# The $m$-competition indices of symmetric primitive digraphs with loop*

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**Abstract**

For a positive integer $m$, where $1 \leq m \leq n$, the $m$-competition index (generalized competition index) of a primitive digraph $D$ of order $n$ is the smallest positive integer $k$ such that for every pair of vertices $x$ and $y$, there exist $m$ distinct vertices $v_1, v_2, \ldots, v_m$ such that there exist walks of length $k$ from $x$ to $v_i$ and from $y$ to $v_i$ for $1 \leq i \leq m$. In this paper, we study the generalized competition indices of symmetric primitive digraphs with loop. We determine the generalized competition index set and characterize completely the symmetric primitive digraph in this class such that the generalized competition index is equal to the maximum value.

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**Keywords:** Competition index; $m$-Competition index; Scrambling index; Generalized competition index.

# 1 Introduction

For terminology and notation used here we follow [1, 4]. Let $D = (V, E)$ denote a digraph with vertex set $V = V(D)$, arc set $E = E(D)$ and order $n$. Loops are permitted but multiple arcs are not. A digraph $D$ is called *primitive* if for some positive integer $k$, there is a walk of length exactly $k$ from each vertex $u$ to each vertex $v$ (possibly $u$ again). The smallest such $k$ is called the *exponent* of $D$, and it is denoted by $\exp(D)$. It is well known that $D$ is primitive if and only if $D$ is strongly connected and the greatest common divisor of all the cycle lengths of $D$ is 1.

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The distance from vertex $u$ to vertex $v$ in $D$, is the length of a shortest walk from $u$ to $v$, and denoted by $d_D(u,v)$ (for short, $d(u,v)$). The notation $u \xrightarrow{k} v$ is used to indicate that there is a walk of length $k$ from $u$ to $v$.

Let $D$ be a primitive digraph of order $n$. For a positive integer $m$ where $1 \leq m \leq n$, we define the *m-competition index (generalized competition index)* of $D$, denoted by $k_m(D)$, as the smallest positive integer $k$ such that for every pair of vertices $x$ and $y$, there exist $m$ distinct vertices $v_1, v_2, \ldots, v_m$ such that $x \xrightarrow{k} v_i$ and $y \xrightarrow{k} v_i$ for $1 \leq i \leq m$ in $D$.

Akelbek and Kirkland [2, 3] introduced the scrambling index of a primitive digraph $D$, denoted by $k(D)$. Kim [4] introduced the $m$-competition index as a generalization of the competition index. In the case of primitive digraphs, the definitions of the scrambling index and 1-competition index are identical. We have $k(D) = k_1(D)$.

For a positive integer $k$ and a primitive digraph $D$, we define the *k-step outneighborhood* of a vertex $x$ as

$$N^+(D^k : x) = \{v \in V(D) | x \xrightarrow{k} v\}.$$

We define the *k-step common outneighborhood* of vertices $x$ and $y$ as

$$N^+(D^k : x, y) = N^+(D^k : x) \cap N^+(D^k : y).$$

We define the *local m-competition index* of vertices $x$ and $y$ as

$$k_m(D : x, y) = \min\{k : |N^+(D^t : x, y)| \geq m \text{ where } t \geq k\}.$$

We also define the *local m-competition index* of $x$ as

$$k_m(D : x) = \max_{y \in V(D)} \{k_m(D : x, y)\}.$$

Then, we have

$$k_m(D) = \max_{x \in V(D)} k_m(D : x) = \max_{x,y \in V(D)} k_m(D : x, y).$$

The $m$-competition index is a generalization of the scrambling index and the exponent of a primitive digraph. It was known that for $1 \leq m \leq n$ (For example see [4]):

$$k(D) = k_1(D) \leq k_2(D) \leq \cdots \leq k_n(D) = \exp(D).$$

A symmetric digraph is a digraph such that for any vertices $u$ and $v$, $(u, v)$ is an arc if and only if $(v, u)$ is an arc. An undirected graph (possibly with loops) can be regarded as a symmetric digraph.

There has been interest recently in generalized competition index [4, 5, 6, 7]. Let $S_n$ denote the set of all primitive graphs of order $n$ with one loop at least. In this paper, we study $m$-competition indices of $S_n$. We determine $m$-competition index set for $S_n$, and characterize completely the symmetric primitive digraph in this class such that the $m$-competition index is equal to the maximum value.

# 2 The generalized competition indices for special graphs

In this section, we study the generalized competition indices for some special graphs. First, we consider the following graphs,

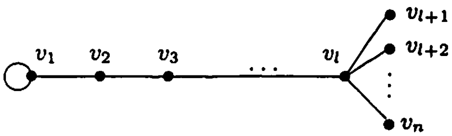


Fig. 1 Graphs $G_{n,l}$

where $1 \le l \le n-1$.

**Theorem 2.1** *For $2 \le l \le n-1$ and $2 \le m \le n-1$,*

$$k_m(G_{n,l}) = \begin{cases} l+m-1, & \text{if } 2 \le m \le l, \\ 2l, & \text{if } m \ge l+1. \end{cases}$$

**Proof** Case 1. $2 \le m \le l$.

For any $v_i$, noticing that $d(v_i, v_1) \le l$ and $v_1$ is a loop vertex, then $\{v_1, v_2, \ldots, v_m\} \subseteq N^+(G_{n,l}^{l+m-1} : v_i)$. Then

$$k_m(G_{n,l}) \le l+m-1.$$

On the other hand, $N^+(G_{n,l}^{l+m-2} : v_{l+1}) =$

$$\begin{cases} V(G_{n,l})\backslash\{v_{l-t} \mid 0 \le t \le l-m, \text{ and } t \text{ is even}\}, & \text{if } l+m \text{ is even}, \\ \{v_1, v_2, \cdots, v_l\}\backslash\{v_{l-t} \mid 0 \le t \le l-m, \text{ and } t \text{ is odd}\}, & \text{if } l+m \text{ is odd}, \end{cases}$$

and $N^+(G_{n,l}^{l+m-2} : v_l) =$

$$\begin{cases} \{v_1, v_2, \cdots, v_l\}\backslash\{v_{l-t} \mid 0 \le t \le l-m-1, \text{ and } t \text{ is odd}\}, & \text{if } l+m \text{ is even}, \\ V(G_{n,l})\backslash\{v_{l-t} \mid 0 \le t \le l-m-1, \text{ and } t \text{ is even}\}, & \text{if } l+m \text{ is odd}. \end{cases}$$

Then $|N^+(G_{n,l}^{l+m-2} : v_l, v_{l+1})| = |N^+(G_{n,l}^{l+m-2} : v_l) \cap N^+(G_{n,l}^{l+m-2} : v_{l+1})| = |\{v_1, v_2, \cdots, v_{m-1}\}| = m-1$, and $k_m(G_{n,l}) > l+m-2$. So

$$k_m(G_{n,l}) = l+m-1.$$

Case 2. $m \geq l+1$.

It is easy to see that for each vertex $v_i$,

$$N^+(G_{n,l}^{2l} : v_i) = V(G_{n,l}),$$

so

$$k_m(G_{n,l}) \leq 2l.$$

On the other hand, since

$$N^+(G_{n,l}^{2l-1} : v_{l+1}) = \{v_1, v_2, \cdots, v_l\},$$

we have $k_m(G_{n,l}) > 2l-1$, and $k_m(G_{n,l}) = 2l$.

Theorem 2.1 holds now. □

**Theorem 2.2** *For* $2 \leq m \leq n-1$, $k_m(G_{n,1}) = 2$.

**Proof** It is clear that $N^+(G_{n,1}^2 : v_i) = V(G_{n,1})$ for each vertex $v_i$, so $k_m(G_{n,1}) \leq 2$.

On the other hand, for any $v_i \neq v_1$, $N^+(G_{n,1}^1 : v_i) = \{v_1\}$. Therefore $k_m(G_{n,1}) > 1$, and $k_m(G_{n,1}) = 2$. □

Next, we consider the following graphs,

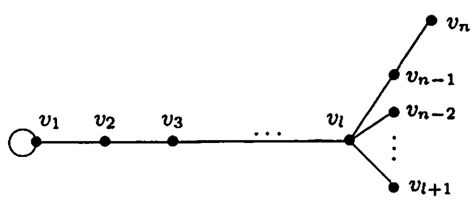


Fig. 2 Graphs $\overline{G}_{n,l}$

where $1 \leq l \leq n-2$.

**Theorem 2.3** *For* $2 \leq l \leq n-2$ *and* $2 \leq m \leq n-1$,

$$k_m(\overline{G}_{n,l}) = \begin{cases} l+m, & \text{if } 2 \leq m \leq l, \\ 2l+1, & \text{if } l+1 \leq m \leq n-1. \end{cases}$$

**Proof** Case 1. $2 \le m \le l$.

For any $v_i$, noticing that $d(v_i, v_1) \le l+1$ and $v_1$ is a loop vertex, then $\{v_1, v_2, \ldots, v_m\} \subseteq N^+(\overline{G}_{n,l}^{l+m} : v_i)$. Then

$$k_m(\overline{G}_{n,l}) \le l + m.$$

On the other hand, $N^+(\overline{G}_{n,l}^{l+m-1} : v_n) =$

$$\begin{cases} \{v_1, v_2, \cdots, v_{n-1}\} \backslash \{v_{l-t} \mid 0 \le t \le l-m, \text{ and } t \text{ is even}\}, & \text{if } l+m \text{ is even}, \\ \{v_1, v_2, \cdots, v_l, v_n\} \backslash \{v_{l-t} \mid 0 \le t \le l-m, \text{ and } t \text{ is odd}\}, & \text{if } l+m \text{ is odd}, \end{cases}$$

and $N^+(\overline{G}_{n,l}^{l+m-1} : v_{n-1}) =$

$$\begin{cases} \{v_1, v_2, \cdots, v_l, v_n\} \backslash \{v_{l-t} \mid 0 \le t \le l-m-1, \text{ and } t \text{ is odd}\}, & \text{if } l+m \text{ is even}, \\ \{v_1, v_2, \cdots, v_{n-1}\} \backslash \{v_{l-t} \mid 0 \le t \le l-m-1, \text{ and } t \text{ is even}\}, & \text{if } l+m \text{ is odd}. \end{cases}$$

Then $|N^+(\overline{G}_{n,l}^{l+m-1} : v_n, v_{n-1})| = |N^+(\overline{G}_{n,l}^{l+m-1} : v_n) \cap N^+(\overline{G}_{n,l}^{l+m-1} : v_{n-1})| = |\{v_1, v_2, \cdots, v_{m-1}\}| = m-1$, and $k_m(\overline{G}_{n,l}) > l+m-1$. So

$$k_m(\overline{G}_{n,l}) = l + m.$$

Case 2. $l+1 \le m \le n-1$.

For any $v_i \ne v_n$, noticing that $d(v_i, v_1) \le l$, $d(v_n, v_1) = l+1$, and $v_1$ is a loop vertex, then $N^+(\overline{G}_{n,l}^{2l+1} : v_i) = V(\overline{G}_{n,l})$ and $N^+(\overline{G}_{n,l}^{2l+1} : v_n) = \{v_1, v_2, \ldots, v_{n-1}\}$. Then

$$k_m(\overline{G}_{n,l}) \le 2l + 1.$$

On the other hand, $N^+(\overline{G}_{n,l}^{2l} : v_n) = \{v_1, v_2, \cdots, v_l, v_n\}$ and $N^+(\overline{G}_{n,l}^{2l} : v_{n-1}) = \{v_1, v_2, \cdots, v_{n-1}\}$. Then $|N^+(\overline{G}_{n,l}^{2l} : v_n, v_{n-1})| = |N^+(\overline{G}_{n,l}^{2l} : v_n) \cap N^+(\overline{G}_{n,l}^{2l} : v_{n-1})| = |\{v_1, v_2, \cdots, v_l\}| = l$, and $k_m(\overline{G}_{n,l}) > 2l$. So

$$k_m(\overline{G}_{n,l}) = 2l + 1.$$

Theorem 2.3 holds now. □

**Theorem 2.4** *For* $2 \le m \le n-1$, $k_m(\overline{G}_{n,1}) = 3$.

**Proof** For any vertex $v_i \ne v_n$, $N^+(\overline{G}_{n,1}^3 : v_i) = V(\overline{G}_{n,1})$. For vertex $v_n$, $N^+(\overline{G}_{n,1}^3 : v_n) = V(\overline{G}_{n,1}) \backslash \{v_n\}$. So for $2 \le m \le n-1$, $k_m(\overline{G}_{n,1}) \le 3$.

On the other hand, $N^+(\overline{G}_{n,1}^2 : v_n) = \{v_1, v_n\}$, $N^+(\overline{G}_{n,1}^2 : v_{n-1}) = V(\overline{G}_{n,1}) \backslash \{v_n\}$. So $|N^+(\overline{G}_{n,1}^2 : v_n) \cap N^+(\overline{G}_{n,1}^2 : v_{n-1})| = |v_1| = 1$. Therefore $k_m(\overline{G}_{n.1}) > 2$, and $k_m(\overline{G}_{n,1}) = 3$. □

The result of Theorm 2.5 is clear.

**Theorem 2.5** *Let $\overline{K}_n$ be the complete graph of order $n$ adding a loop in each vertex. Then for $2 \leq m \leq n-1$,*

$$k_m(\overline{K}_n) = 1.$$

# 3 The generalized competition index set of $S_n$

For $1 \leq m \leq n$, let $E_m = \{k_m(G) \mid G \in S_n\}$. It was known that $E_1 = \{1, 2, \ldots, n-1\}$ (see Theorem 3.3 in [8]) and $E_n = \{1, 2, \ldots, 2n-2\} \setminus S$, where $S$ is the set of all odd numbers in $\{n, n+1, \ldots, 2n-2\}$ (see [9]).

In this section, we show that $E_m = \{1, 2, \ldots, n+m-2\}$ for $2 \leq m \leq n-1$. We also characterize the graph in $S_n$ such that the generalized competition index is equal to the maximum value.

**Theorem 3.1** *Let $G \in S_n$. For $2 \leq m \leq n-1$,*

$$k_m(G) \leq n+m-2.$$

*The equality holds if and only if the graph $G$ is isomorphic to $G_{n,n-1}$.*

**Proof** Let $v_1$ be a loop vertex of $G$. It is easy to see that for any $v_i \in V(G)$, $d(v_i, v_1) \leq n-1$, $|N^+(G^{m-1} : v_1)| \geq m$, and $N^+(G^{m-1} : v_1) \subseteq N^+(G^{n+m-2} : v_i)$. Then for any $v_i, v_j \in V(G)$,

$$|N^+(G^{n+m-2} : v_i, v_j)| \geq |N^+(G^{m-1} : v_1)| \geq m,$$

and so

$$k_m(G) \leq n+m-2.$$

Let $G \in S_n$ such that $k_m(G) = n+m-2$ for $2 \leq m \leq n-1$. For a vertex $v$ with loop, we denote $d_v = \max_{v_i \in V(G)}\{d(v_i, v)\}$. Suppose that there exists a vertex $v$ with loop such that $d_v \leq n-2$. Then for any $v_i \in V(G)$, $d(v_i, v) \leq n-2$, and $N^+(G^{m-1} : v) \subseteq N^+(G^{n+m-3} : v_i)$. Then for any $v_i, v_j \in V(G)$,

$$|N^+(G^{n+m-3} : v_i, v_j)| \geq |N^+(G^{m-1} : v)| \geq m,$$

and so $k_m(G) \leq n+m-3$. It is a contradiction. Then $d_v = n-1$ for each vertex $v$ with loop. This means that $G$ is isomorphic to $G_{n,n-1}$. □

**Theorem 3.2** *For $2 \leq m \leq n-1$, $E_m = \{1, 2, \ldots, n+m-2\}$.*

**Proof** For $m = 2$, taking $2 \leq l \leq n-1$, by Theorem 2.1, $\{3, 4, \ldots, n\} \subseteq E_m$. Combining the results of Theorems 2.2 and 2.5, we have $E_m = \{1, 2, \ldots, n\}$.

For $3 \leq m \leq n-1$, taking $m \leq l \leq n-1$, by Theorem 2.1, $\{2m-1, 2m, \ldots, n+m-2\} \subseteq E_m$. Taking $2 \leq l \leq m-1$, by Theorem 2.1, $\{4, 6, \ldots, 2m-2\} \subseteq E_m$. Taking $2 \leq l \leq m-2$, by Theorem 2.3, $\{5, 7, \ldots, 2m-3\} \subseteq E_m$. Combining the results of Theorems 2.2, 2.4 and 2.5, we have $E_m = \{1, 2, \ldots, n+m-2\}$. □

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