Some Discussions on Super Edge-Magic Labelings of $St(a_1, ..., a_n)$

Gao Zhenbin   Fan Chongjin

ABSTRACT. Lee and Kong conjecture that if $n \geqslant 1$ is an odd number, then $St(a_1, a_2, ..., a_n)$ would be super edge-magic, and meanwhile they proved that the following graphs are super edge-magic: $St(m, n)$ ($n \equiv 0 mod(m+1)$), $St(1, k, n)$ ($k = 1, 2$ or n), $St(2, k, n)$ ($k = 2, 3$), $St(1, 1, k, n)$ ($k = 2, 3$), $St(k, 2, 2, n)$ ($k = 1, 2$). In this paper, the conjecture is further discussed and it is proved that $St(1, m, n)$, $St(3, m, m+1)$, $St(n, n+1, n+2)$ are super edge-magic, and under some conditions $St(a_1, a_2, ..., a_{2n+1})$, $St(a_1, a_2, ..., a_{4n+1})$, $St(a_1, a_2, ..., a_{4n+3})$ are also super edge-magic.

1. INTRODUCTION

In 1970, Kotzig and Rosa [1] defined an edge-magic total labeling of a graph $G(V, E)$; the super edge-magic labeling was introduced by Enomoto et al [2], and they proved the following: C_n is super edge-magic if and only if n is odd, and caterpillars are super edge-magic; Lee and Shan [3] proved that trees with up to 17 vertices with a computer are super edge-magic. Figueroa-Centeno et al [4] proved $K_{1,m} \cup K_{1,n}$ is super edge-magic under some conditions. MacDougall and Wallis [5] discussed the super edge-magic labeling of C_v^t graph. Enomoto et al [6] proved that every graph can be embedded in a connected super edge-magic graph as an induced subgraph. Slamin et al [7] proved that the friendship graph consisting of n triangles is super edge-magic if and only if n is 3, 4, 5 or 7. Fukuchi [8] showed that the generalized Petersen Graph $P(n, 2)$ is super edge-magic if $n \geqslant 3$ is odd. Ngurah et al [9] showed that nP_3 is super edge-magic if $n \geqslant 4$ is even. Figueroa-Centeno et al [10] proved that a graph is super edge-magic if and only if it is strongly 1-harmonious. Chen [11] proved that kP_2 is super edge-magic if and only if k is odd. Figueroa-Centeno et al [12] proved that when k is odd kP_n is super edge-magic. More results of super edge-magic graphs, can be found in the survey paper Gallian [13].

$St(a_1, a_2, ..., a_n)$ is a disjoint union of some star graphs $St(a_1), ..., St(a_n)$. Lee and Kong [14] proved that the following graphs are super edge-magic:

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$St(m,n)$ ($n \equiv 0 mod(m+1)$), $St(1,k,n)$ ($k = 1,2$ or n), $St(2,k,n)$ ($k = 2,3$), $St(1,1,k,n)$ ($k = 2,3$), $St(k,2,2,n)$ ($k = 1,2$). They conjectured that if $n \geqslant 1$ is an odd number $St(a_1, a_2, ..., a_n)$ would be super edge-magic. In this paper, we further discussed $St(a_1, a_2, ..., a_n)$ and got some super edge-magic labelings of $St(a_1, a_2, ..., a_n)$.

2. BASIC NOTATIONS

Definition 1. Edge-magic total graph: For a graph $G(V,E)$, a bijective function $f : V \cup E \to \{1, 2, \cdots, |V \cup E|\}$ is called an edge-magic total labeling of G if $f(u) + f(v) + f(uv)$ is a constant – the value is independent on the choice of any edge uv of G. If such a labeling exists, then G is called an edge-magic total graph.

Definition 2. Super edge-magic graph: Let a graph $G(V,E)$ be edge-magic total, and for the above function f, if $f(V) = \{1, 2, \cdots, |V|\}$, then f is called a super edge-magic labeling of the graph, thus a super edge-magic graph is a graph that admits a super edge-magic labeling.

Definition 3. Star: a star is a graph with $n+1$ vertices and one vertex, called the central vertex, is connected to each of the other vertices with a single edge and these are the only edges in the graph, we use $St(n)$ to denote the star.

Definition 4. $St(a_1, a_2, ..., a_n)$: $St(a_1, a_2, ..., a_n)$ is a disjoint union of some star graphs $St(a_1), ..., St(a_n)$. The central vertex of $St(a_i)$ is denoted by $x_{i,0}$, and the other vertices are denoted by $x_{i,1}, x_{i,2}, \cdots, x_{i,a_i}$ respectively.

2. MAIN RESULTS

Theorem 1. When $n = 2m+1$, and $St(a_1, a_2, ..., a_n)$ satisfies the following items, then $St(a_1, a_2, ..., a_n)$ is super edge-magic:

(1) $a_i = a$ $(1 \leqslant i \leqslant m+1)$.

(2) $a_i = b$ $(m+2 \leqslant i \leqslant 2m+1)$.

Proof No harm to generality, we suppose $a \geqslant b$. And we have

$$|V| = (a+b+2)m + a + 1, \quad |E| = (a+b)m + a.$$

First, we construct the labeling f of the vertices as follows:

$f : \{x_{1,0}, x_{2,0}, \cdots, x_{2m+1,0}\} \to \{2m+1, 2m-1, \cdots, 1, 2m, 2m-2, \cdots, 2\}$.

$f : \{x_{1,k}, x_{2,k}, \cdots, x_{2m+1,k}\} \to \{(2m+1)+1, k(2m+1)+2, \cdots, (k+1)(2m+1)\}$ $(1 \leqslant k \leqslant b)$,

$f : \{x_{1,k}, x_{2,k}, \cdots, x_{m+1,k}\} \to \{(b+1)(2m+1)+(k-b-1)m+k-b, (b+1)(2m+1)+(k-b-1)m+k-b+1, \cdots, (b+1)(2m+1)+(k-b)m+k-b\}$ $(b < k \leqslant a)$;

therefore the labeling numbers of the vertices in $St(a_1, a_2, ..., a_n)$ are distinct, the maximum is $|V|$ and the minimum is 1.

Now, we compute $f(x_{i,0}) + f(x_{i,k})$. First we have

$f(x_{i,0})+f(x_{i,k}) \in \{(k+1)(2m+1)+1, (k+1)(2m+1), \cdots, k(2m+1)+m+2, (k+1)(2m+1)+m+1, (k+1)(2m+1)+m, \cdots, k(2m+1)+2\}$ $(1 \leqslant k \leqslant b)$,

$f(x_{i,0})+f(x_{i,k}) \in \{(b+2)(2m+1)+(k-b-1)m+k-b, (b+2)(2m+1)+(k-b-1)m+k-b-1, \cdots, (b+2)(2m+1)+(k-b)m+k-b+1\}$ $(b < k \leqslant a)$; by the above computation, we see that $f(x_{i,0})+f(x_{i,k})$ are $|E|$ distinct numbers. The maximum of them is $|E|+3m+2$ and the minimum is $3m+3$.

Now, we can label the edges of $St(a_1, a_2, ..., a_n)$. By the increasing order of $f(x_{i,0})+f(x_{i,k})$, label the edges with $|V \cup E|, |V \cup E|-1, \cdots, |V|+1$. From the labelings of the vertices and the edges, for any i, k, we have

$f(x_{i,0})+f(x_{i,k})+f(x_{i,0}x_{i,k}) = |V \cup E|+3m+3$.

So, $St(a_1, a_2, ..., a_n)$ is super edge-magic.

Example 1 When $m=4, a=5, b=2$, $St(a_1, ..., a_n)$ is illustrated in Figure 1.

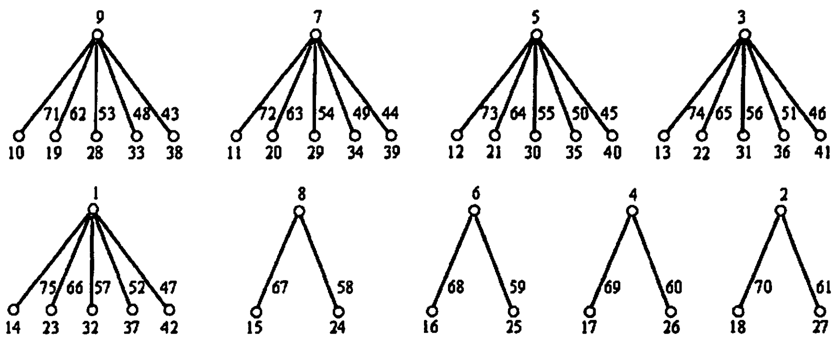

Figure 1

Theorem 2. When $n = 4m+1$, and $St(a_1, a_2, ..., a_n)$ satisfies the following items, then $St(a_1, a_2, ..., a_n)$ is super edge-magic:
(1) $a_i = a_{2m+2-i} = a$ $(i = 1, 2, \cdots, s)$, $a_i = b$ $(i = s+1, s+2, \cdots, 2m+1-s)$ $(b > a, s \leq m)$.
(2) $a_i = a_{6m+3-i} = c$ $(i = 2m+2, 2m+3, \cdots, 2m+1+t)$, $a_i = d(i = 2m+2+t, 2m+3+t, \cdots, 4m+1-t)$ $(d > c, t \leq m)$.
Proof. We label the vertices. let

$f: \{x_{1,0}, x_{2,0}, \cdots, x_{4m+1,0}\} \rightarrow \{4m+1, 4m-1, \cdots, 1, 4m, 4m-2, \cdots, 2\}$.

$f: \{x_{1,k}, x_{2,k}, \cdots, x_{2m+1,k}\} \rightarrow \{4m+2+(k-1)(2m+1), 4m+3+(k-1)(2m+1), \cdots, 4m+1+k(2m+1)\} (1 \leqslant k \leqslant a)$.

$f: \{x_{s+1,k}, x_{s+2,k}, \cdots, x_{2m+1-s,k}\} \rightarrow \{4m+2+a(2m+1)+(k-a-1)(2m+1-2s), 4m+3+a(2m+1)+(k-a-1)(2m+1-2s), \cdots, 4m+1+a(2m+1)+(k-a)(2m+1-2s)\}$ $(a+1 \leq k \leq b)$.

$f: \{x_{2m+2,k}, x_{2m+3,k}, \cdots, x_{4m+1,k}\} \rightarrow \{(2b+4)m+b+2-2s(b-a)+$

$2(k-1)m, (2b+4)m+b+3-2s(b-a)+2(k-1)m, \cdots, (2b+4)m+b+2-2s(b-a)+2km\}$ $(1 \le k \le c)$.

$f: \{x_{2m+2+t,k}, x_{2m+3+t,k}, \cdots, x_{4m+1-t,k}\} \rightarrow \{(2b+4)m+b+2-2s(b-a)+2mc+2(k-c-1)(m-t), (2b+4)m+b+3-2s(b-a)+2mc+2(k-c-1)(m-t), \cdots, (2b+4)m+b+1-2s(b-a)+2mc+2(k-c)(m-t)\}$ $(c-1 \le k \le d)$.

So the labeling numbers of the vertices in $St(a_1, a_2, ..., a_n)$ are distinct, the maximum of them is $|V|$ and the minimum is 1. And we have

$f(x_{i,0}) + f(x_{i,j}) \in \{6m+3, 6m+4, \cdots, 6m+2+|E|\}$.

Now we label the edges of $St(a_1, a_2, ..., a_n)$. By the increasing order of $f(x_{i,0}) + f(x_{i,j})$, label the edges with $|V \cup E|, |V \cup E| - 1, \cdots, |V| + 1$. So, for any i, j, we have

$f(x_{i,0}) + f(x_{i,j}) + f(x_{i,0}x_{i,j}) = |V \cup E| + 6m + 3$.

Finally, we know that $St(a_1, a_2, ..., a_n)$ is super edge-magic.

Example 2 When $m = 3, a = 3, b = 4, c = 2, d = 5$, $St(a_1, \dots, a_n)$ is illustrated in Figure 2.

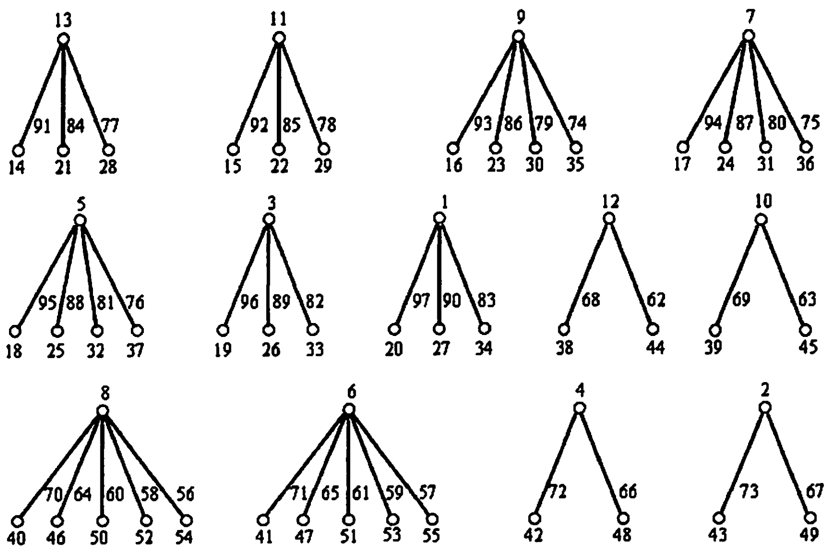

Figure 2

Theorem 2′ . When $n = 4m + 3$, and $St(a_1, a_2, ..., a_n)$ satisfies the following items, then $St(a_1, a_2, ..., a_n)$ is super edge-magic:

(1) $a_i = a$ $(i = 1, 2, \cdots, k, 2m+3-k, 2m+4-k, \cdots, 2m+2)$, $a_i = b$ $(i = k+1, k+2, \cdots, 2m+2-k)$ $(b > a, k \le m)$.

(2) $a_i = c$ $(i = 2m+3, 2m+4, \cdots, 2m+2+l, 4m+4-l, 4m+5-l, \cdots, 4m+3)$, $a_i = d$ $(i = 2m+3+l, 2m+4+l, \cdots, 4m+3-l)$ $(d > c, l \le m)$.

Proof. The proof is similar to Theorem 2, omitted.

Theorem 3. $St(1,m,n)$ is super edge-magic.
Proof. First, we have $|V| = m+n+4$, $|E| = m+n+1$.
We label the vertices as follows:
$f(x_{1,0}) = m+2$, $f(x_{2,0}) = m+4$, $f(x_{3,0}) = m+1$, $f(x_{1,1}) = m+3$,
$f : x_{2,j} \to j\ (j = 1,2,\cdots,m)$,
$f : x_{3,j} \to m+4+j\ (j = 1,2,\cdots,n)$.
The labeling numbers of the vertices of $St(1,m,n)$ are distinct, the maximum is $|V|$ and the minimum is 1. Next, we label the edgesas follows:
$f(x_{1,0}x_{1,1}) = m+2n+5$,
$f : x_{2,0}x_{2,j} \to 2m+2n+6-j\ (j = 1,2,\cdots,m)$,
$f : x_{3,0}x_{3,j} \to m+2n+5-j\ (j = 1,2,\cdots,n)$.
So, for any i, j, we have
$f(x_{i,0}) + f(x_{i,j}) + f(x_{i,0}x_{i,j}) = 3m+2n+10$.
Finally, we see that $St(1,m,n)$ is super edge-magic.

Theorem 4. $St(3,m,m+1)$ is super edge-magic.
Proof. Now, $|V| = 2m+7$, $|E| = 2m+4$.
We label the vertices as follows:
$f(x_{1,0}) = 1$, $f(x_{2,0}) = m+4$, $f(x_{3,0}) = m+3$,
$f : x_{1,j} \to 2m+4+j\ (j = 1,2,3)$,
$f : x_{2,j} \to m+4+j\ (j = 1,2,\cdots,m)$,
$f : x_{3,j} \to 1+j\ (j = 1,2,\cdots,m+1)$.
The labeling numbers of the vertices of $St(1,m,n)$ are distinct, the maximum is $|V|$ and the minimum is 1. Next, we label the edges as follows:
$f : x_{1,0}x_{1,j} \to 3m+11-j\ (j = 1,2,3)$,
$f : x_{2,0}x_{2,j} \to 3m+8-j\ (j = 1,2,\cdots,m)$,
$f : x_{3,0}x_{3,j} \to 4m+12-j\ (j = 1,2,\cdots,m+1)$.
So, for any i, j, we have
$f(x_{i,0}) + f(x_{i,j}) + f(x_{i,0}x_{i,j}) = 5m+16$.
Hence, $St(3,m,m+1)$ is super edge-magic.

Theorem 5. $St(m,m+1,m+2)$ is super edge-magic.
Proof. Here, $|V| = 3m+6$, $|E| = 3m+3$.
The labeling of the vertices as follows:
$f(x_{1,0}) = 2m+4$, $f(x_{2,0}) = m+3$, $f(x_{3,0}) = 1$,
$f : x_{1,j} \to m+3+j\ (j = 1,2,\cdots,m)$,
$f : x_{2,j} \to 1+j\ (j = 1,2,\cdots,m+1)$,
$f : x_{3,j} \to 2m+4+j\ (j = 1,2,\cdots,m+2)$.
The labeling of the edges as follows:
$f : x_{1,0}x_{1,j} \to 4m+7-j\ (j = 1,2,\cdots,m)$,
$f : x_{2,0}x_{2,j} \to 6m+10-j\ (j = 1,2,\cdots,m+1)$,

$f: x_{3,0}x_{3,j} \to 5m+9-j$ $(j=1,2,\cdots,m+2)$.
$f(x_{i,0})+f(x_{i,j})+f(x_{i,0}x_{i,j})=7m+14$.
As the above, $St(m,m+1,m+2)$ is super edge-magic.

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Gao Zhenbin Fan Chongjin , College of Science, Harbin Engineering University,
Harbin 150001, Heilongjiang Province, P.R. China
E-mail address: gaozenbin@YaHoo.com.cn