Lower bound on the number of the maximum genus embedding of $K_{n,n}$ †

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Abstract

In this paper, we provide a method to obtain the lower bound on the number of the distinct maximum genus embedding of the complete bipartite graph $K_{n,n}$ (n is an odd number), which, in some sense, improves the results of S. Stahl and H. Ren.

Key Words: graph embedding; maximum genus; v-type-edge
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1. Introduction

Graphs considered here are all connected and finite. A *surface* S means a compact and connected two-manifold without boundaries. A *cellular embedding* of a graph G into a surface S is a one-to-one mapping $\psi : G \to S$ such that each component of $S - \psi(G)$ is homomorphic to an open disc. The maximum genus $\gamma_M(G)$ of a connected graph G is the maximum integer k such that there exists an embedding of G into the orientable surface of genus k. By Euler's polyhedron formula, if a cellular embedding of a graph G with n vertices, m edges and r faces is on an orientable surface of genus γ, the $n - m + r = 2 - 2\gamma$. Since $r \geqslant 1$, we have $\gamma(G) \leq \lfloor \frac{\beta(G)}{2} \rfloor$, where $\beta(G) = m - n + 1$ is called the *Betti number* (or *cycle rank*) of the graph G. It follows that $\gamma(G) \leq \lfloor \frac{\beta(G)}{2} \rfloor$. If $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$, then the graph is called *upper embeddable*. Since the introductory investigations on

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the maximum genus of graphs by Nordhaus, Stewart, and White[1], this parameter has attracted considerable attention from mathematicians and computer scientists. Up to now, the research about the maximum genus of graphs mainly focus on the aspects as characterizations and complexity, the upper embeddability, the lower bound, the enumeration of the distinct maximum genus embedding, *etc.*. For more detailed information, the reader can be found in a survey in [2].

It is well known that the enumeration of the distinct maximum genus embedding plays an important role in the study of the genus distribution problem, which may be used to decide whether two given graphs are isomorphic. It was S. Stahl[3] who provides the first result about the lower bound on the number of the distinct maximum genus embedding, which is states as the following:

Lemma 1[3] A connected graph (loops and multi-edges are allowed) of order n with degree sequence $d_1, d_2, \ldots, d_n$ has at least $(d_1-5)!(d_2-5)!(d_3-5)!(d_4-5)!\prod_{i=5}^{n}(d_i-2)!$ distinct orientable embeddings with at most two facial walks, where $m! = 1$ whenever $m \leqslant 0$.

But up to now, except [3] and [4], there is little result concerning the number of the maximum genus embedding of graphs. In this paper, we will provide a result concerning the complete bipartite graph $K_{n,n}$ (n is an odd number) which is better than that of S. Stahl[3] and H. Ren[4] in some sence. Furthermore, the enumerative method below can be used to any maximum genus embedding, other than the method in [3] which is restricted to upper embeddable graphs. Terminologies and notations not explained here can be seen in [5], [6] and [7].

2. Main results

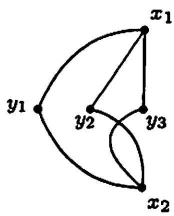
A *complete bipartite graph* G with bipartition X and Y is denoted by $G_{[X][Y]}$. A *2-path* is called a *v-type-edge*, and is denoted by $\mathcal{V}$. Let $\psi(G)$ be an embedding of a graph G. We say that a *v-type-edge* are inserted into $\psi(G)$ if the three endpoints of the *v-type-edge* are inserted into the corners of the faces in $\psi(G)$, yielding an embedding of $G+\mathcal{V}$. The embedding $\psi(G)$ of G is called a *one-face-embedding* (or *two-face-embedding*) if the total face number of $\psi(G)$ is one (or two). The following observation can be easily obtained and is essential in the proof of the Theorem A.

Observation Let $\psi(G)$ be an embedding of a graph G. We can insert a *v-type-edge* $\mathcal{V}$ to $\psi(G)$ to get an embedding $\rho(G+\mathcal{V})$ of $G+\mathcal{V}$ so that the face number of $\rho(G+\mathcal{V})$ is not more than that of $\psi(G)$.

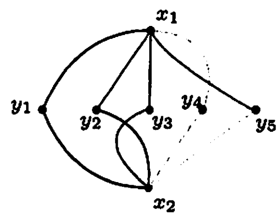
Theorem A For $n \equiv 1$ (*mod* 2), the number of the distinct maximum genus embedding of the complete bipartite graph $K_{n,n}$ is at least

$$2^{\frac{n-1}{2}} \times \left((n-2)!!\right)^n \times \left((n-1)!\right)^n.$$

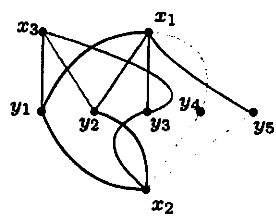
Proof Let $n = 2s+1$ and $V(K_{n,n}) = \{x_1, x_2, \dots, x_n\} \cup \{y_1, y_2, \dots, y_n\}$, where $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ are the two independent set of $K_{n,n}$. We denote the *v-type-edge* $y_{2i}x_jy_{2i+1}$ by $\mathcal{V}_{ji}$, where $i \in \{1, 2, \dots, s\}$ and $j \in \{1, 2, \dots, n\}$.



$G_{[x_1,x_2][y_1,y_2,y_3]}$



$G_{[x_1,x_2][y_1,y_2,\dots,y_5]}$



$G_{[x_1,x_2][y_1,y_2,\dots,y_5]} \cup x_3y_1 \cup \mathcal{V}_{3,1}$

Claim 1: For $G_{[x_1,x_2][y_1,y_2,\dots,y_n]}$, the number of the distinct *one-face-embedding* is at least $2^s \times ((2s-1)!!)^2$.

There are 2 different ways to embed $G_{[x_1,x_2][y_1,y_2,y_3]}$ on an orientable surface so that the embedding is a *one-face-embedding*. Select any one of them and denote its face boundary by W_0. In W_0, there are three *face-corner* containing x_1 and x_2 respectively. So, there are 3 different ways to put $\mathcal{V}_{1,2}$ in W_0, and 3 different ways to put $\mathcal{V}_{2,2}$ in W_0. Therefore, the total number of ways to put $\mathcal{V}_{1,2} \cup \mathcal{V}_{2,2}$ in W_0 is $3 \times 3 = 9$. For each of the above 9 ways, there are 2 different ways to make the embedding of $G_{[x_1,x_2][y_1,y_2,\dots,y_5]}$ being a *one-face-embedding*. So, for each of the *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,y_3]}$, there are $3 \times 3 \times 2$ different ways to add $\mathcal{V}_{1,2} \cup \mathcal{V}_{2,2}$ to $G_{[x_1,x_2][y_1,y_2,y_3]}$ to get a *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\dots,y_5]}$.

Similarly, for each of the *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\dots,y_5]}$ there are $5 \times 5 \times 2$ different ways to add $\mathcal{V}_{1,3} \cup \mathcal{V}_{2,3}$ to $G_{[x_1,x_2][y_1,y_2,\dots,y_5]}$ to get a *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\dots,y_7]}$.

In general, for each of the *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\dots,y_{2k-1}]}$, there are $(2k-1) \times (2k-1) \times 2$ different ways to add $\mathcal{V}_{1,k} \cup \mathcal{V}_{2,k}$ to $G_{[x_1,x_2][y_1,y_2,\dots,y_{2k-1}]}$ to get a *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\dots,y_{2k+1}]}$.

From the above we can get that the number of the distinct *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\dots,y_n]}$ is at least

$$2 \times (3 \times 3 \times 2) \times (5 \times 5 \times 2) \times (7 \times 7 \times 2) \times \cdots \times ((2s-1) \times (2s-1) \times 2)$$
$$= 2^s \times ((2s-1)!!)^2.$$

Claim 2: For each of the *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\dots,y_n]}$, there are at least $2 \times (2s-1)!! \times 2^{2s}$ different ways to make $G_{[x_1,x_2,x_3][y_1,y_2,\dots,y_n]}$ being a *one-face-embedding*.

Let $\mathcal{E}_1$ be an arbitrary *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\dots,y_n]}$. In $\mathcal{E}_1$, there are two different *face-corner* containing y_i ($i = 1, 2, 3$). So, there are $2 \times 2 \times 2(= 8)$ different ways to add $y_1x_3 \cup \mathcal{V}_{3,1}$ to $\mathcal{E}_1$ to make $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]} \cup y_1x_3 \cup \mathcal{V}_{3,1}$ being a *one-face-embedding*. For each of the above 8 *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]} \cup y_1x_3 \cup \mathcal{V}_{3,1}$, there are 3 different *face-corner* containing x_3 and 2 different *face-corner* containing y_i $(i = 4, 5)$. So, for each of the above 8 *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]} \cup y_1x_3 \cup \mathcal{V}_{3,1}$, there are $3\times2\times2$ different ways to add $\mathcal{V}_{3,2}$ to $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]} \cup y_1x_3 \cup \mathcal{V}_{3,1}$ to make $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]} \cup y_1x_3 \cup \mathcal{V}_{3,1} \cup \mathcal{V}_{3,2}$ being a *one-face-embedding*.

In general, for each of the *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]} \cup y_1x_3 \cup \mathcal{V}_{3,1} \cup \mathcal{V}_{3,2} \cup \cdots \cup \mathcal{V}_{3,k-1}$, there are $(2k-1) \times 2 \times 2$ different ways to add $\mathcal{V}_{3,k}$ to $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]} \cup y_1x_3 \cup \mathcal{V}_{3,1} \cup \mathcal{V}_{3,2} \cup \cdots \cup \mathcal{V}_{3,k-1}$ to get a *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]} \cup y_1x_3 \cup \mathcal{V}_{3,1} \cup \mathcal{V}_{3,2} \cup \cdots \cup \mathcal{V}_{3,k-1} \cup \mathcal{V}_{3,k}$.

From the above we can get that for each of the *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]}$, there are at least

$$(2 \times 2 \times 2) \times (3 \times 2 \times 2) \times (5 \times 2 \times 2) \times \cdots \times ((2s-1) \times 2 \times 2)$$
$$= 2 \times (2s-1)!! \times 2^{2s}$$

different ways to make $G_{[x_1,x_2,x_3][y_1,y_2,\ldots,y_n]}$ being a *one-face-embedding*.

Claim 3: For each of the *one-face-embedding* of $G_{[x_1,x_2,x_3][y_1,y_2,\ldots,y_n]}$, there are at least $3\times(2s-1)!!\times3^{2s}$ ways to make $G_{[x_1,x_2,x_3,x_4][y_1,y_2,\ldots,y_n]}$ being a *one-face-embedding*.

Let $\mathcal{E}_2$ be an arbitrary *one-face-embedding* of $G_{[x_1,x_2,x_3][y_1,y_2,\ldots,y_n]}$. In $\mathcal{E}_2$, there are three different *face-corner* containing y_i $(i = 1, 2, 3)$. So, there are $3 \times 3 \times 3(= 27)$ different ways to add $y_1x_4 \cup \mathcal{V}_{4,1}$ to $\mathcal{E}_2$ to make $G_{[x_1,x_2,x_3][y_1,y_2,\ldots,y_n]} \cup y_1x_4 \cup \mathcal{V}_{4,1}$ being a *one-face-embedding*. For each of the above 27 *one-face-embedding* of $G_{[x_1,x_2,x_3][y_1,y_2,\ldots,y_n]} \cup y_1x_4 \cup \mathcal{V}_{4,1}$, there are 3 different *face-corner* containing x_4 and 3 different *face-corner* containing y_i $(i = 4, 5)$. So, for each of the above 27 *one-face-embedding* of $G_{[x_1,x_2,x_3][y_1,y_2,\ldots,y_n]} \cup y_1x_4 \cup \mathcal{V}_{4,1}$, there are $3 \times 3 \times 3$ different ways to add $\mathcal{V}_{4,2}$ to $G_{[x_1,x_2,x_3][y_1,y_2,\ldots,y_n]} \cup y_1x_4 \cup \mathcal{V}_{4,1}$ to make $G_{[x_1,x_2,x_3][y_1,y_2,\ldots,y_n]} \cup y_1x_4 \cup \mathcal{V}_{4,1} \cup \mathcal{V}_{4,2}$ being a *one-face-embedding*.

In general, for each of the *one-face-embedding* of $G_{[x_1,x_2,x_3][y_1,y_2,\ldots,y_n]} \cup y_1x_4 \cup \mathcal{V}_{4,1} \cup \mathcal{V}_{4,2} \cup \cdots \cup \mathcal{V}_{4,k-1}$, there are $(2k-1)\times3\times3$ different ways to add $\mathcal{V}_{4,k}$ to $G_{[x_1,x_2,x_3][y_1,y_2,\ldots,y_n]} \cup y_1x_4 \cup \mathcal{V}_{4,1} \cup \mathcal{V}_{4,2} \cup \cdots \cup \mathcal{V}_{4,k-1}$ to get a *one-face-embedding* of $G_{[x_1,x_2,x_3][y_1,y_2,\ldots,y_n]} \cup y_1x_4 \cup \mathcal{V}_{4,1} \cup \mathcal{V}_{4,2} \cup \cdots \cup \mathcal{V}_{4,k-1} \cup \mathcal{V}_{4,k}$.

From the above we can get that for each of the *one-face-embedding* of $G_{[x_1,x_2,x_3][y_1,y_2,\ldots,y_n]}$, there are at least

$$(3 \times 3 \times 3) \times (3 \times 3 \times 3) \times (5 \times 3 \times 3) \times \cdots \times ((2s-1) \times 3 \times 3)$$
$$= 3 \times (2s-1)!! \times 3^{2s}$$

different ways to make $G_{[x_1,x_2,x_3,x_4][y_1,y_2,\dots,y_n]}$ being a *one-face-embedding.*

Similarly, we can get the following general result.

Claim 4: For each of *one-face-embedding* of $G_{[x_1,x_2,\dots,x_{k-1}][y_1,y_2,\dots,y_n]}$, there are at least $(k-1)\times(2s-1)!!\times(k-1)^{2s}$ different ways to make $G_{[x_1,x_2,\dots,x_{k-1},x_k][y_1,y_2,\dots,y_n]}$ being a *one-face-embedding.*

Noticing that a *one-face-embedding* of a graph must be its maximum genus embedding, we can get, from Claim 1 - Claim 4, that the number of the distinct maximum genus embedding of $K_{n,n}$ is at least

$$\begin{aligned}&\{2^s\times((2s-1)!!)^2\}\times\{2\times(2s-1)!!\times 2^{2s}\}\times\{3\times(2s-1)!!\\&\times 3^{2s}\}\times\cdots\times\{2s\times(2s-1)!!\times(2s)^{2s}\}\\&=2^s\times((2s-1)!!)^{2s+1}\times((2s)!)^{2s+1}\\&=2^{\frac{n-1}{2}}\times((n-2)!!)^n\times((n-1)!)^n.\end{aligned}$$ □

Remark Through a comparison we can get that the result in Theorem A is much better than that of Lemma 1[3] when $n\leqslant 9$.

In [4], the second author of the present paper obtained that a connected loopless graph of order n has at least $\frac{1}{4^{\gamma_M(G)}}\prod_{v\in V(G)}(d(v)-1)!$ distinct maximum genus embedding. Let $f_1(n)=2^{\frac{n-1}{2}}\times((n-2)!!)^n\times((n-1)!)^n$, $f_2(n)=\frac{1}{4^{\gamma_M(G)}}\prod_{v\in V(G)}(d(v)-1)!=\frac{1}{4^{\frac{(n-1)(n-1)}{2}}}\times((n-1)!)^{2n}$. Through a computation we can get $f_1(3)-f_2(3)=16$, $f_1(5)-f_2(5)=6772211712$. So, when $n\leqslant 5$ the result obtained in Theorem A is much better than that of [4].

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