A Relationship Between Balanced Modular Tableaux and k-Ribbon Shapes

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Abstract

We give a new combinatorial bijection between a certain set of balanced modular tableaux of Gusein-Zade, Luengo and Melle-Hernandez and k-ribbon shapes. In addition we also use the Schensted algorithm for the rim hook tableaux of Stanton and White to write down an explicit generating function for these balanced modular tableaux.

1 Introduction

Young tableaux are well-known combinatorial objects that have deep connections with representation theory, geometry and algebra. Standard Young tableaux are intimately connected with symmetric function theory and the irreducible representations of the symmetric group. Study of symmetric functions has also led to a study of ribbon tableaux, first introduced by Lascoux, Leclerc and Thibon [7].

In [4] Gusein-Zade, Luengo and Melle-Hernandez showed that the number of tableaux with kn boxes such that cell (i, j) is filled with the integer $aj + bi$ (mod k) and the entries attain each of the residues $(0, 1, ..., k-1)$ n times, is equal to the Euler characteristic of the Hilbert scheme of points on the quotient stack $[\mathbb{A}^2/G]$, where G is the cyclic group $\mathbb{Z}_k$ acting on

$\mathbb{A}^2$ as $(x, y) \to (r^a x, r^b y)$ and r is a primitive k-root of unity. In this paper we give a simple direct combinatorial bijection between the tableaux of Gusein-Zade, Luengo and Melle-Hernandez and k-ribbon tableaux. As an interesting Corollary, we show that Fibonacci triples satisfy the sufficient conditions on the parameters required for the bijection between balanced modular tableaux and k-ribbon shapes to hold. We use the results of Stanton and White [6] to give a generating function for the number of such tableaux, which we call *balanced modular tableaux*, in a special case.

2 Background and Definitions

We say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a **partition** of n if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and $\sum_{i=1}^{k} \lambda_i = n$. A partition is described pictorially by its **Ferrers diagram,** an array of n cells into k left-justified rows with row i containing λ_i cells for $1 \leq i \leq k$. The k**th diagonal** of a Ferrers diagram is the set of cells (i, j) such that $j - i = k$. The **outer rim** of a partition is the set of cells (i, j) such that the cell $(i+1, j+1)$ is not in the partition.

For example, the Ferrers diagram for the partition $\lambda = (6, 5, 3, 3, 1)$ is:

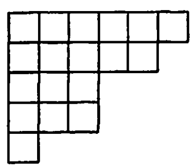

Figure 1: Ferrers Shape for partition $\lambda = (6, 5, 3, 3, 1)$

A *Young tableau* is a filling of the Ferrers diagram with numbers such that each square in the diagram contains a number. We define *modular* and *balanced modular* tableaux below.

Definition 2.1. Given $n, k \in \mathbb{N}$ and $a, b \in \mathbb{Z}$, we define a **modular tableau** $T^k_{a,b}$ of shape λ to be a filling of the partition $\lambda \vdash n$ such that the (i, j) cell of the Ferrers diagram of λ is filled with number $aj + bi$ mod k.

Definition 2.2. Let $n, k \in \mathbb{N}$ and $a, b \in \mathbb{Z}$. A modular tableau $T^k_{a,b}$ of shape $\lambda \vdash n$ is called **balanced** with respect to k, a, b if it contains exactly n/k cells labelled j for $0 \leq j \leq k-1$.

For a partition $\lambda \vdash n$ to be balanced with respect to k, a, b it is clearly necessary but not sufficient that n be an integer multiple of k. For example, the tableau in Figure 2 is a **balanced modular tableau** of size $n = 20$

with $k = 5, a = 2$ and $b = 3$. Note that there are $20/5 = 4$ occurrences of $0 \le j \le 4$ in the tableau.

0	2	4	1	3
3	0	2	4	1
1	3	0		
4	1	3		
2	4			
0	2			

Figure 2: Balanced Modular Tableau for $\lambda = (5, 5, 3, 3, 2, 2)$ with $k = 5, a = 2, b = 3$.

Definition 2.3. A **k-ribbon** is defined as k adjacent cells in the Ferrers diagram none of which lie on the same diagonal. A **k-ribbon shape** is a partition which can be completely decomposed into k-ribbons. (See Figure 3.)

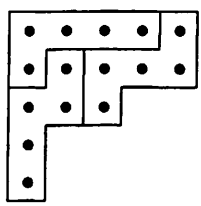

Figure 3: A 5-ribbon shape with three 5-ribbons

Definition 2.4. A **k-ribbon tableau** is a filling of a k-ribbon shape with the numbers $1, 2, \dots n$ such that entries in each ribbon are identical, each number appears in exactly one ribbon and the rows and columns are weakly increasing. (See Figure 4.)

By definition a k-ribbon tableau has shape λ which is a k-ribbon shape. However, decomposition of a k-ribbon shape into k-ribbons is not necessarily unique. The shape $\lambda = (5, 5, 3, 1, 1)$ in the example above admits six distinct 5-ribbon decompositions.

3 Results

In [2], Li proves several theorems relating the cellular decomposition of certain Hilbert Schemes and $(a, b; m)-$*admissable Young diagrams.* The

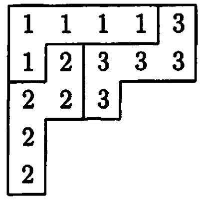

Figure 4: A 5-ribbon tableau with three 5-ribbons

$(a, b; m)$–admissable Young diagrams of Li are precisely our *balanced modular tableaux.* Li indicates that in the case $(a, b) = (1, -1)$, the number of admissible Young diagrams (balanced modular tableaux) can be computed using the a formula of Göttsche for computing the Betti numbers of an appropriate Hilbert Scheme, see [3]. In this section we construct a direct combinatorial bijection between k-ribbon shapes and this same set of balanced modular tableaux that does not require the deep results of Li and the formula of Göttsche. First we prove two technical Lemmas about balanced modular tableaux.

Lemma 3.1. *Let $T^k_{a,b}$ be a modular tableau of shape $\lambda \vdash n$ such that $a+b = k$. Then the entries on a given diagonal are constant.*

Proof. Let $T^k_{a,b}$ be a modular tableau of shape $\lambda \vdash n$ such that $a + b = k$. Consider the cells on diagonal ℓ. These cells are indexed by $(i, i + \ell)$ and are filled with $a(i + \ell) + bi \pmod k$ by the definition of modularity. It is easy to check that the labels of the cells on diagonal ℓ are:

$$\begin{aligned} a(i+\ell) + bi \pmod k &= (a+b)i + a\ell \pmod k \\ &= ki + a\ell \pmod k \\ &= a\ell \pmod k \end{aligned}$$

which is independent of the indices i and j and is therefore constant. □

Lemma 3.2. *Let $T^k_{a,b}$ be a balanced modular tableau of shape $\lambda \vdash n$ such that $a + b = k$ and $gcd(a, b, k) = 1$. If the last cell in the first row contains an s then there exists a cell containing $t = s - b \pmod k$ on the outer rim of $T^k_{a,b}$. Furthermore, this cell has no cell immediately below it.*

Proof. Let $T^k_{a,b}$ be a balanced modular tableau of shape $\lambda \vdash n$ such that $a+b = k$ and $gcd(a, b, k) = 1$. Let s be the label in the rightmost cell of the top row of $T^k_{a,b}$. Define $t = s - b \pmod k$. Then every cell in $T^k_{a,b}$ labelled t either has an s immediately below it or the cell is necessarily on the outer rim of $T^k_{a,b}$.

We can pair all of the cells in the tableau labelled s with the cell above labelled t unless the cell containing an s is in the first row. Since s is in the rightmost cell of the first row, then there must exist a cell labelled t with no cell below it, therefore that cell is on the outer rim of $T^k_{a,b}$.

□

Theorem 3.1. *A modular tableau $T^k_{a,b}$ such that $a+b = k$ and $gcd(a,b,k) = 1$ is balanced if and only if its shape is a k-ribbon shape.*

Proof. Let $T^k_{a,b}$ be a balanced modular tableau of shape $\lambda \vdash n$ such that $a + b = k$ and $gcd(a, b, k) = 1$. We proceed by induction on the size of $T^k_{a,b}$. If $T^k_{a,b}$ has size $n = k$ then by virtue of being balanced, it contains each j for $0 \le j \le k$ exactly once and is therefore a k-ribbon.

Now consider $T^k_{a,b}$ with size mk. Then by Lemma 3.1 the entries on a given diagonal are constant modulo k. In addition, since $gcd(a, b, k) = 1$ then any k consecutive diagonals contain cells filled with the numbers 0, 1, ..., $k-1$, although not necessarily in that order. Let s be the label in the rightmost cell of the top row of $T^k_{a,b}$. By Lemma 3.2 there exists at least one cell on the outer rim of $T^k_{a,b}$ labelled with t. Choose the northeast most cell on the outer rim that is labelled with t that has no cell below it. Construct a k-ribbon by selecting k adjacent cells on the outer rim beginning with this cell t and moving northeast. The final cell in such a k-ribbon will be labelled with an s. This cell labelled s cannot have a cell labelled t immediately to its right since this would contradict our choice of the cell labelled t. Therefore the k-ribbon that results can be removed to leave a balanced modular tableau of size $(m-1)k$. By induction on the size of the tableau, we obtain a k-ribbon shape.

To prove the other direction, begin with a partition λ of $n = mk$ that is a k-ribbon shape. Fill the cells of λ such that cell (i, j) contains $j - i \pmod k$. Since λ is a k-ribbon shape of size mk, it can be decomposed into m k-ribbons each containing cells from k consecutive diagonals and this each contains exactly one cell labelled $0, 1, \ldots, k-1$, thus the resulting tableau is a balanced modular tableau. □

Example 3.3. For example, in Figure 2, the last cell in the first row is labelled 3. The construction above indicates that we can find a 5−ribbon whose tail is labelled $3 - 3 = 0$ and whose head is labelled 3 that contains each of the numbers $0, 1, 2, 3, 4$ exactly once. Moreover, this 5−ribbon is on the outer rim and can be removed leaving a valid 5−ribbon shape. In Figure 2, the desired ribbon begins with the 0 in cell $(6, 1)$ and ends with the cell $(4, 3)$.

The following corollary is a direct result of Theorem 3.1.

Corollary 3.4. *If F_i is the ith Fibonacci number, then a modular tableau T of shape $\lambda \vdash n$ is balanced with respect to (F_i, F_{i+1}, F_{i+2}) if and only if λ is an F_{i+2}-ribbon shape.*

Proof. The result follows from the fact that that any three consecutive Fibonacci numbers are pairwise relatively prime and $F_i + F_{i+1} = F_{i+2}$. □

Theorem 3.1 implies that counting balanced modular tableaux of size n in the special case where $a+b=k$ and $gcd(a,b,k)=1$ reduces to counting the distinct k-ribbon *shapes* of size n (not the k-ribbon tableaux themselves as there may be multiple such tableaux for a given shape). Stanton and White developed a bijection between k-ribbon tableaux and k-tuples of tableaux [6]. We note that the bijection of Stanton and White implicitly defines a bijection between k-ribbon *shapes* of size n and a k-tuples of partitions $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $|\lambda_1| + |\lambda_2| + \cdots + |\lambda_k| = n$ simply by ignoring the content of the tableaux. For details of this interpretation of Stanton and White's bijection, see [5]. Therefore we have the following generating function for balanced modular tableaux.

Theorem 3.2. *Let $n \in \mathbb{N}$ and $\mathfrak{T}^k_{a,b}(n)$ be the set of balanced modular tableaux $T^k_{a,b}$ of size $n = mk$ such that $a+b=k$ and $gcd(a,b,k)=1$. Then for a fixed integer $k > 1$,*

$$\sum_{n\geq 0} |\mathfrak{T}^k_{a,b}(n)| q^n = \Big(\prod_{j\geq 1} \frac{1}{1-q^j}\Big)^k$$

where $(\prod_{j\geq 1} \frac{1}{1-q^j})^k$ is the generating functions for k-colored partitions of n.

4 Remarks

The condition that $gcd(a,b,k)=1$ in Theorem 3.1 is necessary as without this condition not all of the numbers 0 through $k-1$ will appear in a modular tableau. This condition can be relaxed if we expand the notion of *balanced.* Given a modular tableau $T^k_{a,b}$ such that $a+b=k$, $gcd(a,b,k)=d$ and for which every integer that *does* appear in the tableau appears the same number of times (call this *quasi-balanced*), we can create a balanced modular tableau $T^{k'}_{a',b'}$ where $a' = a/d$, $b' = b/d$ and $k' = k/d$. Thus the set of quasi-balanced modular tableaux $T^k_{a,b}$ with $a+b=k$, $gcd(a,b,k)=d$ is in bijection with the set of k'-ribbon shapes.

The condition that $a+b=k$, however, appears quite strict. Without this condition, we lose the fact that each diagonal in a tableau contains the same number on it, thus losing the important condition that any k consecutive diagonals contains the numbers 0 through $k-1$. It is an open question to determine an efficient way of counting balanced modular tableaux $T^k_{a,b}$ where $a+b \neq k$.

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