On odd graceful graphs

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Abstract. We determine all connected odd graceful graphs of order ≤ 6, we show that if G is an odd graceful graph, then $G \cup K_{m,n}$ is odd graceful for all $m, n \geq 1$, we give an analogous statement to graceful graphs statement, and we show that some families of graphs are odd graceful.

0. Introduction.

Gnanajothi [2] defined a graph G with q edges to be *odd graceful* if there is an injection f from $V(G)$ to $\{0, 1, 2, \ldots, 2q-1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, 2q - 1\}$. She proved that the class of odd graceful graphs lies between the class of graphs with α-labelings and the class of bipartite graphs by showing that every graph with an α-labeling has an odd graceful labeling and every graph with an odd cycle is not odd graceful.

She conjectured that all trees are odd graceful and proved the conjecture for all trees with order up to 10. She also proved that many families of graphs such as P_n,

C_n if and only if n is even, $K_{m,n}$, books, crowns, caterpillars and rooted trees of height 2 are odd-graceful. Sekar [5] showed also that many families of graphs such as all n polygonal snakes with n even, lobsters, banana trees, regular bombo trees, graphs obtained from even cycles by identifying a vertex of the cycle with the endpoint of a star and the splitting graphs of P_n and C_n , n is even, are odd graceful.

This paper is divided into two sections. Section 1, in which we determine all connected odd-graceful graphs of order ≤ 6, we show that if G is odd graceful, then $G \cup K_{m,n}$ is odd graceful for all $m, n \geq 1$, and we show that if G is an odd graceful Eulerian graph of q edges, then $q \equiv 0$ or 2 (mod 4). This result corresponds to the result in case G is graceful and Eulerian, which had been stated and proved by Rosa [4]. The proofs are identical. In section 2, we show that several families of graphs are odd graceful.

Throughout this paper, we use the standard notations and conventions as in [1] and [3].

1. General Results

Theorem 1.1.

All connected graphs of order $\leq$ 6 are not odd graceful, except the following 28 graphs

(i) $K_{3,3}$, $K_{2,4}$,C_6 ,C_4,

(ii) all non-isomorphic trees of order ≤ 6,

(iii) the following graphs of order ≤ 6:

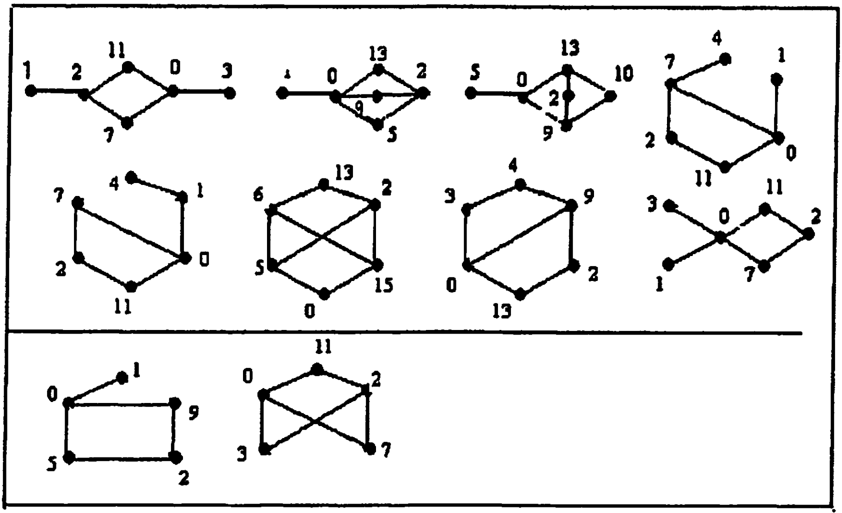


Figure (1)

Proof.

The graphs in (i) are odd-graceful by theorems due to Gnanajothi [2]; the graphs in case (ii) are odd-graceful: note that there are exactly (14) non-isomorphic trees of order ≤ 6 and these are odd-graceful by the theorem: "All trees of order ≤ 10 are odd-graceful" which is also due to Gnanajothi [2]. And those in case (iii) are shown to be odd-graceful by giving specific odd-graceful labeling assigned to the vertices of each graph as indicated in Figure(1). According to Harary [3] the remaining (115) connected graphs of order ≤ 6 are not odd-graceful by the theorem: "Every graph with an odd cycle is not odd-graceful" which is also due to Gnanajothi [2].

Theorem 1.2.

If G is an odd graceful Eulerian graph of q edges, then $q \equiv 0$ or $2 \pmod 4$.

Proof.

Let G be an odd-graceful Eulerian graph, and let f: $V(G) \to \{0, 1, 2, \ldots, 2q\text{-}1\}$ be an odd-graceful labeling for G.

Since G is an Eulerian graph then $\sum(f(v_i)-f(v_j))=2k$, k is a constant. For each v_i, $v_j \in V(G)$

$$\sum|f(v_i)-f(v_j)|=2k', \text{ so } 1+3+5+\ldots+2q-1=2k'$$

this implies that $\frac{q}{2}(1+2q-1)=2k'$. Hence $q \equiv 0$ or 2 (mod 4).

Theorem 1.3.

Let G be an odd-graceful graph, then $G \cup K_{m,n}$ is odd graceful for all $m, n \geq 1$.

Proof.

Let f be an odd graceful labeling of the graph G. Let $q = |E(G)|$ and $V(K_{m,n}) = V_1 \cup V_2$, where V_i, $i = 1, 2$ are two independent sets of vertices, such that $V_1 = \{u_1, u_2, u_3, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, v_3, \ldots, v_n\}$, $m, n \geq 1$.

Define the function :

$$\bar{f}: V(G \cup k_{m,n}) \to \{0,1,\ldots,2(q+mn)-1\}$$

as follows

$$\bar{f}(u_i) = 2(q+mn-i)+1 \quad , \quad 1 \leq i \leq m$$

$$\bar{f}(v_i) = 2m(i-1) \quad , \quad 1 \leq i \leq n$$

$$\bar{f}\,|_{V(G)} = f + (2m(n-1)+1).$$

We observe that $\bar{f}$ is injective.

<u>The edge labels will be as follows:</u>

- The vertices u_1 and v_i, $1 \le i \le n$ induce the edge labels $\{2q + 2mn - 1, 2q + 2mn - 2m - 1, 2q + 2mn - 4m - 1, \ldots, 2q + 2m - 1\}$.
- The vertices u_2 and v_i, $1 \le i \le$ n induce the edge labels $\{2q + 2mn - 3, 2q + 2mn - 2m - 3, 2q + 2mn - 4m - 3, \ldots, 2q + 2m - 3\}$.
- The vertices u_3 and v_i, $1 \le i \le n$ induce the edge labels $\{2q + 2mn - 5, 2q + 2mn - 2m - 5, 2q + 2mn - 4m - 5, \ldots, 2q + 2m - 5\}$.

and so on until the vertices u_m and v_i, $1 \le i \le$ n induce the edge labels $\{2q + 2mn - 2m + 1, 2q + 2mn - 4m + 1, 2q + 2mn - 6m + 1, \ldots, 2q + 1\}$.

- The remaining edge labels $\{1, 3, 5, \ldots, 2q - 1\}$ come from the edge labels of the graph G, since G is odd-graceful and we added a constant number on its vertex labels. So, we have the same edge labels $\{1, 3, 5, \ldots, 2q - 1\}$. Hence $\overline{f^*}$ is injective as required.

Corollary 1.4.

According to Theorem 1.3 numerous families of disconnected graphs are odd-graceful, e.g.

1. $\bigcup_{i=1}^{r} K_{m_i,n_i}$, $r \ge 1$, $n_i, m_i \ge 1$.
2. $C_r \cup K_{m,n}$ if and only if r is an even integer, $m, n \ge 1$.
3. $P_n \cup K_{r,s}$, $n, r, s \ge 1$.

   ($K_{m,n}$, $m, n \ge 1$, C_r, r is even and P_n, $n \ge 1$ are odd-graceful [3].)

2. Some odd-graceful graphs

In the following theorems we did not consider the trivial cases,
e.g. in Theorem 2.4, if $n = 1$, then $P_n \times P_3 = P_1 \times P_3$ which is isomorphic to P_3, and so it is odd-graceful. Also in Theorem 2.8 (1) if $m = 1$, $n = 1$, then $S^1(K_{m,n}) = S^1(K_{1,1})$ is isomorphic to P_4 , and it is also odd-graceful, and so on .... .

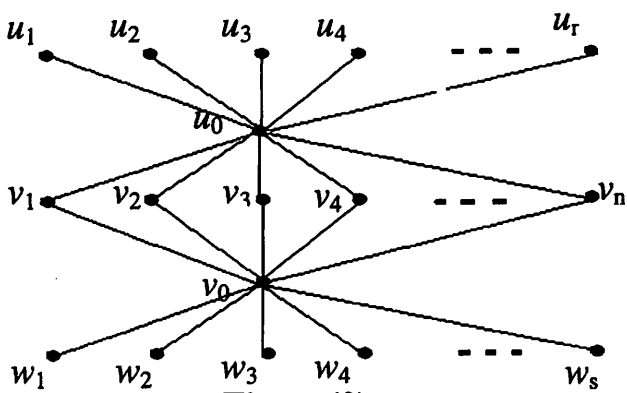
The graph $K_{2,n}(r, s)$ is obtained from $K_{2,n}$, ($n \geq 2$) by adding r and s ($r, s \geq 1$) pendent edges out from the two vertices of degree n.

Theorem 2.1.

The graphs $K_{2,n}(r, s)$ are odd-graceful for all $n, r, s \geq 1$.

Proof.

Let $K_{2,n}(r, s)$ be described as indicated in Figure(2)



Figure(2)

The number of edges of the graph $K_{2,n}(r, s)$ is $2n + r + s$. We define the labeling function :

$$f: V(K_{2,n}(r, s)) \rightarrow \{0, 1, 2, \ldots 2(2n + r + s) - 1\}$$

as follows

$$f(u_i) = 2(q - n - i) + 1 \quad , \qquad 1 \leq i \leq r$$

$f(u_0) = 2n$

$f(v_i) = 2(q - i) + 1 \quad , \quad 1 \le i \le n$

$f(v_0) = 0$

$f(w_i) = 2i - 1 \quad , \quad 1 \le i \le s$

The edge labels will be as follows:

- The vertices v_0 and w_i , $1 \le i \le s$, induce the edge labels {1, 3, 5, …, 2s - 1}.
- The vertices u_0 and u_i , $1 \le i \le r$, induce the edge labels {2s + 1, 2s + 3, 2s + 5, …, 2s + 2r - 1}.
- The vertices u_0 and v_i , $1 \le i \le n$, induce the edge labels {2s + 2r + 1, 2s + 2r + 3, 2s + 2r + 5, …, 2n + 2r + 2s - 1}.
- The vertices v_0 and v_i , $1 \le i \le n$, induce the edge labels {2n + 2r + 2s +1, 2n + 2r + 2s + 3, 2n +2r + 2s + 5,…, 4n +2r +2s–1}.

Hence the result follows.

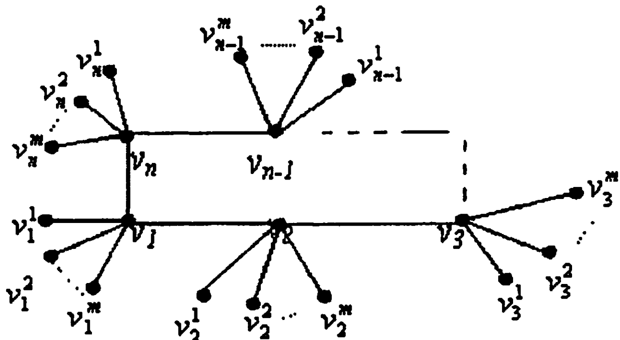
Let G_1 and G_2 be two disjoint graphs. *The corona* ($G_1 \Theta G_2$) of G_1 and G_2 is the graph obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 , and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

Theorem 2.2.

The graphs $C_n \Theta \overline{K_m}$, for n = 4, 6, 8, 10, 12 and $m \ge$ 4, are odd-graceful.

Proof.

Let $C_n \Theta \overline{K_m}$ be described as indicated in Figure(3):



Figure(3)

It is clear that the number of edges of the graph $C_n \Theta \overline{K_m}$ is $n(m + 1)$. We define the labeling function:

$$f: V(C_n \Theta \overline{K_m}) \rightarrow \{0, 1, 2, 3, \ldots, 2n(m+1) - 1\}$$

as follows

$$f(v_j) = \begin{cases} (j-1) & , j = 1, 3, 5, \ldots, n-1 \\ 2nm + 2n - j + 1 & , j = 2, 4, 6, \ldots, n-2 \end{cases}$$

$f(v_n) = 2nm + 3$

$f(v_1^i) = 2m + 2i - 1 \quad , 1 \leq i \leq m$

$$f(v_2^i) = \begin{cases} 2n - 4 + 2i & , i = 1, 2, 3, \ldots, (n/2 - 2) \\ 2n - 2 + 2i & , i = n/2 - 1,\ n/2,\ n/2 + 1, \ldots, m \end{cases}$$

$f(v_3^i) = (2n - 2)m - 2i + 3 \quad , 1 \leq i \leq m$

$f(v_4^i) = 4m + 2i + 2n - 4 \quad , 1 \leq i \leq m$

$f(v_5^i) = (2n - 6)m - 2i + 5 \quad , 1 \leq i \leq m$

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$f(v_h^i) = (2h - 4)m + 2i + (2n - h) \quad , 4 \leq h(\text{even}) \leq n - 2,$
$1 \leq i \leq m$

$f(v_k^i) = [2n - (2k - 4)]\, m - 2i + k \quad , 5 \leq k(\text{odd}) \leq n - 3,$

$1 \leq i \leq m$

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$f(v^i_{n-1}) = 6m - 2i + (n - 1)$, $1 \leq i \leq m$

$f(v^i_n) = (2n - 2)m + 2i + 2$, $1 \leq i \leq m.$

<u>The edge labels will be as follows:</u>

- The path $v_1v_2v_3\ldots v_{n-1}$ induces the edge labels $\{2nm + 2n - 1, 2nm + 2n - 3, 2nm + 2n - 5, \ldots, 2nm + 5\}$,
- $f^*(v_1v_n) = 2nm + 3$
- $f^*(v_{n-1}v_n) = 2nm - n + 5$
- The vertices v_2 and v_2^i, $1 \leq i \leq (n/2 - 2)$, induce the edge labels $\{2nm + 1, 2nm - 1, 2nm - 3, \ldots, 2nm - n + 7\}$.
- The vertices v_2 and v_2^i, $n/2 - 1 \leq i \leq m$, induce the edge labels $\{2nm - n + 3, 2nm - n + 1, 2nm - n - 1, \ldots, 2nm - 2m + 1\}$.
- The vertices v_3 and v_3^i, $1 \leq i \leq m$, induce the edge labels $\{2nm - 2m - 1, 2nm - 2m - 3, 2nm - 2m - 5, \ldots, 2nm - 4m + 1\}$.
- The vertices v_4 and v_4^i, $1 \leq i \leq m$, induce the edge labels $\{2nm - 4m - 1, 2nm - 4m - 3, 2nm - 4m - 5, \ldots, 2nm - 6m + 1\}$.
- The vertices v_5 and v_5^i, $1 \leq i \leq m$, induce the edge labels $\{2nm - 6m - 1, 2nm - 6m - 3, 2nm - 6m - 5, \ldots, 2nm - 8m + 1\}$.

  and so on until the vertices v_{n-1} and v_{n-1}^i, $1 \leq i \leq m$, induce the edge labels $\{6m - 1, 6m - 3, 6m - 5, \ldots,$

$4m + 1\}$.

- The vertices v_1 and v_1^i, $1 \leq i \leq m$, induce the edge labels $\{4m - 1, 4m - 3, 4m - 5, \ldots, 2m + 1\}$.
- Finally, the vertices v_n and v_n^i, $1 \leq i \leq m$, induce the edge labels $\{2m - 1, 2m - 3, 2m - 5, \ldots 3, 1\}$. So we obtain all the edge labels. Hence $C_n \Theta \overline{K_m}$ is odd-graceful for $n = 4, 6, 8, 10, 12$ and $m \geq 4$.

In the following theorems we mention only the vertices labels, the reader can fulfill the proof as we did in the previous theorems.

Let G_1 and G_2 be two disjoint graphs. *The union* $(G_1 \cup G_2)$ of G_1 and G_2 is the graph having vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

Theorem 2.3.

The graphs $P_n \cup C_4$, $n \geq 2$ are odd-graceful.

Proof.

Let the path P_n have the consecutive vertices u_1, u_2, u_3, …, u_n and C_4 have the consecutive vertices v_1, v_2, v_3, v_4. The number of edges of the graph $P_n \cup C_4$ is $n + 3$. Now we define the labeling function

$$f: V(P_n \cup C_4) \rightarrow \{0, 1, 2, \ldots, 2n + 5\}$$

as follows:

$$f(u_i) = \begin{cases} 2n + 5 - i & , \quad i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ 5 + i & , \quad i = 2, 4, 6, \ldots, n - 1 \text{ or } n \end{cases}$$

$f(v_1, v_2, v_3, v_4) = (0, 2n + 5, 2, 2n + 1)$.

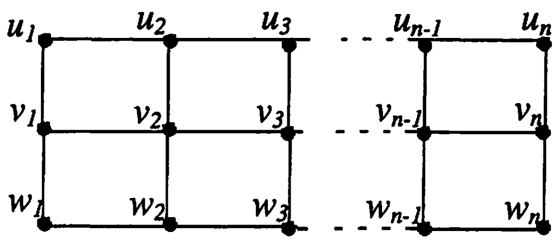
Let G_1 and G_2 be two disjoint graphs. *The cartesian product* $(G_1 \times G_2)$ of G_1 and G_2 is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set$\{(u_1, v_1)(u_2, v_2)$: $(u_1 = u_2$ and $v_1v_2 \in E(G_2))$ or $(v_1 = v_2$ and $u_1u_2 \in E(G_1))\}$.

Theorem 2.4.

The graphs $P_n \times P_3$, $n \geq 2$ are odd-graceful.

Proof.

Let $P_n \times P_3$ be described as indicated in Figure(4):



Figure(4)

It is clear that the number of edges of the graph $P_n \times P_3$ is $5n - 3$. We define the labeling function

$$f: V(P_n \times P_3) \to \{0, 1, 2, \ldots, 10n - 7\}$$

as follows:

$$f(u_i) = \begin{cases} 10n - 5i - 2 & , \quad i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ 5i - 6 & , \quad i = 2, 4, 6, \ldots, n - 1 \text{ or } n \end{cases}$$

$$f(v_i) = \begin{cases} 5(i - 1) & , \quad i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ 10n - 5i - 3 & , \quad i = 2, 4, 6, \ldots, n - 1 \text{ or } n \end{cases}$$

$$f(w_i) = \begin{cases} 10n - 5i - 4 & , \quad i = 1, 3, 5, \dots, n \text{ or } n-1 \\ 5i - 4 & , \quad i = 2, 4, 6, \dots, n-1 \text{ or } n. \end{cases}$$

Let G_1 and G_2 be two disjoint graphs. *The conjunction* $(G_1 \wedge G_2)$ of G_1 and G_2 is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set$\{(u_1, v_1)(u_2, v_2): u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$.[3]

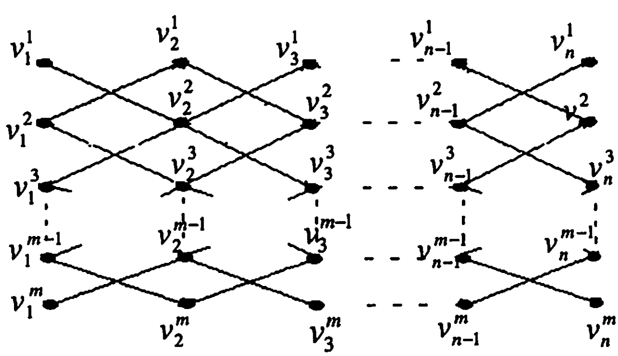
Theorem 2.5.

The following graphs are odd-graceful

1. $P_m \wedge P_n$, $m = 3$ or m is even , $n \geq 2$
2. $P_n \wedge C_3$, $n \geq 2$
3. $P_n \wedge C_4$, $n \geq 2$
4. $P_n \wedge S_m$, $n, m \geq 2$.

Proof.

(1) Let $P_m \wedge P_n$ be described as indicated in Figure(5):



Figure(5)

It is clear that the number of edges of the graph $P_m \wedge P_n$ is $2(m-1)(n-1)$. We define the labeling function

$f: V(P_m \wedge P_n) \rightarrow \{0, 1, 2, \dots, 4(m-1)(n-1) - 1\}$

as follows:

$$f(v_i^1) = \begin{cases} i - 1 & , i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ 4(m-1)(n-1) - 2(n-2) - i, & i = 2, 4, 6, \ldots, n - 1 \text{ or } n, \end{cases}$$

and we have two cases for labeling $f(v_i^j)$

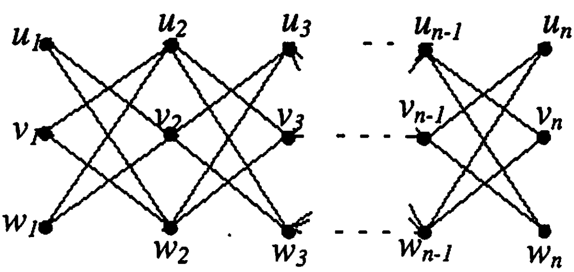
- <u>If $j = 3, 5, \ldots, m - 1$</u>

$$f(v_i^j) = \begin{cases} (j + 1)(n-1) + (i - 1), & i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ (4m-3j-1)\, n-i-(4m-3j-3), & i = 2, 4, 6, \ldots, n - 1 \text{ or } n \end{cases}$$

- <u>If $j = 2, 4, \ldots, m$</u>

$$f(v_i^j) = \begin{cases} (j - 2)(n - 1) + i & , i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ (4m-3j+2)n-\mathrm{i}-(4m-3j+1), & i = 2, 4, 6, \ldots, n - 1 \text{ or } n \end{cases}$$

(2) Let $P_n \wedge C_3$ be described as indicated in Figure(6):



Figure(6)

It is clear that the number of edges of the graph $P_n \wedge C_3$ is $6(n-1)$.

We define the labeling function

$$f: V(P_n \wedge C_3) \rightarrow \{0, 1, 2, \ldots, 12n - 13\}$$

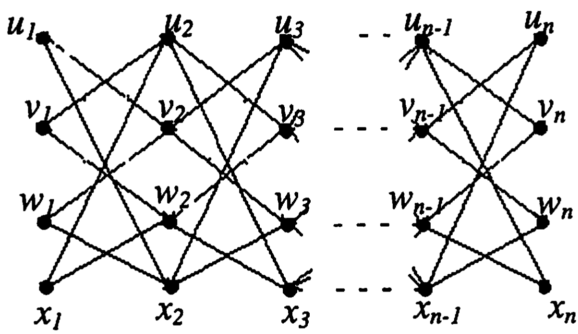
as follows:

$$f(u_i) = \begin{cases} i - 1 & , \quad i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ 8n - i - 7 & , \quad i = 2, 4, 6, \ldots, n - 1 \text{ or } n \end{cases}$$

$$f(v_i) = \begin{cases} 2n + i - 3 & , \quad i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ 12n - i - 11 & , \quad i = 2, 4, 6, \ldots, n - 1 \text{ or } n \end{cases}$$

$$f(w_i) = \begin{cases} 8n + i - 9 & , \quad i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ 10n - i - 9 & , \quad i = 2, 4, 6, \ldots, n - 1 \text{ or } n . \end{cases}$$

(3) Let $P_n \wedge C_4$ be described as indicated in Figure(7):



Figure(7)

The number of edges of the graph $P_n \wedge C_4$ is $8(n - 1)$. We define the labeling function :

$$f: V(P_n \wedge C_4) \rightarrow \{0, 1, 2, \ldots, 16n - 17\}$$

as follows:

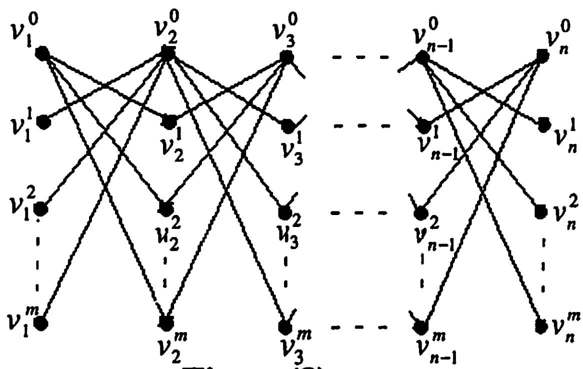
$$f(u_i) = \begin{cases} i - 1 & , \quad i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ 8n - i - 6 & , \quad i = 2, 4, 6, \ldots, n - 1 \text{ or } n \end{cases}$$

$$f(v_i) = \begin{cases} i & , \quad i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ 16n - i - 15 & , \quad i = 2, 4, 6, \ldots, n - 1 \text{ or } n \end{cases}$$

$$f(w_i)=\begin{cases} 2n+i-3 & , \quad i=1,3,5,\ldots,n \text{ or } n-1 \\ 4n-i-2 & , \quad i=2,4,6,\ldots,n-1 \text{ or } n \end{cases}$$

$$f(x_i)=\begin{cases} 2n+i-2 & , \quad i=1,3,5,\ldots,n \text{ or } n-1 \\ 12n-i-11 & , \quad i=2,4,6,\ldots,n-1 \text{ or } n\,. \end{cases}$$

(4) Let $P_n \wedge S_m$ be described as indicated in Figure(8):



Figure(8)

It is clear that the number of edges of the graph $P_n \wedge S_m$ is $2(n-1)m$. We define the labeling function:

$$f: V(P_n \wedge S_m) \rightarrow \{0, 1, 2, \ldots, 4(n-1)m-1\}$$

as follows:

$$f(v_i^0)=\begin{cases} i-1 & , \quad i=1,3,5,\ldots,n \text{ or } n-1 \\ 2(n-1)m-i+2 & , \quad i=2,4,6,\ldots,n-1 \text{ or } n \end{cases}$$

$$f(v_i^j)=\begin{cases} 2(n-1)(j-1)+i & , \quad i=1,3,5,\ldots,n \text{ or } n-1 \\ & , \quad j=1,2,3,4,\ldots,m \\ 2(n-1)(2m-j+1)-i+1 & , \quad i=2,4,6,\ldots,n-1 \text{ or } n \\ & , \quad j=1,2,3,4,\ldots,m. \end{cases}$$

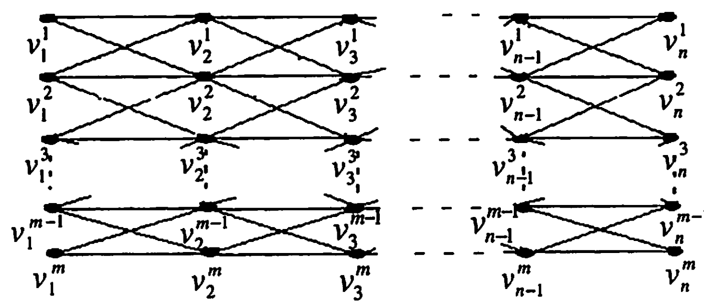
Let G_1 and G_2 be two disjoint graphs. *The* *symmetric product* $(G_1 \oplus G_2)$ of G_1 and G_2 is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1, v_1)(u_2, v_2): u_1u_2 \in E(G_1)$ or $v_1v_2 \in E(G_2)$ but not both$\}$.

Theorem 2.6.

The graphs $P_n \oplus \overline{K_m}$, m , $n \geq 2$ are odd-graceful.

Proof.

Let $P_n \oplus \overline{K_m}$ be described as indicated in Figure(9):



Figure(9)

It is clear that the number of edges of the graph $P_n \oplus \overline{K_m}$ is $(3m\text{-}2)(n\text{-}1)$. We define the labeling function

$f: V(P_n \oplus \overline{K_m}) \rightarrow \{0, 1, 2, \ldots, 2(3m - 2)(n - 1) - 1\}$

as follows:

$$f(v_i^j) = \begin{cases} 2(j-1)(n-1) + (i-1)\,, & i = 1, 3, 5, \ldots, n \text{ or } n\text{-}1 \\ & j = 1, 2, 3, \ldots, m \\ 2(3m-2j)(n-1) - i+1, & i = 2, 4, 6, \ldots, n\text{-}1 \text{ or } n \\ & j = 1, 2, 3, \ldots, m. \end{cases}$$

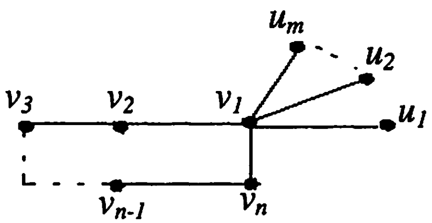
Let $G_n \circ S_m$ be the graph obtained by identifying a vertex of G_n with the centre of S_m.

Theorem 2.7.

The graphs $C_n \circ S_m$, n is even and $m \geq 1$ are odd-graceful.

Proof.

Let $C_n \circ S_m$ be described as indicated in Figure(10):



Figure(10)

The number of edges of the graph $C_n \circ S_m$ is $(m + n)$. We define the labeling function

$$f: V(C_n \circ S_m) \rightarrow \{0, 1, 2, \ldots, 2(m+n) - 1\}$$

as follows:

$$f(v_i) = \begin{cases} i - 1 & , \quad i = 1, 3, 5, \ldots, n - 1 \\ 2(m + n) + 1 - i & , \quad i = 2, 4, 6, \ldots, n - 2 \end{cases}$$

$f(v_n) = 2m + 3$

$f(u_i) = 2i + 1$, $1 \leq i < |m - n/2 + 2|$

and $|m - n/2 + 2| < i \leq m$.

For a graph G, *the splitting graph* of G, $S^1(G)$, is obtained from G by adding for each vertex v of G a new vertex v^1 so that v^1 is adjacent to every vertex that is adjacent to v.

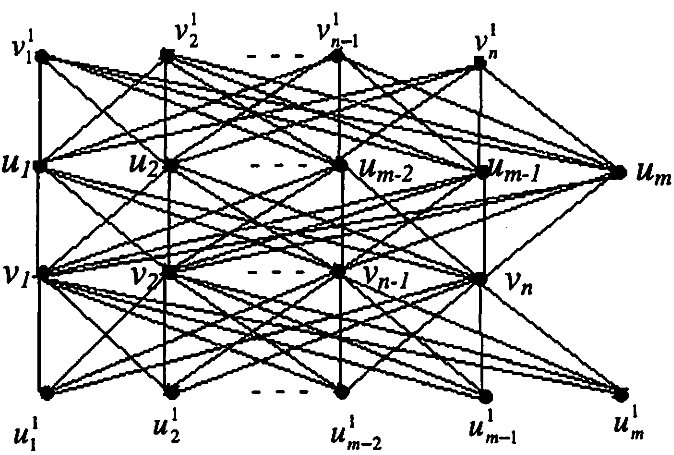
Theorem 2.8.

The following graphs are odd-graceful:

1. $S^1(K_{m,n})$, $m, n \geq 2$
2. $S^1(P_n \wedge P_2)$, $n \geq 2$
3. $S^1(S_n \wedge P_2)$, $n \geq 2$
4. $S^1(P_n \oplus \overline{K_2})$, $n \geq 2$

Proof.

(1) Let $(K_{m,n}) = \{u_1, u_2, u_3, \ldots, u_m ; v_1, v_2, v_3, \ldots, v_n\}$, $m, n \geq 2$. It is clear that the number of edges of the graph $S^1(K_{m,n})$ is $3mn$. The graph is indicated in Figure(11):



Figure(11)

We define the labeling function

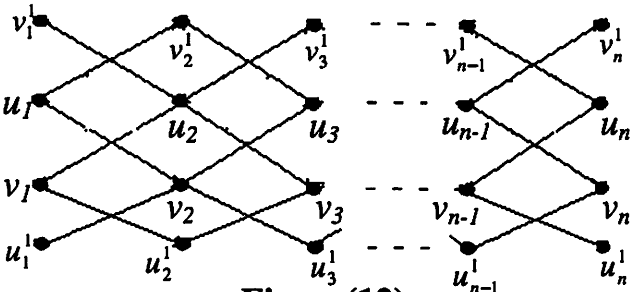
$$f : V(S^1(K_{m,n})) \to \{0, 1, 2, \ldots, 6mn - 1\}$$

as follows:

$$f(u_i) = 2mn + 2n(i - 1) \quad , \quad 1 \leq i \leq m$$
$$f(v_i) = 6mn - 2i + 1 \quad , \quad 1 \leq i \leq n$$
$$f(u_i^1) = 2n(i - 1) \quad , \quad 1 \leq i \leq m$$

$$f(v_i^1) = 4mn - 2i + 1 \quad , \quad 1 \le i \le n$$

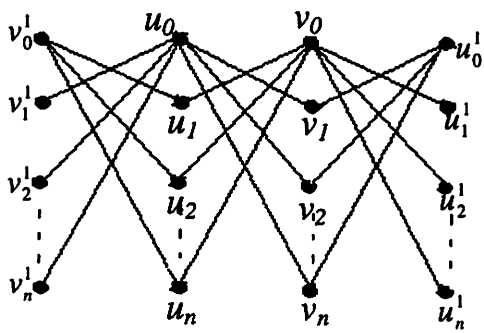
(2)) Let $S^1(P_n \wedge P_2)$ be described as indicated in Figure(12):



Figure(12)

It is clear that $S^1(P_n \wedge P_2) \cong (P_n \wedge P_4)$ which is an odd-graceful graph by Theorem 2.5. Hence $S^1(P_n \wedge P_2)$ is odd-graceful for each $n \ge 2$.

(3) Let $S^1(S_n \wedge P_2)$ be described as indicated in Figure(13):

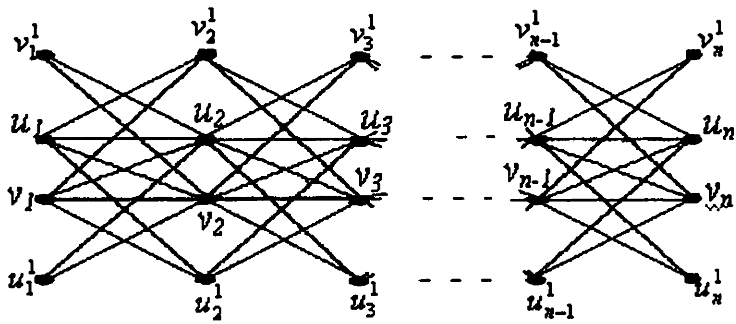


Figure(13)

It is clear that $S^1(S_n \wedge P_2) \cong (S_n \wedge P_4)$ which is an odd-graceful graph by Theorem 2.5. Hence $S^1(S_n \wedge P_2)$ is odd-graceful for each $n \ge 2$.

(4) Let $S^1(P_n \oplus \overline{K_2})$ be described as indicated in

Figure(14):



Figure(14)

It is clear that the number of edges of the graph $S^1(P_n \oplus \overline{K_2})$ is $12(n - 1)$. Now we define the labeling function

$$f : V(S^1(P_n \oplus \overline{K_2}) \to \{0, 1, 2, \ldots, 24n - 25\}$$

as follows:

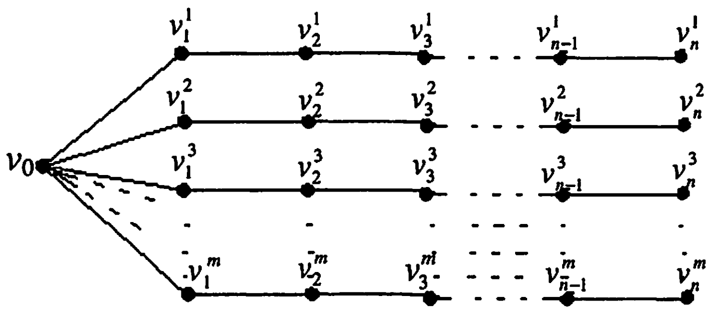
$$f(u_i) = \begin{cases} 2n + i - 3 & , \quad i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ 24n - i - 23 & , \quad i = 2, 4, 6, \ldots, n - 1 \text{ or } n \end{cases}$$

$$f(v_i) = \begin{cases} 4n + i - 5 & , \quad i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ 12n - i - 11 & , \quad i = 2, 4, 6, \ldots, n - 1 \text{ or } n \end{cases}$$

$$f(v_i^1) = \begin{cases} i - 1 & , \quad i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ 18n - i - 17 & , \quad i = 2, 4, 6, \ldots, n - 1 \text{ or } n \end{cases}$$

$$f(u_i^1) = \begin{cases} 6n + i - 7 & , \quad i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ 6n - i - 5 & , \quad i = 2, 4, 6, \ldots, n - 1 \text{ or } n. \end{cases}$$

We recall that the n-stars are the graphs obtained from a star S_m by identifying each pendant vertex of S_m with an end vertex of the path P_n. This is shown in Figure(15):



Figure(15)

Gnanajothi [2] proved the graphs n-stars in case n = 3, 4 are odd-graceful, we extend this result to n = 5, 6, 7, 9, 11 in the next theorem.

Theorem 2.9.

All 5, 6, 7, 9, and 11-star graphs are odd-graceful.

Proof.

- The number of edges of the graph 5-stars is $5m$, where the star is S_m. The graph is defined by taking n = 5 in Figure(15). Now we define the labeling function:

$$f: V(5\text{-stars}) \to \{0, 1, 2, 3, \ldots, 10m - 1\}$$

as follows

$$f(v_0) = 0$$
$$f(v_1^i) = 10m + 1 - 2i \quad , \ 1 \le i \le m$$
$$f(v_2^i) = 4m + 2 - 4i \quad , \ 1 \le i \le m$$
$$f(v_3^i) = 8m + 1 - 2i \quad , \ 1 \le i \le m$$
$$f(v_4^i) = 10m + 2 - 4i \quad , \ 1 \le i \le m$$
$$f(v_5^i) = 6m + 1 - 2i \quad , \ 1 \le i \le m.$$

- The number of edges of the graph 6-stars is $6m$. The graph is defined by taking $n = 6$ in Figure(15). Now we define the labeling function:

$$f: V(6\text{-stars}) \rightarrow \{0, 1, 2, 3, \ldots, 12m - 1\}$$

as follows

$$f(v_0) = 0$$
$$f(v_1^i) = 12m + 1 - 2i \quad , \; 1 \leq i \leq m$$
$$f(v_2^i) = 4m + 2 - 4i \quad , \; 1 \leq i \leq m$$
$$f(v_3^i) = 10m + 1 - 2i \quad , \; 1 \leq i \leq m$$
$$f(v_4^i) = 8m + 2 - 4i \quad , \; 1 \leq i \leq m$$
$$f(v_5^i) = 6m + 1 - 2i \quad , \; 1 \leq i \leq m$$
$$f(v_6^i) = 12m + 2 - 4i \quad , \; 1 \leq i \leq m.$$

- The number of edges of the graph 7-stars is $7m$. The graph is defined by taking $n = 7$ in Figure(15). We define the labeling function:

$$f: V(7\text{-stars}) \rightarrow \{0, 1, 2, 3, \ldots, 14m-1\}$$

as follows

$$f(v_0) = 0$$
$$f(v_1^i) = 14m + 1 - 2i \quad , \; 1 \leq i \leq m$$
$$f(v_2^i) = 4m + 2 - 4i \quad , \; 1 \leq i \leq m$$
$$f(v_3^i) = 12m + 1 - 2i \quad , \; 1 \leq i \leq m$$
$$f(v_4^i) = 8m + 2 - 4i \quad , \; 1 \leq i \leq m$$
$$f(v_5^i) = 10m + 1 - 2i \quad , \; 1 \leq i \leq m$$
$$f(v_6^i) = 12m + 2 - 4i \quad , \; 1 \leq i \leq m$$
$$f(v_7^i) = 4m + 1 - 2i \quad , \; 1 \leq i \leq m.$$

- The number of edges of the graph 9-stars is $9m$. The graph is defined by taking $n = 9$ in Figure(15). We define the labeling function:

$$f: V(9\text{-stars}) \rightarrow \{0, 1, 2, 3, \ldots, 18m - 1\}$$

as follows

$f(v_0) = 0$
$f(v_1^i) = 18m + 1 - 2i \quad , \ 1 \leq i \leq m$
$f(v_2^i) = 4m + 2 - 4i \quad , \ 1 \leq i \leq m$
$f(v_3^i) = 16m + 1 - 2i \quad , \ 1 \leq i \leq m$
$f(v_4^i) = 8m + 2 - 4i \quad , \ 1 \leq i \leq m$
$f(v_5^i) = 14m + 1 - 2i \quad , \ 1 \leq i \leq m$
$f(v_6^i) = 18m + 2 - 4i \quad , \ 1 \leq i \leq m$
$f(v_7^i) = 6m + 1 - 2i \quad , \ 1 \leq i \leq m$
$f(v_8^i) = 12m + 2 - 4i \quad , \ 1 \leq i \leq m$
$f(v_9^i) = 12m + 1 - 2i \quad , \ 1 \leq i \leq m.$

- The number of edges of the graph 11-stars is $11m$. The graph is defined by taking $n = 11$ in Figure(15). We define the labeling function:

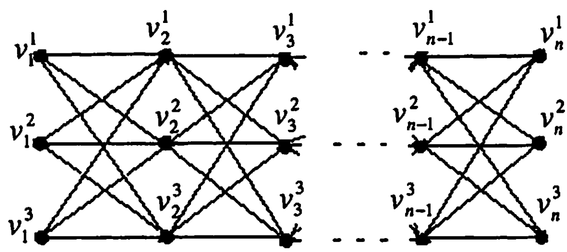
$$f : V(11\text{-stars}) \rightarrow \{0, 1, 2, 3, \ldots, 22m - 1\}$$

as follows

$f(v_0) = 0$
$f(v_1^i) = 22m + 1 - 2i \quad , \ 1 \leq i \leq m$
$f(v_2^i) = 4m + 2 - 4i \quad , \ 1 \leq i \leq m$
$f(v_3^i) = 20m + 1 - 2i \quad , \ 1 \leq i \leq m$
$f(v_4^i) = 8m + 2 - 4i \quad , \ 1 \leq i \leq m$
$f(v_5^i) = 18m + 1 - 2i \quad , \ 1 \leq i \leq m$
$f(v_6^i) = 12m + 2 - 4i \quad , \ 1 \leq i \leq m$
$f(v_7^i) = 6m + 1 - 2i \quad , \ 1 \leq i \leq m$
$f(v_8^i) = 22m + 2 - 4i \quad , \ 1 \leq i \leq m$
$f(v_9^i) = 12m + 1 - 2i \quad , \ 1 \leq i \leq m$
$f(v_{10}^i) = 16m + 2 - 4i \quad , \ 1 \leq i \leq m$
$f(v_{11}^i) = 16m + 1 - 2i \quad , \ 1 \leq i \leq m.$

Theorem 2.10.

The following graph G, shown in Figure(16), is odd-graceful.



Figure(16)

Proof.

It easy to see that the number of edges of the graph G is $9(n-1)$. We define the labeling function:

$$f: V(G) \rightarrow \{0, 1, 2, 3, \ldots, 18n - 19\}$$

as follows

$$f(v_i^1) = \begin{cases} i - 1 & , \quad i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ 18n - i - 17 & , \quad i = 2, 4, 6, \ldots, n - 1 \text{ or } n \end{cases}$$

$$f(v_i^2) = \begin{cases} 2n + i - 3 & , \quad i = 1, 3, 5, \ldots, n \text{ or } n - 1 \\ 14n - i - 13 & , \quad i = 2, 4, 6, \ldots, n-1 \text{ or } n \end{cases}$$

$$f(v_i^3) = \begin{cases} 12n + i - 13 & , \quad i = 1, 3, 5, \ldots, n \text{ or } n-1 \\ 10n - i - 9 & , \quad i = 2, 4, 6, \ldots, n-1 \text{ or } n. \end{cases}$$

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