A Note on a Cube-Packing Problem

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Abstract

For positive integer n, let $f_3(n)$ be the least upper bound of the sums of the lengths of the sides of n cubes packed into a unit cube C in three dimensions in such a way that the smaller cubes have sides parallel to those of C. In this paper, we improve the lower bound of $f_3(n)$.

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1 Introduction and known results

In 1932, Erdős defined a function $f(n)$ which denotes the maximum sum of the side lengths of n squares that can be packed into a unit square S. In [1], P. Erdős and Soifer gave some results concerning $f(n)$. In [3], this kind of packing and covering problem was generalized to the case of equilateral triangle. In [4], we generalized this kind of packing problem to the case in 3 dimensions, and give the definition of the packing function $f_3(n)$:

Definition 1. For positive integer n, let $f_3(n)$ be the least upper bound of the sums of the lengths of the sides of n cubes packed into a unit cube C in three dimensions in such a way that the smaller cubes have sides parallel to those of C.

Definition 2. For a cube C, dissect each of its sides into n equal parts, then through these dissecting points draw parallel surfaces of the surfaces of C, so we get a packing of C by n^3 cubes. Such a configuration is called an n^3−grid. When C is a unit cube, the packing is a standard n^3−packing.

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The following results were obtained in [4].

Theorem 3. (1) $f_3(k^3) = k^2$.

(2) $f_3(2) = 1$; *When* $1 < n \le 7$, $f_3(n) = \frac{n}{2}$.

(3) $f_3(n) \le n^{\frac{2}{3}}$.

(4) $f_3(k^3 - 1) \ge k^2 - \frac{1}{k}$; $f_3(n) \ge (n^{\frac{1}{3}} - 1)^2$.

The upper bound for $f_3(n)$ is sharp but the lower bound is not. In this paper, we give an improvement of this result.

2 The improvement of the lower bound of $f_3(n)$

Theorem 4. *When* $n > 8$ *and* $n \neq k^3$,

$$f_3(n) \ge \frac{n+1}{\left[n^{\frac{1}{3}}\right]} - \left[n^{\frac{1}{3}}\right] - 4. \tag{2.1}$$

Proof. When $n > 8$, for any $n \neq k^3$, there exists an integer $m \ge 2$ such that $m^3 + 1 < n < (m+1)^3$. Note that $(m+1)^3 = m^3 + 3m^2 + 3m + 1 = m^3 + 3m(m+1) + 1 = m^3 + 6m + 6(m-1) + \cdots + 6 + 1$. For any n between $m^3 + 1$ and $(m+1)^3$, we have the following equation:

$$n = m^3 + 6l_1 + 6l_2 + \cdots + 6l_k + l',$$

where $0 \le l' \le 5$, $m \ge l_1 > l_2 > \cdots > l_k \ge 1$.

If $l' = 0$, let $l = l'$, then n can be denoted by:

$$n = m^3 + 6l_1 + 6l_2 + \cdots + 6l_k - l.$$

When $1 \le l' \le 5$, set $l_0 = m+1, l_{k+1} = 0$. There must exist an integer i, $0 \le i \le k+1$, such that $l_i > l_{i+1} + 1$. Otherwise, $k = m$, and $l_1 = m, l_2 = m-1, \cdots, l_m = 1$. Thus

$$n = m^3 + 6\sum_{j=1}^{m} j + l' \ge m^3 + 6\sum_{j=1}^{m} j + 1 = (m+1)^3.$$

This contradicts the fact that $n < (m+1)^3$. If $i < k$, add 1 to the value of l_{i+1}, and let $l = 6 - l'$, then n can be denoted by

$$n = m^3 + 6l_1 + 6l_2 + \cdots + 6l_k - l,$$

where $1 \le l \le 5, m \ge l_1 > l_2 > \cdots > l_k \ge 1$. If $i = k$, still denote $l = 6 - l'$, then n can be denoted by

$$n = m^3 + 6l_1 + \cdots + 6l_k + 6l_{k+1} - l,$$

where $l_1 > l_2 > \cdots > l_k > l_{k+1} = 1, 1 \le l \le 5$.

Combining the case when $l' = 0$ and the case when $1 \le l' \le 5$, n always can be denoted by

$$n = m^3 + 6l_1 + 6l_2 + \cdots + 6l_k - l, \tag{2.2}$$

where $m \ge l_1 > l_2 > \cdots > l_k \ge 1$, $0 \le l \le 5$, $k \le m$. When $l_k = 1$, we can denote n by:

$$n = m^3 + 6l_1 + 6l_2 + \cdots + 6l_k + 1 - (l + 1), \tag{2.3}$$

Case 1: When $l_k \neq 1$, we can pack the n smaller cubes as follows:

First step: From a standard m^3-packing of C, remove an l_1^3-grid in the top left corner of the front and replace it with an $(l_1 + 1)^3$-grid packing the same volume. Then in the obtained $(l_1 + 1)^3$ cubes, remove l_1^3 cubes in the top left corner of the front and replace them with one cube which is denoted by A_1 packing the same volume. At the same time, in the remaining $(l_1+1)^3 - l_1^3 = 3l_1^2+3l_1+1$ cubes remove l cubes. Then according to the following Step 2, pack $n_1 = (l_1 - 1)^3 + 6l_2 + \cdots + 6l_k$ cubes in the cube A_1. Thus we pack

$$\begin{aligned}
& (m^3 - l_1^3) + (l_1 + 1)^3 - l_1^3 - l + (l_1 - 1)^3 + 6l_2 + \cdots + 6l_k \\
= \;& m^3 - l_1^3 + l_1^3 + 3l_1^2 + 3l_1 + 1 - l_1^3 + l_1^3 - 3l_1^2 + 3l_1 - 1 \\
& + 6l_2 + \cdots + 6l_k - l \\
= \;& m^3 + 6l_1 + 6l_2 + \cdots + 6l_k - l = n
\end{aligned}$$

small cubes into the unit cube.

Step 2: Pack n_1 smaller cubes into the cube A_1 as follows:

Make an $(l_1 - 3)^3$-grid of A_1, then remove an l_2^3-grid in the top left corner of the front and replace it with an $(l_2 + 1)^3$-grid packing the same volume. Then in the obtained $(l_2 + 1)^3$ cubes, remove l_2^3 cubes in the top left corner of the front and replace them with one cube which is denoted by A_2 packing the same volume. In A_2, according to the following Step 3, pack $n_2 = (l_2 - 1)^3 + 6l_3 + \cdots + 6l_k$ small cubes. Thus we pack

$$\begin{aligned}
& (l_1 - 1)^3 - l_2^3 + (l_2 + 1)^3 - l_2^3 + (l_2 - 1)^3 + 6l_3 + \cdots + 6l_k \\
= \;& (l_1 - 1)^3 + 6l_2 + \cdots + 6l_k = n_1.
\end{aligned}$$

small cubes into A_1.

Similar to Step 2, we finish Step 3, Step 4, ..., Step $k - 1$.

Step k: Pack $(l_{k-1} - 1)^3 + 6l_k$ smaller cubes in the cube A_{k-1} as follows:

Make an $(l_{k-1} - 1)^3$–grid of A_{k-1}, then remove an l_k^3-grid in the top left corner of the front and replace it with an $(l_k + 1)^3$-grid packing the same volume. Then in the obtained $(l_k + 1)^3$-grid, remove an l_k^3-grid in the top left corner of the front and replace it with an $(l_k-1)^3$-grid packing the same volume. Thus, we pack $(l_{k-1}-1)^3 - l_k^3 + (l_k+1)^3 - l_k^3 + (l_k-1)^3 = (l_{k-1}-1)^3 + 6l_k$ cubes in A_{k-1}.

From the above, it's easy to see:

$$
\begin{aligned}
& f_3(n) \\
\geq\ & \frac{1}{m}\left(m^3 - l_1^3\right) + \frac{1}{m}\frac{l_1}{(l_1+1)}\left[3l_1^2 + 3l_1 + 1 - l\right] \\
& + \frac{1}{m}\frac{l_1^2}{(l_1+1)}\left[\frac{1}{l_1-1}\left[(l_1-1)^3 - l_2^3\right] + \frac{l_2}{(l_1-1)(l_2+1)}(3l_2^2+3l_2+1)\right] \\
& + \frac{1}{m}\frac{l_1^2}{(l_1+1)(l_1-1)}\frac{l_2^2}{(l_2+1)}\left[\frac{1}{l_2-1}\left[(l_2-1)^3 - l_3^3\right] + \right. \\
& \left. \frac{l_3}{(l_2-1)(l_3+1)}(3l_3^2+3l_3+1)\right] \\
& + \cdots + \\
& \frac{1}{m}\left(\prod_{i=1}^{k-2}\frac{l_i^2}{(l_i+1)(l_i-1)}\right)\frac{l_{k-1}^2}{(l_{k-1}+1)}\left[\frac{1}{l_{k-1}-1}\left[(l_{k-1}-1)^3 - l_k^3\right] + \right. \\
& \left. \frac{l_k}{(l_{k-1}-1)(l_k+1)}(3l_k^2+3l_k+1) + \frac{l_k^2}{(l_{k-1}-1)(l_k+1)}\frac{(l_k-1)^3}{l_k-1}\right] \qquad (2.4)
\end{aligned}
$$

So

$$
\begin{aligned}
& mf_3(n) \\
\geq\ & m^3 + \left[\frac{l_1^2}{l_1^2-1}(l_1-1)^3 - l_1^3 + \frac{l_1(3l_1^2+3l_1+1-l)}{l_1+1}\right] + \\
& \frac{l_1^2}{l_1^2-1}\left[\frac{l_2^2}{l_2^2-1}(l_2-1)^3 - l_2^3 + \frac{l_2(3l_2^2+3l_2+1)}{l_2+1}\right] + \cdots + \\
& \prod_{i=1}^{k-2}\frac{l_i^2}{l_i^2-1}\left[\frac{l_{k-1}^2}{l_{k-1}^2-1}(l_{k-1}-1)^3 - l_{k-1}^3 + \frac{l_{k-1}(3l_{k-1}^2+3l_{k-1}+1)}{l_{k-1}+1}\right] + \\
& \prod_{i=1}^{k-1}\frac{l_i^2}{l_i^2-1}\left[\frac{l_k^2}{l_k^2-1}(l_k-1)^3 - l_k^3 + \frac{l_k(3l_k^2+3l_k+1)}{l_k+1}\right] \qquad (2.5)
\end{aligned}
$$

$$
\begin{aligned}
\geq\ & m^3 + \frac{l_1(4l_1+1)}{l_1+1} - l + \frac{l_1^2}{l_1^2-1}\frac{l_2(4l_2+1)}{l_2+1} + \frac{l_1^2}{l_1^2-1}\frac{l_2^2}{l_2^2-1}\frac{l_3(4l_3+1)}{l_3+1} \\
& + \cdots + \prod_{i=1}^{k-1}\frac{l_i^2}{l_i^2-1}\frac{l_k(4l_k+1)}{l_k+1}
\end{aligned}
$$

$$
\begin{aligned}
&\geq m^3 + \frac{l_1(4l_1+1)}{l_1+1} - l + \frac{l_2(4l_2+1)}{l_2+1} + \cdots + \frac{l_k(4l_k+1)}{l_k+1} \\
&= m^3 + 4(l_1 + l_2 + \cdots + l_k) + k - l - \frac{4l_1+1}{l_1+1} - \cdots - \frac{4l_k+1}{l_k+1} \\
&\geq n - 2(l_1 + l_2 + \cdots + l_k) + k - 4 - 4 - \cdots - 4 \\
&= n - 2(l_1 + l_2 + \cdots + l_k) - 3k \\
&\geq n - 2(2 + 3 + \cdots + m) - 3(m-1) \\
&= n - m(m+1) + 2 - 3(m-1) \\
&= (n+5) - m(m+1) - 3m, \qquad (2.6)
\end{aligned}
$$

where we use the fact that $k \leq m, m \geq l_1 > l_2 > \cdots > l_k \geq 2$. So

$$f_3(n) \geq \frac{n+5}{m} - m - 4 = \frac{(n+5)}{\left\lceil n^{\frac{1}{3}} \right\rceil} - \left\lceil n^{\frac{1}{3}} \right\rceil - 4. \qquad (2.7)$$

Case 2: When $l_k = 1$, we can denote n by the formula(2.3). Similar to Case 1, we can pack n small cubes into the unit cube as follows.

First step: From a standard m^3-packing of the unit cube C, remove an l_1^3-grid in the top left corner of the front and replace it with an $(l_1+1)^3$-grid packing the same volume. Then in the obtained $(l_1+1)^3$ cubes, remove l_1^3 cubes in the top left corner of the front and replace them with one cube which is denoted by B_1 packing the same volume. At the same time, in the remaining $(l_1+1)^3 - l_1^3 = 3l_1^2 + 3l_1 + 1$ cubes remove $l+1$ cubes. Then according to the following Step 2, pack
$(l_1-1)^3 + 6l_2 + \cdots + 6l_k + 1$ cubes in the cube B_1. Thus we pack

$$
\begin{aligned}
&m^3 - l_1^3 + (l_1+1)^3 - l_1^3 - (1+l) + (l_1-1)^3 + 6l_2 + \cdots + 6l_k + 1 \\
= \; &m^3 + 6l_1 + 6l_2 + \cdots + 6l_k + 1 - (1+l) = n
\end{aligned}
$$

small cubes in the unit cube.

Step 2: Pack $(l_1-1)^3 + 6l_2 + \cdots + 6l_k + 1$ smaller cubes into the cube B_1 as follows:

Make an $(l_1-3)^3$-grid of B_1, then replace an l_2^3-grid in the top left corner of the front with a cube denoted by B_2 packing the same volume. Then in B_2, according to the following Step 3, pack $(l_2-1)^3 + 6l_3 + \cdots + 6l_k + 1$ small cubes.

Similar to Step 2, we finish Step 3, Step 4, ..., Step $k-1$.

Step k: Pack $(l_{k-1}-1)^3 + 6l_k + 1 = (l_{k-1}-1)^3 + 7$ smaller cubes into the cube B_{k-1} as follows:

Make an $(l_{k-1}-1)^3$−grid of B_{k-1}, then remove a cube in the top left corner of the front and replace it with a 2^3−grid(8 small cubes) packing the same volume. Thus, we pack

$$(l_{k-1}-1)^3 - 1 + 8 = (l_{k-1}-1)^3 + 7$$

cubes in B_{k-1}.

Denote $l' = l + 1$. By the process of packing n small cubes into the unit cube, it's easy to see:

$$
\begin{aligned}
f_3(n) \quad \geq \quad & \frac{1}{m}\left(m^3 - l_1^3\right) + \frac{1}{m}\frac{l_1}{(l_1+1)}\left[3l_1^2 + 3l_1 + 1 - l'\right] + \\
& \frac{1}{m}\frac{l_1^2}{(l_1+1)}\left[\frac{1}{l_1 - 1}\left[(l_1-1)^3 - l_2^3\right]\right. \\
& \qquad \left. + \frac{l_2}{(l_1-1)(l_2+1)}(3l_2^2 + 3l_2 + 1)\right] + \\
& \frac{1}{m}\frac{l_1^2}{(l_1+1)(l_1-1)}\frac{l_2^2}{(l_2+1)}\left[\frac{1}{l_2-1}\left[(l_2-1)^3 - l_3^3\right]\right. \\
& \qquad \left. + \frac{l_3}{(l_2-1)(l_3+1)}(3l_3^2 + 3l_3 + 1)\right] + \cdots + \\
& \frac{1}{m}\left(\prod_{i=1}^{k-2}\frac{l_i^2}{(l_i+1)(l_i-1)}\right)\frac{l_{k-1}^2}{(l_{k-1}+1)} \cdot \\
& \left[\frac{1}{l_{k-1}-1}\left[(l_{k-1}-1)^3 - 1\right] + \frac{8}{2(l_{k-1}-1)}\right] \qquad (2.8)
\end{aligned}
$$

$$
\begin{aligned}
mf_3(n) \geq m^3 + & \left[\frac{l_1^2}{l_1^2-1}(l_1-1)^3 - l_1^3 + \frac{l_1(3l_1^2 + 3l_1 + 1 - l')}{l_1+1}\right] + \\
& \frac{l_1^2}{l_1^2-1}\left[\frac{l_2^2}{l_2^2-1}(l_2-1)^3 - l_2^3 + \frac{l_2(3l_2^2+3l_2+1)}{l_2+1}\right] + \cdots + \qquad (2.9) \\
& \prod_{i=1}^{k-2}\frac{l_i^2}{l_i^2-1}\left[\frac{l_{k-1}^2}{l_{k-1}^2-1}(l_{k-1}-1)^3 - l_{k-1}^3\right. \\
& \left. + \frac{l_{k-1}(3l_{k-1}^2 + 3l_{k-1}+1)}{l_{k-1}+1}\right] + 3\prod_{i=1}^{k-1}\frac{l_i^2}{l_i^2-1} \\
\geq m^3 & + \frac{l_1(4l_1+1)}{l_1+1} - l' + \frac{l_1^2}{l_1^2-1}\frac{l_2(4l_2+1)}{l_2+1} \\
& + \frac{l_1^2}{l_1^2-1}\frac{l_2^2}{l_2^2-1}\frac{l_3(4l_3+1)}{l_3+1} + \cdots \\
& + \prod_{i=1}^{k-2}\frac{l_i^2}{l_i^2-1}\frac{l_{k-1}(4l_{k-1}+1)}{l_{k-1}+1} + 3\prod_{i=1}^{k-1}\frac{l_i^2}{l_i^2-1} \\
\geq m^3 & + \frac{l_1(4l_1+1)}{l_1+1} - l' + \frac{l_2(4l_2+1)}{l_2+1} + \cdots + \frac{l_{k-1}(4l_{k-1}+1)}{l_{k-1}+1} + 3
\end{aligned}
$$

$$
\begin{aligned}
&= m^3 + 4(l_1 + l_2 + \cdots + l_{k-1}) + (k-1) \\
&\quad - l' - \frac{4l_1+1}{l_1+1} - \cdots - \frac{4l_{k-1}+1}{l_{k-1}+1} + 3 \\
&= n - 2(l_1 + l_2 + \cdots + l_{k-1}) - 7 + (k-1) \\
&\quad - \frac{4l_1+1}{l_1+1} - \cdots - \frac{4l_{k-1}+1}{l_{k-1}+1} + 3 \\
&\geq n - 2(l_1 + l_2 + \cdots + l_{k-1}) - 7 + (k-1) - 4(k-1) + 3 \\
&= n - 2(l_1 + l_2 + \cdots + l_{k-1}) - 4 - 3(k-1) \\
&\geq n - 2(2 + \cdots + m) - 4 - 3(m-1) \\
&= (n+1) - m(m+1) - 3m. \qquad (2.10)
\end{aligned}
$$

$$
f_3(n) \geq \frac{n+1}{m} - m - 4 = \frac{n+1}{\lceil n^{\frac{1}{3}} \rceil} - \lceil n^{\frac{1}{3}} \rceil - 4. \qquad (2.11)
$$

From (2.7) and (2.11), we have

$$
f_3(n) \geq \frac{n+1}{\lceil n^{\frac{1}{3}} \rceil} - \lceil n^{\frac{1}{3}} \rceil - 4. \qquad (2.12)
$$

□

It's easy to see that the lower bound here we get is asymptotically of the same order of magnitude as the known upper bound.

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