong K_{1,p-1} + e$ since $(K_{1,p-1}+e)^{+--}$ is hamiltonian by Lemma 3.2. It follows that $d_G(u) \leq p-2$, and $d_{G^{+--}}(e') = p + q - d_G(u) - 2 \geq q$. We can also insert e' into C. We denote the resulting cycle by C'.

Now it remains to insert w into C'. Since $d_{G^{+--}}(w) = q$, if $q \geq p$, we can insert w into C' and obtain a Hamilton cycle of G^{+--}. If $q = p - 1$, then G is a double star.

If there are two consecutive vertices on C', which are adjacent to w in G^{+--}, then C' can be extended to a Hamilton cycle of G^{+--} by inserting w in a obvious way. If there are not, w are adjacent to q vertices which are pairwise independent in C'. However, since e' and v are adjacent in G^{+--}, we still find a Hamilton cycle (with bold lines) of G^{+--} as illustrated in Figure 2.

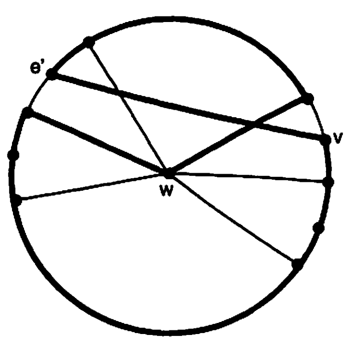

Figure 2. A Hamilton cycle of G^{+--}

Case 2. $q = \Delta'(G)$.

Since $q = \Delta'(G)$ and $d_G(u) + d_G(v) = \Delta'(G)$, there is only edge which is not adjacent to uv in G. We claim that $\delta(G) = 1$. If $\delta(G) \geq 2$ then $G \cong G_2$ or $G \cong G_3$, where G_2 and G_3 are as shown in Figure 1, by Lemma 3.2, G^{+--} is hamiltonian.

First assume G has an isolated edge, say $e' = xy$, and let $H = G - \{x, y\}$. By the choice of G, either $H \in \{K_3, P_3, K_{1,p-3}\}$ or H^{+--} is hamiltonian. It is easy to check that $(K_3 + K_2)^{+--}$, $(P_3 + K_2)^{+--}$ are hamiltonian. We shall see that $(K_{1,p-3} + K_2)^{+--}$ is also hamiltonian in the proof of Theorem 1.1 (Subcase 1.2). For the later case, let C be a Hamilton cycle of H^{+--}. Combining $d_{G^{+--}}(x) = d_{G^{+--}}(y) = q$ and $d_{G^{+--}}(e') = p + q - 3$ with $q = \Delta'(G) \geq p$, we can insert x, y and e' one by one into C, and obtain a Hamilton cycle of G^{+--}.

Now assume G has no isolated edge, and let x be a vertex of G with $d_G(x) = 1$. Let w be the neighbor of x in G and $e' = wx$. As we have seen before, $d_{G^{+--}}(e') = p + q - d_G(w) - 2 \geq q - 1$. If $d_G(w) < p - 1$ then $d_{G^{+--}}(e') \geq q$; if $d_G(w) = p - 1$, then $w = u$ and $d_G(v) \geq 2$, this implies that $q > p$ and thus $d_{G^{+--}}(e') = q - 1$. By the choice of G, $(G - x)^{+--}$ is hamiltonian, and let C be a Hamilton cycle. Since the length of C is $p + q - 2$, e' can be inserted into C. Let C' be the resulting new cycle of length $p + q - 1$ obtained from C by inserting e' in each of above cases. Since $d_G(w) = q$ and $q \geq p$, we can insert w into C' and obtain a Hamilton cycle of G^{+--}. This contradicts the choice of G. □

Proof of Theorem 1.1. If $G \in \{2K_2, K_3, K_{1,1}\}$, it is easy to check that G^{+--} is not hamiltonian. If $G \cong K_{1,p-1}$ for $p \geq 3$, then the number of component of $G^{+--} - S$ is p, and is greater than $p - 1$, the cardinality of S, where S is the set of vertices with degree one in G. Hence G^{+--} is not hamiltonian.

To show its sufficiency, assume G is a graph with no isolated vertices and is not a star and $G \notin \{2K_2, K_3\}$. Let us consider two cases.

Case 1. $p \geq \Delta'(G) + 1$.

Then $\delta(G^{+--}) = min\{q, p + q - \Delta'(G) - 1\} = q$. Since G is not a star and has no isolated vertices, $q \geq max\{\alpha(G), \Delta(G) + 1\} = \alpha(G^{+--})$.

Subcase 1.1. G has no isolated edges.

By Theorem 2.3, $\kappa(G^{+--}) = \delta(G^{+--}) = q \geq max\{\alpha(G), \Delta(G) + 1\} = \alpha(G^{+--})$. By theorem 2.1 G^{+--} is hamiltonian.

Subcase 1.2. G has an isolated edge.

By Theorem 2.3, $\kappa(G^{+--}) = \delta(G^{+--}) - 1 = q - 1$. If $\alpha(G^{+--}) < q$ then $\alpha(G^{+--}) \leq \kappa(G^{+--})$, and by Theorem 2.1, G^{+--} is hamiltonian. Now suppose $\alpha(G^{+--}) = q$. By $\alpha(G^{+--}) = max\{\alpha(G), \Delta(G) + 1\}$, if $q = \Delta(G) + 1$, $G \cong K_{1,\Delta(G)} + K_2$ since G has an isolated edge; if $q = \alpha(G)$, then by Lemma 3.1, G is the union of some stars, one of which is K_2. Therefore, $G \cong mK_2, m \geq 3$ or $G \cong m_1K_2 + K_{1,r_1} + K_{1,r_2} + \cdots + K_{1,r_t}$, where $m_1 \geq 1$ and $r_i \geq 2$ for each $i = 1,2,3,\cdots,t$.

Let us first consider the case $G \cong mK_2$, where $m > 2$. Let $V(G) = \{u_1, v_1, u_2, v_2, \cdots, u_m, v_m\}$, and $E(G) = \{e_1, e_2, \cdots, e_m\}$, where $e_i = u_iv_i$ for $i = 1,2,\cdots,m$. Then we can find a Hamilton cycle of G^{+--} :

$$e_2u_1v_1e_3u_2v_2e_4u_3v_3\cdots e_{i+1}u_iv_i\cdots e_mu_{m-1}v_{m-1}e_1u_mv_me_2.$$

If $G \cong m_1K_2 + K_{1,r_1} + K_{1,r_2} + \cdots + K_{1,r_t}$, let $G' = K_{1,r_1} + K_{1,r_2} + \cdots + K_{1,r_t}$ and $V(K_{1,r_i}) = \{v_0^i, v_1^i, \cdots, v_{r_i}^i\}$, $E(K_{1,r_i}) = \{e_1^i, e_2^i, \cdots, e_{r_i}^i\}$, where $e_j^i = v_0^iv_j^i$ for $i = 1,2,\cdots,t$, $j = 1,2,\cdots,r_i$.

Let P_k be $e_{r_k}^k v_1^k v_0^k v_2^k e_1^k v_3^k e_2^k \cdots v_{i+1}^k e_i^k \cdots v_{r_k}^k e_{r_k-1}^k$. Then P_k is a Hamilton path of K_{1,r_k}^{+--} for $k = 1,\cdots,t$. Then $e_{r_1}^1 P_1 e_{r_1-1}^1 e_{r_2}^2 P_2 e_{r_2-1}^2 \cdots e_{r_t}^t P_t e_{r_t-1}^t$ is a Hamilton path of G'^{+--}. We denote it simply by P.

If $m \geq 3$, $(m_1K_2)^{+--}$ has a Hamilton path P'. By simply connecting one end vertex of P' to $e_{r_1}^1$, and the other end vertex of P' to $e_{r_t-1}^t$, one can obtain a Hamilton cycle of G^{+--}.

If $m = 2$, we label the vertices of $2K_2$ as $\{u_1, v_1, u_2, v_2\}$ and $E(2K_2) = \{e_1, e_2\}$, $e_i = u_iv_i$ for $i = 1,2$. Then $u_1v_1e_2e_1v_2u_2$ is a Hamilton path of $(2K_2)^{+--}$. By the similar way as in the previous argument, one can obtain a Hamilton cycle of G^{+--}.

If $m = 1$, we denote the isolated edge by e, and let u, v be two end vertices of G. Then $P + u + v + e + uv + ue_{r_1}^1 + ve_{r_t-1}^t - v_1^1v_0^1 + v_1^1e + ev_0^1$ is a Hamilton cycle of G^{+--}.

Case 2. $p \leq \Delta'(G)$

By Theorem 2.3, $\kappa(G^{+--}) = \delta(G^{+--})$. Recall that $\delta(G^{+--}) = min\{q, p + q - \Delta'(G) - 1\}$.

Subcase 2.1. $q \geq \Delta'(G) + 1$

Then $p + q - \Delta'(G) - 1 \geq p$, and thus $\delta(G^{+--}) \geq p$. Since $p \geq \alpha(G)$ and $p \geq \Delta(G) + 1$ always hold, we have $\kappa(G^{+--}) = \delta(G^{+--}) \geq p \geq max\{\alpha(G), \Delta(G) + 1\} = \alpha(G^{+--})$, and G^{+--} is hamiltonian by Theorem 2.1.

Subcase 2.2. $q \leq \Delta'(G)$.

Then $max\{p, q\} \leq \Delta'(G)$, by Lemma 3.3, G^{+--} is hamiltonian.

The proof is complete. □

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