On the signless Laplacian spectral radius of digraphs

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April 3, 2012

Abstract

Let $G = (V, E)$ be a digraph with n vertices and m arcs without loops and multiarcs, $V = \{v_1, v_2, \ldots, v_n\}$. Denote the outdegree and average 2-outdegree of the vertex v_i by d_i^+ and m_i^+, respectively. Let $A(G)$ be the adjacency matrix and $D(G) = diag(d_1^+, d_2^+, \ldots, d_n^+)$ be the diagonal matrix with outdegree of the vertices of the digraph G. Then we call $Q(G) = D(G) + A(G)$ signless Laplacian matrix of G. In this paper, we obtain some upper and lower bounds for the spectral radius of $Q(G)$ which is called signless Laplacian spectral radius of G. We also show that some bounds involving outdegrees and the average 2-outdegrees of the vertices of G can be obtained from our bounds.

1 Introduction

Let G be a strongly connected digraph with n vertices and m arcs without loops and multiarcs, $V = \{v_1, v_2, \ldots, v_n\}$. If (v_i, v_j) be an arc of G, then v_i is called the initial vertex and v_j the terminal vertex of this arc. The outdegree d_i^+ of a vertex v_i in the digraph G is defined to be the number of arcs in G with initial vertex v_i. Let t_i^+ be the sum of the outdegrees of all vertices in $N_i^+(v_i) = \{v_j : (v_i, v_j) \in E\}$ and call it the 2-outdegree. Moreover, call $m_i^+ = \frac{t_i^+}{d_i^+}$ the average 2-outdegree, $1 \leq i \leq n$.

The signless Laplacian eigenvalues $q_1, q_2, \ldots, q_n$ of G are the eigenvalues of its signless Laplacian matrix $Q(G)$. In general $Q(G)$ is not symmetric

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and so its eigenvalues can be complex numbers. We usually assume that $|q_1| \geq |q_2| \geq \cdots \geq |q_n|$. The signless Laplacian spectral radius of G is denoted and defined as $q(G) = |q_1|$ i.e., it is the largest absolute value of the signless Laplacian eigenvalues of G. Since $Q(G)$ is a nonnegative matrix, it follows from Perron Frobenius theory that $q(G) = q_1$ is a real number.

The signless Laplacian matrix of a simple, undirected graph G_1 is $Q(G_1) = D(G_1) + A(G_1)$ where $A(G_1)$ and $D(G_1)$ are the adjacency matrix and the diagonal matrix of the vertex degrees of G, respectively. Since $Q(G_1)$ is a real symmetric matrix, all its eigenvalues are real numbers. So the signless Laplacian spectral radius $q(G_1)$ of G_1 is defined to be the largest eigenvalue of $Q(G_1)$. For applications its is crucial to be able to compute or at least estimate $q(G_1)$ for a given simple undirected graph G_1. This a classical problem with numerous results pertaining to it (see [1,2,5,6,7,9]).

In this paper, we study on the signless Laplacian spectral radius $q(G)$ of the digraph G. We obtain some upper and lower bounds for it and also show that some bounds involving outdegrees and the average 2-outdegree of the vertices of G can be obtained from our bounds as:

$$\min\left\{d_i^+ + d_j^+ : (v_i, v_j) \in E\right\} \leq q(G) \leq \max\left\{d_i^+ + d_j^+ : (v_i, v_j) \in E\right\} \quad (1)$$

$$\min\left\{d_i^+ + m_i^+ : v_i \in V(G)\right\} \leq q(G) \leq \max\left\{d_i^+ + m_i^+ : v_i \in V(G)\right\} \quad (2)$$

$$q(G) \leq \max\left\{\frac{d_i^+ + d_j^+ + \sqrt{\left(d_i^+ - d_j^+\right)^2 + 4m_i^+ m_j^+}}{2} : (v_i, v_j) \in E\right\} \quad (3)$$

$$q(G) \geq \min\left\{\frac{d_i^+ + d_j^+ + \sqrt{\left(d_i^+ - d_j^+\right)^2 + 4m_i^+ m_j^+}}{2} : (v_i, v_j) \in E\right\} \quad (4)$$

$$q(G) \leq \max\left\{d_i^+ + \sqrt{\sum_{(v_j, v_i) \in E} d_j^+} : v_i \in V(G)\right\}. \quad (5)$$

The terminology not defined in here can be found in [3,4].

2 Preliminary Lemmas

Now we give the following lemmas which will be used then.

Lemma 2.1 *[8] Let M be an irreducible nonnegative matrix. Then $\rho(M)$ is an eigenvalue of M and there is a positive vector X such that $MX = \rho(M)X$.*

Lemma 2.2 *[8] Let $M = (m_{ij})$ be an $n \times n$ nonnegative matrix and let $R_i(M)$ be the i-th row sum of M, i.e., $R_i(M) = \sum_{j=1}^{n} m_{ij}$ $(1 \le i \le n)$. Then*

$$\min\{R_i(M) : 1 \le i \le n\} \le \rho(M) \le \max\{R_i(M) : 1 \le i \le n\}.$$

If M is irreducible then, each equality holds if and only if $R_1 = R_2 = \cdots = R_n$.

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^+ = \{x : x \in \mathbb{R}, x > 0\}$. Now we present the main results of this paper.

3 Some upper and lower bounds for the signless Laplacian spectral radius of digraphs

The similar techniques in this section have been used to derive upper bounds for the signless Laplacian spectral radius of undirected graphs in [6]. Now we will give a generation of their results on the signless Laplacian spectral radius for digraphs.

Theorem 3.1 *Let $G = (V, E)$ be a digraph on n vertices and $b_i^+ \in \mathbb{R}^+$ $(1 \le i \le n)$. Then*

$$\min\left\{r_i^+ : v_i \in V(G)\right\} \le q(G) \le \max\left\{r_i^+ : v_i \in V(G)\right\} \tag{6}$$

where $r_i^+ = d_i^+ + \frac{1}{b_i^+} \sum_{(v_i,v_j)\in E} b_j^+$ $(1 \le i \le n)$. Moreover, if G is strongly connected digraph equality holds on both sides of (6) if and only if $r_1^+ = r_2^+ = \cdots = r_n^+$.

Proof. Let $B = diag\left(b_1^+, b_2^+, \ldots, b_n^+\right)$ be an $n \times n$ diagonal matrix. Considering the matrix $B^{-1}Q(G)B$, it is easy to see that

$$R_i\left(B^{-1}Q(G)B\right) = d_i^+ + \frac{1}{b_i^+} \sum_{(v_i,v_j)\in E} b_j^+.$$

We note that

$$q(G) = q\left(B^{-1}Q(G)B\right).$$

Then from Lemma 2.2, (6) follows and we conclude that if G is strongly connected digraph the equality holds on both sides of (6) if and only if $r_1^+ = r_2^+ = \cdots = r_n^+$. ■

Theorem 3.2 *Let $G = (V, E)$ be a strongly connected digraph on n vertices and $b_i^+ \in \mathbb{R}^+$ $(1 \le i \le n)$. Then*

$$q(G) \le \max\left\{\frac{d_i^+ + d_j^+ + \sqrt{\left(d_i^+ - d_j^+\right)^2 + 4c_i^+ c_j^+}}{2} : (v_i, v_j) \in E\right\}. \quad (7)$$

If

$$q(G) > \max\left\{\frac{d_i^+ + d_j^+ - \sqrt{\left(d_i^+ - d_j^+\right)^2 + 4c_i^+ c_j^+}}{2} : (v_i, v_j) \in E\right\},$$

then

$$q(G) \ge \min\left\{\frac{d_i^+ + d_j^+ + \sqrt{\left(d_i^+ - d_j^+\right)^2 + 4c_i^+ c_j^+}}{2} : (v_i, v_j) \in E\right\} \quad (8)$$

where $c_i^+ = \frac{1}{b_i^+} \sum_{(v_i, v_j) \in E} b_j^+$ $(1 \le i \le n)$.

Proof. *Upper bound:* From Lemma 2.1, there exist a positive eigenvector $X = (x_1, x_2, \ldots, x_n)$ of $B^{-1}Q(G)B$ corresponding to the eigenvalue $q\left(B^{-1}Q(G)B\right)$. We assume that one of the eigencomponents, say x_i, is equal to 1 and the other eigencomponents are less than or equal to 1, i.e., $x_i = 1$ and $0 < x_k \le 1$ for all $1 \le k \le n$. Also let $x_j = \max\{x_k : (v_i, v_k) \in E\}$. Since

$$B^{-1}Q(G)BX = q(G)X, \quad (9)$$

we have

$$q(G) = d_i^+ + \frac{1}{b_i^+} \sum_{(v_i, v_k) \in E} b_k^+ x_k \le d_i^+ + \frac{1}{b_i^+} \sum_{(v_i, v_k) \in E} b_k^+ x_j \quad (10)$$

and

$$q(G)\,x_j = d_j^+ x_j + \frac{1}{b_j^+}\sum_{(v_j,v_k)\in E} b_k^+ x_k \le d_j^+ x_j + \frac{1}{b_j^+}\sum_{(v_j,v_k)\in E} b_k^+. \tag{11}$$

From (10) and (11) we get

$$\left(q(G) - d_i^+\right)\left(q(G) - d_j^+\right) \le \frac{1}{b_i^+ b_j^+}\sum_{(v_i,v_k)\in E} b_k^+ \sum_{(v_j,v_k)\in E} b_k^+ = c_i^+ c_j^+.$$

Therefore we obtain

$$q(G) \le \frac{d_i^+ + d_j^+ + \sqrt{\left(d_i^+ - d_j^+\right)^2 + 4c_i^+ c_j^+}}{2}.$$

Hence (7) holds.

Lower Bound: In order to prove lower bound we assume that one of the eigencomponents, say x_i, is equal to 1 and the other eigencomponents are greater than or equal to 1, i.e., $x_i = 1$ and $x_k \ge 1$ for all $1 \le k \le n$. Also let $x_j = \min\{x_k : (v_i, v_k) \in E\}$. From (9) we have

$$q(G) = d_i^+ + \frac{1}{b_i^+}\sum_{(v_i,v_k)\in E} b_k^+ x_k \ge d_i^+ + \frac{1}{b_i^+}\sum_{(v_i,v_k)\in E} b_k^+ x_j \tag{12}$$

and

$$q(G)\,x_j = d_j^+ x_j + \frac{1}{b_j^+}\sum_{(v_j,v_k)\in E} b_k^+ x_k \ge d_j^+ x_j + \frac{1}{b_j^+}\sum_{(v_j,v_k)\in E} b_k^+. \tag{13}$$

From (12) and (13) we obtain

$$\left(q(G) - d_i^+\right)\left(q(G) - d_j^+\right) \ge \frac{1}{b_i^+ b_j^+}\sum_{(v_i,v_k)\in E} b_k^+ \sum_{(v_j,v_k)\in E} b_k^+ = c_i^+ c_j^+.$$

By solving the above inequality with respect to the condition

$$q(G) > \max\left\{\frac{d_i^+ + d_j^+ - \sqrt{\left(d_i^+ - d_j^+\right)^2 + 4c_i^+ c_j^+}}{2} : (v_i, v_j) \in E\right\},$$

we arrive at

$$q(G) \geq \frac{d_i^+ + d_j^+ + \sqrt{\left(d_i^+ - d_j^+\right)^2 + 4c_i^+ c_j^+}}{2}.$$

This completes the proof of theorem. ■

Theorem 3.3 *Let $G = (V, E)$ be a strongly connected digraph on n vertices and $b_i^+ \in \mathbb{R}^+$ $(1 \leq i \leq n)$. Then*

$$q(G) \leq \max\left\{ d_i^+ + \sqrt{\sum_{(v_j, v_i) \in E} s_j^+} : v_i \in V(G) \right\} \tag{14}$$

where $s_i^+ = \frac{1}{b_i^{+2}} \sum_{(v_i, v_j) \in E} b_j^{+2}$ and if the equality holds then $d_i^+ + \sqrt{\sum_{(v_j, v_i) \in E} s_j^+}$ $(1 \leq i \leq n)$ is a constant.

Proof. Let $X = (x_1, x_2, \ldots, x_n)$ be an eigenvector of $B^{-1}Q(G)B$ corresponding to the eigenvalue $q\left(B^{-1}Q(G)B\right)$, where $B = diag\left(b_1^+, b_2^+, \ldots, b_n^+\right)$. We assume that $x_i = 1$ and $0 < x_k \leq 1$ for all $1 \leq k \leq n$. Also let $x_j = \max\{x_k : (v_i, v_k) \in E\}$. From (9), we have

$$\left(q(G) - d_k^+\right) x_k = \sum_{(v_k, v_h) \in E} \frac{b_h^+}{b_k^+} x_h.$$

Using Cauchy-Schwartz inequality we get

$$\begin{aligned} \left[\left(q(G) - d_k^+\right) x_k\right]^2 &\leq \sum_{(v_k, v_h) \in E} \frac{b_h^{+2}}{b_k^{+2}} \sum_{(v_k, v_h) \in E} x_h^2 \\ &= s_k^+ \sum_{(v_k, v_h) \in E} x_h^2. \end{aligned}$$

Thus we obtain

$$\sum_{k=1}^{n} \left(q(G) - d_k^+\right)^2 x_k^2 \leq \sum_{k=1}^{n} s_k^+ \sum_{(v_k, v_h) \in E} x_h^2 = \sum_{k=1}^{n} \left(\sum_{(v_h, v_k) \in E} s_h^+ \right) x_k^2$$

and

$$\sum_{k=1}^{n} \left[\left(q(G) - d_k^+\right)^2 - \sum_{(v_h, v_k) \in E} s_h^+ \right] x_k^2 \leq 0. \tag{15}$$

Therefore we conclude that there exist some k such that

$$\left(q(G)-d_k^+\right)^2-\sum_{(v_h,v_k)\in E} s_h^+ \le 0.$$

Then we obtain that

$$q(G)\le d_k^+ + \sqrt{\sum_{(v_h,v_k)\in E} s_h^+}.$$

Hence, (14) holds. If the equality holds in (14), then there exist some k, say $k=1$, such that $q(G)=d_1^+ + \sqrt{\sum_{(v_j,v_1)\in E} s_j^+}$. From (15) we have

$$\sum_{k=2}^{n}\left[\left(q(G)-d_k^+\right)^2-\sum_{(v_h,v_k)\in E} s_h^+\right]x_k^2 \le 0.$$

By similar reasoning as above we get

$$q(G)=d_k^+ + \sqrt{\sum_{(v_h,v_k)\in E} s_h^+},\ 1\le k\le n.$$

That means $d_k^+ + \sqrt{\sum_{(v_h,v_k)\in E} s_h^+}$ $(1\le k\le n)$ is a constant. Hence the result.
■

Remark 3.4 *From Theorem 3.1, 3.2 and 3.3, we obtain the following bounds.*

1) Taking $b_i^+ = d_i^+$ in (6), we have the bounds in (2).

2) Taking $b_i^+ = 1$ in (7) and (8), we have the upper and lower bounds in (1), respectively.

3) Taking $b_i^+ = d_i^+$ in (7) and (8), we have the upper bound in (3) and the lower bound in (4), respectively.

4) Taking $b_i^+ = 1$ in (14), we have the upper bound in (5).

Remark 3.5 *Obviously, for the digraph G, we can define its in-degrees, 2-indegrees and average 2-indegrees, etc. So we can easily obtain some similar results on $q(G)$ as in Theorem 3.1, 3.2 and 3.3.*

Acknowledgment. The authors thank the referees for their helpful comments and suggestions concerning the presentation of this paper. They are also thankful to TUBITAK and the Office of Selçuk University Scientific Research Project (BAP).

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