Paths in the Square Unit Lattice

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Abstract

An n−bit binary string $(x_1, x_2, \ldots, x_n), x_i = \pm 1$ and its partial sums $s_j = x_1 + x_2 + \ldots + x_j$ determine paths in the unit lattice $\{(x, y) \mid x, y \text{ integers}\}$ with "steps" / \ . We enumerate classes of these paths satisfying restrictions. These include well-known counts (e.g., Catalan numbers and Chung-Feller numbers) and new counts, some presented as problems with solutions.

1 Introduction

A lower case Greek letter denotes a binary string (bits ± 1). Let $x_i = x_i(\delta)$ denote the i^{th} entry of the string δ, and $\ell(\delta)$ the length of the string. If $\ell(\delta) = n \geq 1$,

$$\delta = (x_1, \ldots, x_n), \quad x_i = x_i(\delta) \in \{1, -1\}, \quad i = 1, \ldots, n, \ n \geq 1. \tag{1}$$

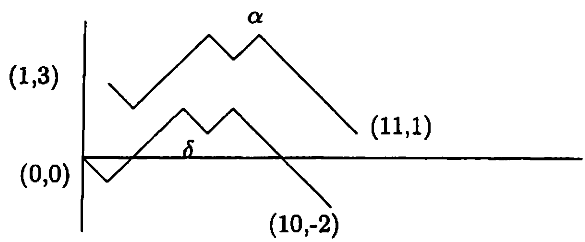
Let $s_j = s_j(\delta)$ denote the j^{th} partial sum, the sum of the first j entries of δ:

$$s_j = x_1 + x_2 + \ldots + x_j, \qquad j = 1, 2, \ldots, n. \tag{2}$$

Corresponding to a string δ there is a path in the square unit lattice $\{(x, y) \mid x, y \text{ integers}\}$ which starts at the point $(0, 0)$, proceeds by *steps* (straight line segments from (x, y) to $(x+1, y+1)$ or to $(x+1, y-1)$) to $(1, s_1(\delta))$, then in turn to $(2, s_2(\delta)), \ldots, (n, s_n(\delta))$. The congruent-by-translation path α which starts at (a, u), ends at $(a+n, u+s_n(\delta))$(Fig. 1).

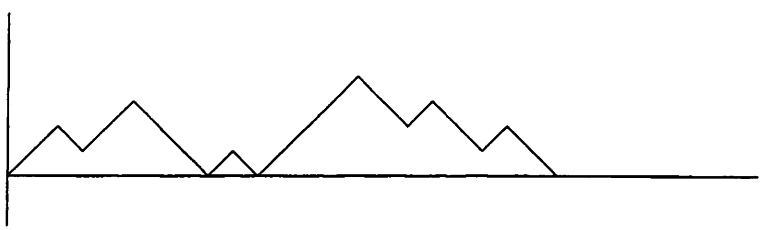
For example, a path δ which starts at $(0, 0)$ and ends at $(2n, 0)$ and satisfies $s_i(\delta) \geq 0, 0 \leq i \leq 2n$ (no point of the path lies below the x-axis) is called a Catalan path of length $2n$ (Figure 2). $C(n)$ denotes the cardinality of the set of Catalan paths of length $2n$; it is well-known that

$$C(n) = \frac{1}{n+1}\binom{2n}{n}, \qquad n \geq 1. \tag{3}$$



$\delta = (-1,1,1,1,-1,1,-1,-1,-1,-1)$, $s(\delta) = (-1,0,1,2,1,2,1,0-1,-2)$

Figure 1



$\delta = (1,1,-1,1,1,-1,-1,-1,1,-1,1,1,1,1,-1,-1,1,-1,-1,1,-1,-1)$

A Catalan Path

Figure 2

Our objective is to enumerate, in a simple way, several classes of paths. Some of what we write appears in some form in the literature ([1],[2], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]), published over many decades and in different locations. It seemed to us desirable to simplify presentations, add some new proofs and new counts, and make it all accessible in one location. In Section 2 we state more definitions and prove basic properties of strings and paths. In Section 3 we prove theorems, including (3), the remarkable Raney's theorem, and enumerations of subsets of paths in the context of voting patterns in a two-person contest. In Section 4 we find new enumerations (of classes of paths satisfying restrictions), posing them as problems with solutions.

2 Preliminaries

For convenience we take

$$\binom{n}{k} = \begin{cases} n!/k!(n-k)! & \text{if } 0 \le k \le n, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

(so that $\binom{n}{k} = 0$ when $k < 0$ or $n < 0$ or $0 \le n < k$), and occasionally implicitly use the equality

$$\binom{n}{k} = \binom{n-1}{k-1}\frac{n}{k} .$$

Lemma 1 ([11] p. 70) *For given $p \ge 0, q \ge 0$, the number of strings which have p bits 1 and q bits -1 is*

$$\binom{p+q}{p} = \binom{p+q}{q}. \tag{5}$$

Proof. Place $p+q$ symbols -1 in a row; choose p of them ($\binom{p+q}{p}$ choices), and delete from them the minus signs (making them 1's). The $\binom{p+q}{p}$ strings we have constructed are precisely those we want. □

The string $\delta = (x_1, x_2, \ldots, x_n)$ has length n. If it has p bits 1 and q bits -1, then

$$\begin{aligned} n &= p+q \\ s_n &= p-q \end{aligned} \qquad \text{and} \qquad \begin{aligned} p &= (n+s_n)/2 \\ q &= (n-s_n)/2 . \end{aligned} \tag{6}$$

Consequently n and $s_n(\delta)$ are both even or both odd.

Let $f(n,k)$ denote the number of paths which start at $(0,0)$ and end at (n,k), $n \ge 1$, $-n \le k \le n, k \equiv n \pmod 2$. Lemma 1 has the equivalent version

$$f(n,k) = \binom{n}{\frac{n+k}{2}} = \binom{p+q}{p} \text{ where } \begin{matrix} p+q=n \\ p-q=k \end{matrix} .$$

Another useful equivalent version is

Lemma 2 *Given two lattice points $(a,u), (b,v)$ with $a < b$, $-(b-a) \le v-u \le b-a$, $v-u \equiv b-a \pmod 2$, the number of paths from (a,u) to (b,v) is*

$$f(b-a, v-u) = \binom{b-a}{\frac{b-a+v-u}{2}} = \binom{b-a}{\frac{b-a-(v-u)}{2}} . \qquad \square$$

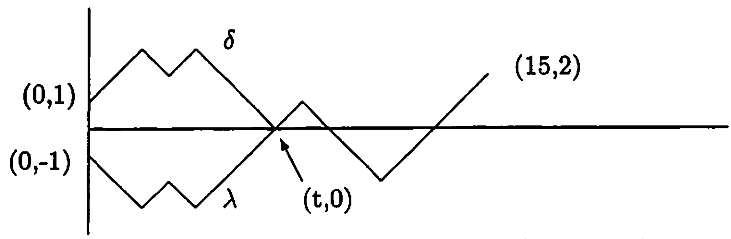
Note that the condition $-(b-a) \le v-u \le b-a$ on $v-u$ is unnecessary because of the definition (4) of $\binom{n}{k}$.

Lemma 3 *The number of strings with length n (paths starting at arbitrary point (a,u) and length n) is*

$$\sum_{k\equiv n (\text{mod } 2)} f(n,k) = \sum_{k\equiv n (\text{mod } 2)} \binom{n}{\frac{n+k}{2}} = \sum_{p\ge 0}\binom{n}{p} = 2^n. \qquad \square$$

Lemma 4 (A useful reflection, first seen by Aebly [1] and Mirimanoff [13]; used by D. André [2], and in [10] and many others.) *For given* $0 \le k \le n, n \ge 1, n \equiv k \pmod 2$, *the number of paths from* $(0,1)$ *to* $(n, k+1)$ (*the paths have length* n *and non-negative rise* k) *which meet (cross or touch) the* x*-axis is*

$$\binom{n}{\frac{n+k+2}{2}}.$$

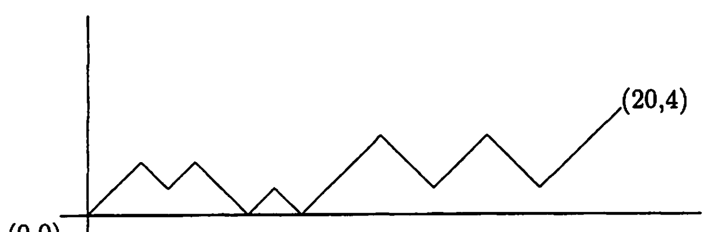


$\delta = (1,1,-1,1,-1,-1,-1,1,-1,-1,-1,1,1,1,1), n = 15, k = 1, t = 7$

Figure 3

Proof. Consider a path $\delta = (x_1, \ldots, x_n)$ which joins $(0,1)$ to $(n, k+1)$ and which meets the x-axis (see Figure 3). Let t be the abscissa of the first point on the path with ordinate 0. Corresponding to δ there is the unique path λ from $(0,-1)$ to $(n, k+1)$ obtained by reflecting across the x-axis the part of the path δ from $(0,1)$ to $(t,0)$ (See Figure 3). The number of the paths we seek is the number of paths λ, which (by Lemma 2) is

$$\binom{n-0}{\frac{(n-0)+((k+1)-(-1))}{2}} = \binom{n}{\frac{n+k+2}{2}}. \qquad \square$$



$n = 20, k = 4, (0,0)$ to (n,k), no point below $x-$axis

Figure 4

3 Theorems

Theorem 1 *For* $n \ge 1, k \ge 0, k \equiv n \pmod 2$, *let* $g(n,k)$ *denote the number of paths from* $(0,0)$ *to* (n,k) *with no point below the* x*-axis (see Figure 4* *for example*);

$$g(n,k)=\binom{n}{\frac{n+k}{2}}\frac{2k+2}{n+k+2}=\binom{p+q}{p}\frac{p-q+1}{p+1}\text{where}\begin{matrix}p+q=n\geq 1\\ p-q=k\geq 0\end{matrix}. \quad (7)$$

In terms of strings this is the number of binary strings $\delta=(x_1,x_2,\ldots,x_n)$ *satisfying*

$$s_1=1, s_2\geq 0, s_3\geq 0,\ldots,s_{n-1}\geq 0, s_n=k\geq 0.$$

Proof. Apply the translation $(x,y)\to(x,y+1)$ to the paths; the translated copies are paths from $(0,1)$ to $(n,k+1)$ (these paths have length n and rise $k\geq 0$) with no point below the line $y=1$ (see Figure 5 for example). The number of these paths is the number of paths from $(0,1)$ to $(n,k+1)$ less the number of paths from $(0,1)$ to $(n,k+1)$ which meet the x-axis which by Lemma 2 and Lemma 4, is

$$\binom{n}{\frac{n+k}{2}}-\binom{n}{\frac{n+k+2}{2}}=\binom{n}{\frac{n+k}{2}}\frac{2k+2}{n+k+2}.$$

(1,1,-1,1,-1,-1,1,1,-1,1,1,-1-1,-1,1,1,1,1)

$n=17, k=3, (0,1)$ to $(n,k+1)$ no point below $y=1$

Figure 5

For another proof see [5]. For a generalization to n dimensions see [18]. This theorem may be considered a generalization of

Theorem 2

$$C(m)=\binom{2m}{m}\frac{1}{m+1},\qquad m\geq 1.$$

Proof. The Catalan number $C(m)$, $m\geq 1$, is the number of paths from $(0,0)$ to $(2m,0)$ with no point below the x-axis. Take $n=2m, k=0$ in (7), Theorem 1. □

The translation $(x,y)\to(x+1,y+1)$ takes a path, from $(0,0)$ to (n,k), $n\geq 1, k\geq 0$, no point below the x-axis (a path counted in Theorem

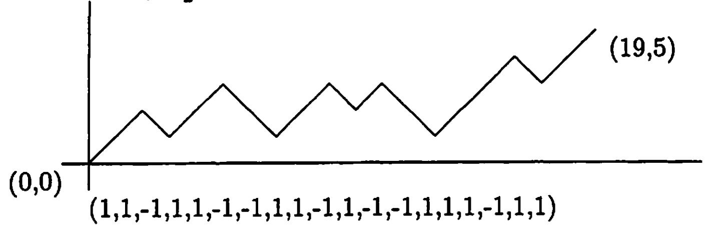
1) to a path from $(1,1)$ to $(n+1,k+1)$, $n \geq 1, k \geq 0$, no point below line $y = 1$, so there are $g(n,k)$ of these paths. Replacing n by $n-1$ and k by $k-1$ we have

Theorem 3 *For given $n \geq 1, k \geq 1, k \equiv n$ (mod 2), the number of paths from $(0,0)$ to (n,k) with string $\delta = (x_1, x_2, \ldots x_{n-1}, x_n)$ satisfying*

$$s_1 = 1, s_2 \geq 1, s_3 \geq 1, \ldots, s_{n-1} \geq 1, s_n = k \geq 1$$

(see example in Figure 6) is

$$g(n-1,k-1) = \binom{n-1}{\frac{n+k-2}{2}} \frac{2k}{n+k} = \binom{p+q}{p} \frac{p-q}{p+q} \text{ where } \begin{matrix} p+q=n \geq 1 \\ p-q=k \geq 0 \end{matrix}.$$



$n = 19, k = 5, (0,0)$ to $(n,k), s_i \geq 1$ for $1 \leq i \leq n$

Figure 6

Let $G(n)$ denote the number of paths from $(0,0)$ with length $n \geq 1$ and no point below the x-axis:

$$G(n) = \sum_{\substack{k \equiv n (mod\ 2) \\ k \geq 0}} g(n,k) = \sum_{\substack{k \equiv n (mod\ 2) \\ k \geq 0}} \binom{n}{\frac{n-k}{2}} \frac{2k+2}{n+k+2}.$$

Calculating $G(n)$ for small values of n we find that

n	1	2	3	4	5	6	7	8
$G(n)$	1	2	3	6	10	20	35	70

We recognize that for $0 \leq n \leq 8$, the numbers $G(n)$ are the "central" elements $\binom{n}{\lfloor n/2 \rfloor}$ in Pascal's triangle of binomial coefficients! Hence we are led to

Theorem 4 ([5],[11])

$$G(n) = \binom{n}{\lfloor n/2 \rfloor}, \qquad n \geq 1.$$

Proof. Case: n is even, say $n = 2m$.

$$G(n) = \sum_{\substack{k\equiv n(\text{mod } 2)\\ k\geq 0}} \binom{n}{\frac{n-k}{2}} \frac{2k+2}{n+k+2} = \sum_{i=0}^{m} \frac{4i+2}{2m+2i+2} \binom{2m}{\frac{2m-2i}{2}}$$

$$= \sum_{i=0}^{m} \frac{2i+1}{m+i+1} \binom{2m}{m-i} = \sum_{i=0}^{m} \frac{2i+1}{2m+1} \binom{2m+1}{m-i}$$

(and letting $j = m - i$)

$$= \sum_{j=0}^{m} \frac{2m-2j+1}{2m+1} \binom{2m+1}{j} = \sum_{j=0}^{m} \binom{2m+1}{j} - 2\sum_{j=1}^{m} \frac{j}{2m+1} \binom{2m+1}{j}$$

$$= 1 + \sum_{j=1}^{m} \left[\binom{2m}{j-1} + \binom{2m}{j} \right] - 2\sum_{j=1}^{m} \binom{2m}{j-1}$$

$$= 1 + \sum_{j=1}^{m} \left[\binom{2m}{j} - \binom{2m}{j-1} \right] = \binom{2m}{m}.$$

The case n is odd is similar. □

Theorem 5 *For given $n \geq 1$, the number of strings*

$$s_j = x_1 + x_2 + \ldots + x_j, \qquad j = 1, 2, \ldots, n. \tag{8}$$

of length n satisfying

$$s_1 = 1, s_2 \geq 1, s_3 \geq 1, \ldots, s_{n-1} \geq 1, s_n \geq 1$$

is

$$G(n-1) = \binom{n-1}{\lfloor (n-1)/2 \rfloor}.$$ □

Strings (1) and their corresponding paths from $(0, 0)$ to (n, k) describe a voting pattern in an election with two competing candidates A and B. The votes are recorded, as they come in, with a 1 if A is chosen and -1 if B is chosen. The record of an electorate of n voters is a string (1); there are 2^n such sequences.

In terms of voting preferences in such a contest the number of sequences (binary strings of length n) in which:

(i) (Theorem 1, [7]) A always has at least as many votes as B and at the last vote (n^{th} voter) has $k \geq 0$ votes more than B is $g(n, k)$, with probability

$$g(n,k) \Big/ \binom{n}{\frac{n+k}{2}} = \binom{n}{\frac{n+k}{2}} \frac{2k+2}{n+k+2} \Big/ \binom{n}{\frac{n+k}{2}} = \frac{p-q+1}{p+1};$$

(ii) (Theorem 4) A always has at least as many votes as B is $G(n)$, with probability

$$G(n) \Big/ 2^n = \binom{n}{\lfloor n/2 \rfloor} \Big/ 2^n;$$

(iii) (Theorem 3, [2], [4], [8], [10], [14], [15]) A always has more votes than B and at the last vote (n^{th} voter) A has $k > 0$ votes more than B is $g(n-1, k-1)$, with probability

$$g(n-1, k-1) \Big/ \binom{n}{\frac{n+k}{2}} = \binom{n-1}{\frac{n+k-2}{2}} \frac{2k}{n+k} \Big/ \binom{n}{\frac{n+k}{2}} = \frac{k}{n} = \frac{p-q}{p+q};$$

(iv) (Theorem 5) A always has more votes than B is $G(n-1)$, with probability

$$G(n-1) \Big/ 2^n = \binom{n-1}{\lfloor (n-1)/2 \rfloor} \Big/ 2^n. \qquad \square$$

The *circular shift* operator R maps δ onto δ^R whose j^{th} entry is $x_j(\delta^R) = x_{j+1}(\delta)$ ($x_m(\delta) = x_{m-n}(\delta)$ when $m > n$), i.e.,

$$\delta^R = (x_2(\delta), x_3(\delta), \ldots, x_n(\delta), x_1(\delta)).$$

Note that $\{E, R, R^2, \ldots, R^{n-1}\}$ form a cyclic group of order n, where $R^0 = E$ is the identity element and $R^i = R^{i-n}$ if $i \geq n$. Iterating $x_j(\delta^R) = x_{j+1}(\delta)$, we can state succinctly

$$x_j(\delta^{R^i}) = x_{j+i}(\delta), \qquad 0 \leq i \leq n-1, \quad 1 \leq j \leq n. \tag{9}$$

We will need the following two numbers determined by a string δ:

$$m(\delta) = \min\{s_1(\delta), s_2(\delta), \ldots, s_n(\delta)\} \tag{10}$$

and

$$t(\delta) = \max\{r \mid s_r(\delta) = m(\delta), 1 \leq r \leq n\}. \tag{11}$$

Note that $m(\delta) \leq s_1(\delta) = x_1(\delta) \leq 1$ and that

$$s_j(\delta) \begin{cases} \geq m(\delta) & \text{for } 1 \leq j \leq t(\delta), \\ > m(\delta) & \text{for } t(\delta) < j \leq n. \end{cases}$$

Theorem 6 (Raney's Lemma, [12] pp. 345–346) *If δ is any n-bit binary (bits ± 1) string* (1) *with $s_n(\delta) = 1$, then exactly one of δ's n circular shifts has all partial sums positive, i.e.,*

(i) (*existence*) *there exists* k $(0 \le k \le n-1)$ *such that* $s_j(\delta^{R^k}) \ge 1$ *for all* $1 \le j \le n$;

(ii) (*uniqueness*) *for every* $d \ne k$ $(0 \le d \le n-1)$, δ^{R^d} *has at least one partial sum which is not positive, i.e.,* $s_j(\delta^{R^d}) \le 0$ for at least one $j \in \{1, 2, \ldots, n\}$.

Proof. We claim that $k = t(\delta)$ satisfies (i) and (ii). For

$$
\begin{aligned}
s_j(\delta^{R^k}) = s_j(\delta^{R^{t(\delta)}}) &= x_1(\delta^{R^{t(\delta)}}) + x_2(\delta^{R^{t(\delta)}}) + \ldots + x_j(\delta^{R^{t(\delta)}}) \\
&= x_{1+t(\delta)}(\delta) + x_{2+t(\delta)}(\delta) + \ldots + x_{j+t(\delta)}(\delta) \\
&= \begin{cases} s_{j+t(\delta)}(\delta) - s_{t(\delta)}(\delta) & \text{if } 1 \le j \le n - t(\delta), \\ s_n(\delta) - s_{t(\delta)}(\delta) + s_{j+t(\delta)-n}(\delta) & \text{if } n - t(\delta) + 1 \le j \le n. \end{cases} \qquad (12) \\
&\ge \begin{cases} (1 + m(\delta)) - m(\delta) & \text{if } 1 \le j \le n - t(\delta), \\ 1 - m(\delta) + m(\delta) & \text{if } n - t(\delta) < j \le n. \end{cases}
\end{aligned}
$$

Thus

$$s_j(\delta^{R^k}) \ge 1 \text{ for all } 1 \le j \le n,$$

so $k = t(\delta)$ satisfies (i). (An example in Figure 7)

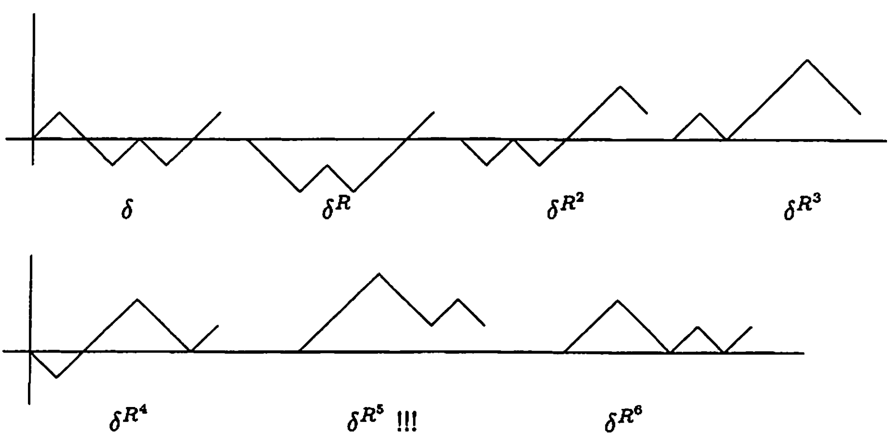


Figure 7

We prove (ii) in two cases.

<u>First case</u>: $0 \le d < k \le n, k = t(\delta)$. In this case we have

$$\begin{aligned} s_{k-d}(\delta^{R^d}) &= x_1(\delta^{R^d}) + x_2(\delta^{R^d}) + \ldots + x_{k-d}(\delta^{R^d}) \\ &= x_{1+d}(\delta) + x_{2+d}(\delta) + \ldots + x_{k-d+d}(\delta) \\ &= s_k(\delta) - s_d(\delta) = m(\delta) - s_d(\delta) \le 0 \end{aligned}$$

(the last inequality holds because $s_j(\delta) \ge m(\delta)$ for every $1 \le j \le n$).

<u>Second case</u>: $0 \le k < d \le n$. In this case we have

$$\begin{aligned} s_{n-d+k}(\delta^{R^d}) &= x_1(\delta^{R^d}) + x_2(\delta^{R^d}) + \ldots + x_{n-d}(\delta^{R^d}) \\ &\quad + x_{n-d+1}(\delta^{R^d}) + x_{n-d+2}(\delta^{R^d}) + \ldots + x_{n-d+k}(\delta^{R^d}) \\ = x_{1+d}(\delta) &+ x_{2+d}(\delta) + \ldots + x_n(\delta) + x_1(\delta) + x_2(\delta) + \ldots + x_k(\delta) \\ &= s_n(\delta) - s_d(\delta) + s_k(\delta) = 1 - s_d(\delta) + m(\delta) \le 0 \end{aligned}$$

(the last inequality holds because $s_d(\delta) \ge m(\delta) + 1$ when $0 \le k < d \le n$).

A pleasing generalization of Raney's Lemma is

Theorem 7 Generalization of Raney's Lemma *If n and k are (fixed) integers, $1 \le k \le n$, $n \equiv k \pmod 2$, and δ is any binary string (bits ± 1) of length n with $s_n(\delta) = k$ then exactly k of δ's n circular shifts have all partial sums positive.*

It turns out that the k circular shifts are $\delta^{R^{t_i}}, i = 1, 2, \ldots, k$, where

$$t_1 = t(\delta), t_2 = t(\delta^{R^{t_1}}), \ldots, t_k = t(\delta^{R^{t_{k-1}}}).$$

(See [9] for a proof.) □

The reflection of the string $\alpha = (x_1(\alpha), x_2(\alpha), \ldots, x_n(\alpha))$ across the x-axis is the string $-\alpha = (-x_1(\alpha), -x_2(\alpha), \ldots, -x_n(\alpha))$; the concatenation of α and $\beta = (x_1(\beta), x_2(\beta), \ldots, x_m(\beta))$ is the string
$\alpha \cdot \beta = (x_1(\alpha), x_2(\alpha), \ldots, x_n(\alpha), x_1(\beta), x_2(\beta), \ldots, x_m(\beta))$.

Let $\mathcal{H}(2n, 2k), 0 \le k \le n, n \ge 1$, denote the set of paths from $(0,0)$ to $(2n, 0)$ with $2k$ steps above the x-axis (and $2n - 2k$ steps below the x-axis).

Theorem 8 (The Chung-Feller Numbers, [11] p. 72, [5], [6], [7])
For $n \in \{1, 2, 3, \ldots\}$, and $k = 0, 1, 2, \ldots, n$,

$$\#\mathcal{H}(2n, 2k) = C(n) = \frac{1}{n+1}\binom{2n}{n} \quad \text{(independent of } k!!) .$$

($\#\mathcal{A}$ denotes the cardinality of the set $\mathcal{A}$.)

Solution (following the proof in [6], cf. [10] p. 341). We describe a bijection between $\mathcal{H}(2n, 2k)$ and $\mathcal{H}(2n, 2k+2)$ for $0 \le k \le n-1$. From this it will follow that

$$\#\mathcal{H}(2n, 0) = \#\mathcal{H}(2n, 2) = \#\mathcal{H}(2n, 4) = \ldots = \#\mathcal{H}(2n, 2n) = \frac{1}{n+1}\binom{2n}{n} .$$

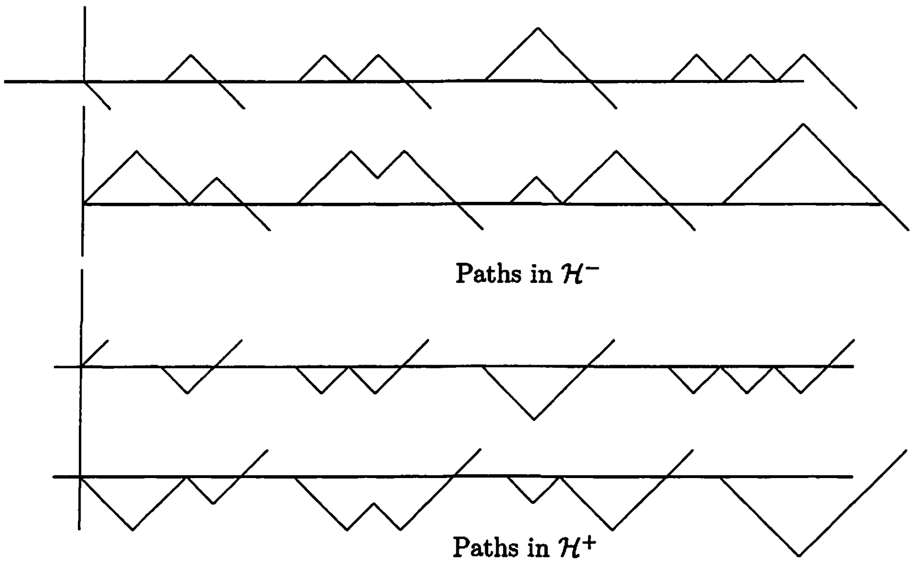


Figure 8

The bijection is determined as follows. Let $\mathcal{H}^-$ denote the set of paths obtained by extending a Catalan string (path) by the bit -1 (the empty string is Catalan of length 0), and $\mathcal{H}^+ = \{\delta | -\delta \in \mathcal{H}^-\}$. See Figure 8 for examples of such paths. Consider a string $\delta \in \mathcal{H}(2n, 2k), 0 \leq k \leq n-1$. It is not difficult to see that

$$\delta = \alpha \cdot \beta \cdot \gamma,$$

where α, β, γ are unique strings: $\alpha \in \mathcal{H}^-$, $\beta \in \mathcal{H}^+$, and γ is the remainder of the string (possibly empty), respectively. Then the mapping

$$T(\delta) = \beta \cdot \alpha \cdot \gamma$$

is the bijection we seek, i.e., if $\delta = \alpha \cdot \beta \cdot \gamma$ has $2k$ $(0 \leq 2k < 2n)$ positive steps (steps above the x-axis) then $T(\delta) = \beta \cdot \alpha \cdot \gamma$ has $2k+2$ positive steps. (See Figure 9 for several examples.)

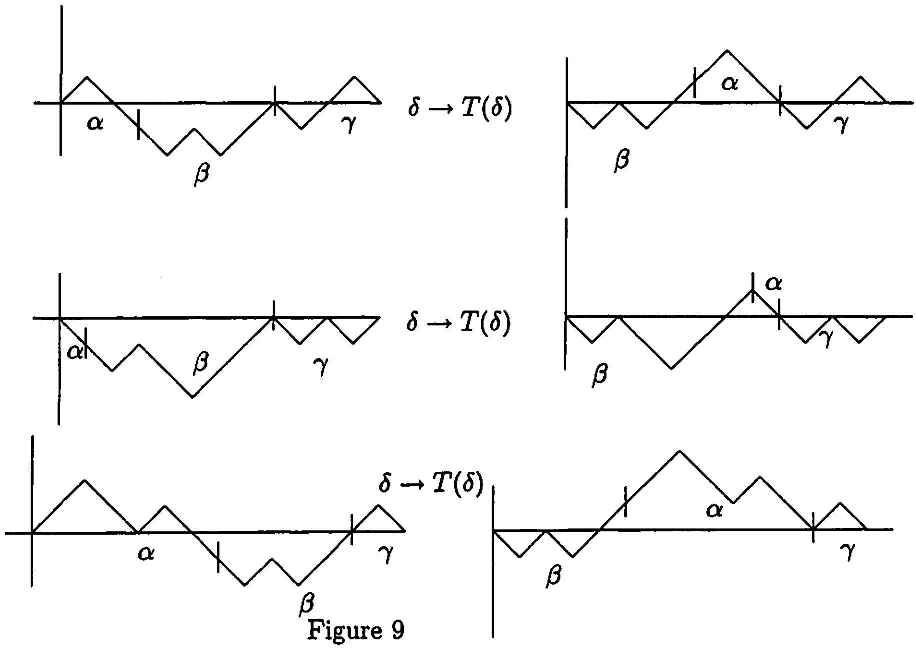


Figure 9

To see that this is so, let $\lambda^{\text{pos}}_{(a,b)}$ denote the number of positive steps in the path with string λ and initial point (a, b). Then

$$\delta^{\text{pos}}_{(0,0)} = (\alpha\beta\gamma)^{\text{pos}}_{(0,0)} = \alpha^{pos}_{(0,0)}\beta^{\text{pos}}_{(\ell(\alpha)-1,-1)}\gamma^{\text{pos}}_{(0,0)} = (\ell(\alpha)-1)+0+\gamma^{\text{pos}}_{(0,0)} \, ,$$

while

$$\begin{aligned}(T(\delta))^{\text{pos}}_{(0,0)} = (\beta\alpha\gamma)^{\text{pos}}_{(0,0)} &= \beta^{\text{pos}}_{(0,0)}\alpha^{\text{pos}}_{(\ell(\beta),1)}\gamma^{\text{pos}}_{(0,0)} \\ &= 1+\ell(\alpha)+\gamma^{\text{pos}}_{(0,0)} = \delta^{\text{pos}}_{(0,0)}+2 \, . \qquad \square\end{aligned}$$

4 Problems and solutions

We will have occasion to use the following more or less obvious

Lemma 5 *The number of distributions of $n \geq 1$ indistinguishable objects into $k \geq 1$ distinguishable boxes is*

$$\binom{n+k-1}{k-1} = \binom{\#\text{objects} + \#\text{boxes} - 1}{\#\text{boxes} - 1}. \qquad (13)$$

To see this, observe that each distribution can be seen as a linear display of n symbols $\bullet$ and $k-1$ symbols $/$:

$$\underbrace{\bullet\bullet\cdots\bullet}_{\text{box 1}}/\underbrace{\bullet\bullet\cdots\bullet}_{\text{box 2}}/\cdots/\underbrace{\bullet\bullet\cdots\bullet}_{\text{box k}}$$

All such displays are created by lining up $n+k-1$ $\bullet$'s, choosing $k-1$ of them in $\binom{n+k-1}{k-1}$ ways, and changing the chosen $\bullet$'s into strokes. □

Problem 1 (*The view of the side of a mountain range*). *Given* $n \geq 1$, $0 \leq k \leq n$, $k \equiv n \pmod 2$, *find the number of paths* $\delta = (x_1, x_2, \ldots, x_n)$ *from* $(0,0)$ *to* (n,k), *with* $x_1 = 1$ *and every bit* -1 *followed by at least one bit* 1.

(Note. We make the condition $x_1 = 1$ so that the path will look like the side of a mountain range: the first step should be an up step.)

Solution. We want to count all the strings δ with $p = \frac{n+k}{2}$ symbols 1 and $q = \frac{n-k}{2}$ symbols -1, with x_1=1 and every bit -1 followed by at least 1 bit 1. Construct these δ's as follows. Place q bits -1 in a row, creating $q+1$ boxes: the two at the ends and the $q-1$ "in-between" boxes. Place a single 1 into each of these q+1 boxes. Distribute the remaining p-(q+1) 1's into the q+1 boxes without restriction in

$$\binom{(p-q-1)+(q+1)-1}{(q+1)-1} = \binom{p-1}{q} = \binom{n-q-1}{q}$$

ways, and we have the strings we want. □

Corollary The number of paths δ of length n from $(0,0)$ with x_1=1 and every bit -1 followed by at least one bit 1 is

$$a_n = \sum_q \binom{n-q-1}{q}, \tag{14}$$

which are known ([16] p. 14) and easily seen to satisfy the recurrence

$$a_n = a_{n-1} + a_{n-2}, \quad n \geq 3, \qquad a_1 = 1, \quad a_2 = 1,$$

so they are the Fibonacci numbers: $a_n = F_n$. The inequality $0 \leq q \leq (n-1)/2$ on the running index q is unnecessary because of our agreement to take $\binom{n}{m} = 0$ if $n < 0$ or $m < 0$ or $m > n \geq 0$. □

(Note. If the condition $x_1 = 1$ is deleted the number of paths is $\sum \binom{n-q}{q}$.)

Problem 2 (Generalization of Problem 1) *For given* $n \geq 1, -n \leq k \leq n$, $n \equiv k \pmod 2, w \geq 0$, *find the number of paths* $\delta = (x_1, x_2, \ldots, x_n)$ *from* $(0,0)$ *to* (n,k), *with* $x_1 = 1$ *and every bit* -1 *followed (immediately) by at least* w *bits 1.*

Solution. We want to count all the strings δ with $p = \frac{n+k}{2}$ symbols 1 and $q = \frac{n-k}{2}$ symbols -1, with $x_1 = 1$ and every bit -1 followed (immediately) by at least w bits 1. Construct these δ's as follows. Place q bits -1 in a row, creating $q+1$ boxes: the two at the ends and the $q-1$ "in-between" boxes, labeled $B_1, B_2, \ldots, B_{q+1}$ from left to right. Place a single 1 into B_1 and w 1's into the other q boxes. Distribute the remaining $p - 1 - qw$ 1's into the $q+1$ boxes without restriction in

$$\binom{(p-1-qw)+(q+1)-1}{(q+1)-1} = \binom{n-qw-1}{q}$$

ways, and we have the strings we want. □

Corollary The number of paths δ of length n from $(0,0)$ with $x_1 = 1$ and every bit -1 followed immediately by w 1's

$$a_n^{(w)} = \sum_q \binom{n-qw-1}{q}. \tag{15}$$

These numbers are easily seen to satisfy the recurrence

$$a_n^{(w)} = a_{n-1}^{(w)} + a_{n-w-1}^{(w)}, \qquad n \geq w+2,$$

$$a_1^{(w)} = a_2^{(w)} = \ldots = a_{w+1}^{(w)} = 1.$$

Of course $a_n^{(0)} = 2^{n-1}$, while $a_n^{(1)} = a_n = F_n$ (Problem 1). The Table shows the values of $a_n^{(w)}$ for small values of n, w.

n	1	2	3	4	5	6	7	8	9	10	11	12
$a_n^{(0)}$	1	2	4	8	16	32	64	128	256	512	1024	2048
$F_n = a_n^{(1)}$	1	1	2	3	5	8	13	21	34	55	89	144
$a_n^{(2)}$	1	1	1	2	3	4	6	9	13	19	28	41
$a_n^{(3)}$	1	1	1	1	2	3	4	5	7	10	14	19
$a_n^{(3)}$	1	1	1	1	1	2	3	4	5	6	8	11

$F_n = a_n^{(1)}, a_n^{(2)}, a_n^{(3)}$ and $a_n^{(3)}$ are respectively the sequences A000045, A000930, A003269 and A003520 in Sloane's *The On-Line Encyclopedia of Integer Sequences* [17].

A substring of like bits not contained in a longer substring of like bits is called a *block*, e.g.,

$$(\ldots, 1\underbrace{-1,-1,-1,-1,-1,-1}_{\text{block of } -1\text{'s}}, 1\ldots) \qquad (\ldots,-1,\underbrace{1,1,1,1,1,1}_{\text{block of } 1\text{'s}},-1\ldots)$$

Note the intimate relation between binary n-strings and compositions of n. If an n-bit binary string δ has d blocks , then the lengths $a_1, a_2, \ldots, a_d$ of the blocks sum to n:

$$a_1 + a_2 + \ldots + a_d = n; \tag{16}$$

this is a composition of n into d positive parts. Furthermore, a composition (16) of n determines two n-strings, both with blocks of lengths $a_1, a_2, \ldots, a_d$, whose first block consists of 1's or -1's and succeeding blocks alternate between -1's and 1's or 1's and -1's. Of course the n-strings $(-1,\ -1, \ldots, -1)$ and $(1,\ 1, \ldots, 1)$ have only one block.

A bit is said to be *isolated* if it is a block of length one. Thus, in $(\cdots 1,\ 1,\ -1,\ 1, \cdots)$ the -1 is isolated, and in $(1,\ -1,\ -1,\ -1, \cdots)$ the 1 is isolated.

In n-strings which contain both -1 bits and 1 bits, the blocks of -1's alternate with blocks of 1's.

Problem 3 *For fixed* $n \geq 2,\ 0 \leq r \leq n$, *find the number* $(n|I(r))$ *of* $n-$*strings with exactly* r *isolated bits*

Solution First, in the case where $r = n$ every bit is isolated, the only n-strings are the alternating strings $(1, -1, 1, -1, \ldots)$ and $(-1, 1, -1, 1, \ldots)$, so $(n|I(n)) = 2$. Note that if the n-string consists entirely of -1's or entirely of 1's then it has no isolated bits. We move on to the case $n \geq 2,\ r < n$, and construct these n-strings in subsets according to the number of blocks, say $b \geq 2$. Place $b-1$ strokes in a row, creating b distinguishable boxes. Choose r of the boxes, in $\binom{b}{r}$ ways, and place a single symbol x into each of the chosen boxes. (x's will shortly become -1's and 1's.) There are $b-r$ empty boxes. Into these, first put two x's and then distribute into them $n-r-2(b-r)$ symbols x without restriction, in $\binom{(n-r)-2(b-r)+(b-r)-1}{(b-r)-1}$ ways. We have $\binom{n-b-1}{b-r-1}$ linear displays of $b-1$ strokes (b boxes); r of these boxes each contain a single x and $b-r$ boxes each contain ≥ 2 x's. Choose -1 or 1 (2 ways), replace all the x's in the first box by the chosen symbol; all the x's are now determined. The answer is

$$(n|I(r)) = 2\sum_{b\geq 1} \binom{b}{r}\binom{n-b-1}{b-r-1}. \qquad \square$$

Taking $r = 0$ we have the

Corollary The number of n-strings with no isolated bits is

$$(n|I(0)) = 2\sum_{b\geq 1} \binom{n-b-1}{b-1} \tag{17}$$

whose values for $n = 2, 3, 4, \ldots, 13$ are

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$(n\|I(0))$	0	2	2	4	6	10	16	26	42	68	110	178	288

□

Problem 4 *For given $n \geq 1$ and $m \geq 1$, find the number of $n-$strings each with all its blocks of length $\geq m$.*

Solution. Create b boxes, and place m x's into each box. Distribute the remaining $n - bm$ x's into the b boxes (without restriction), in $\binom{n-bm+b-1}{b-1}$ ways. Finally, convert the x's into ± 1's; the answer to the problem is

$$2\sum_{b\geq 1}\binom{n-(m-1)b-1}{b-1}.$$ □

Corollary. Strings with no isolated bits are precisely those with $m = 2$:

$$2\sum_{b\geq 1}\binom{n-(2-1)b-1}{b-1} = 2\sum_{b\geq 1}\binom{n-b-1}{b-1}.$$ □

Problem 5 (*no block has length* ≥ 3) *Find the number of strings of length n and having no occurrence of* $1, 1, 1$ *nor* $-1, -1, -1$, *i.e., all blocks have length 1 or 2.*

Solution We construct the n-strings, $n \geq 3$, in subsets according to the number of blocks, say $b, b \geq 1$, each having length 1 or 2. If α of the blocks have length 1 and β have length 2, then $\alpha + \beta = b, \alpha + 2\beta = n$, so $\alpha = 2b - n$ and $\beta = n - b$. Now we construct the strings as follows. Place $b-1$ strokes in a line, creating b distinguishable boxes. Put a single symbol x into each of the b boxes, choose $n - b$ of the b boxes, in $\binom{b}{n-b}$ ways, and put another symbol x into each of these chosen boxes, so that they contain two x's each. Now choose either -1 or 1 (2 choices). If your choice is -1 (resp. 1), replace all the x's in the first box by -1's (resp. 1's), replace all the x's in the next box of x's by 1's (resp. -1's), continue replacing x's in succeeding boxes alternately by -1's and 1's. Delete the strokes, they are no longer needed. At this point we have: the number of n-strings with b blocks ($b \geq 1$) all of length at most 2 is

$$2\binom{b}{n-b}, \quad n \geq 2,\ b \geq 1, \tag{18}$$

and the number of n-strings is

$$2\sum_{b=1}\binom{b}{n-\ell} = 2F_{n+1}, \quad n \geq 1.$$

(cf. [17, A006355)]) □

Some of the solutions to these problems have appeared as solutions to combinatorial problems in other contexts. (See for example [3]).

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