$\{C_4, K_3 + e\}$-metamorphosis of $S_\lambda(2, 4, n)$

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Abstract

Let $(X, \mathcal{B})$ be a λ-fold G-decomposition and let G_i, $i = 1, \ldots, \mu$, be nonisomorphic proper subgraphs of G without isolated vertices. Put $\mathcal{B}_i = \{B_i \mid B \in \mathcal{B}\}$, where B_i is a subgraph of B isomorphic to G_i. A $\{G_1, G_2, \ldots, G_\mu\}$-metamorphosis of $(X, \mathcal{B})$ is a rearrangement, for each $i = 1, \ldots, \mu$, of the edges of $\bigcup_{B \in \mathcal{B}}(E(B) \setminus E(B_i))$ into a family $\mathcal{F}_i$ of copies of G_i with a leave L_i, such that $(X, \mathcal{B}_i \cup \mathcal{F}_i, L_i)$ is a maximum packing of λH with copies of G_i. In this paper, we give a complete answer to the existence problem of an $S_\lambda(2, 4, n)$ having a $\{C_4, K_3 + e\}$-metamorphosis.

1 Preliminaries

Let G and H be simple finite graphs. A λ-fold *G-decomposition of λH* (λ copies of H) is a pair $(X, \mathcal{B})$ where $X = V(H)$, the vertex set of H, and $\mathcal{B}$ is a collection of copies of G (blocks), which partitions the multiset $E(\lambda H)$, the multiset of edges of λH.

Let K_n denote the complete simple graph on n vertices. A λ-fold G-decomposition of λK_n is said a λ-fold *G-design* or *G-system of order n*. A λ-fold K_k-design of order n is well-known as an $S_\lambda(2, k, n)$, a *balanced incomplete block design of order n, block size k and index λ*. A λ-fold K_k-decomposition of the complete multigraph on u_i parts of size g_i, $i = 1, 2, \ldots, h$, is well-known as a *k-GDD (group divisible design) of index λ and type $g_1^{u_1} g_2^{u_2} \ldots g_h^{u_h}$*. If some blocks are isomorphic to K_r and the other are isomorphic to K_s, we have an *$\{r, s\}$-GDD of index λ and type $g_1^{u_1} g_2^{u_2} \ldots g_h^{u_h}$*. If $\lambda = 1$, we drop "of index 1".

A *packing* of λH with copies of G is a triple $(X, \mathcal{B}, L)$, where $X = V(H)$, $\mathcal{B}$ is a collection of copies of G from $E(\lambda H)$ and L, called the *leave*, is the graph induced by the edges of λH not belonging to some block of $\mathcal{B}$. If the cardinality of the multiset $\mathcal{B}$ is as large as possible, the packing $(X, \mathcal{B}, L)$ is said to be maximum. When L is empty, a maximum packing of λH with copies of G coincides with a λ-fold G-decomposition of λH.

A *k-path* P_k, $k \geq 2$, is the graph $[a_1, a_2, \ldots, a_k]$ on vertices $a_1, \cdots, a_k$ and edges $\{a_i, a_{i+1}\}$, $i = 1, \ldots, k-1$. We denote by E_2 the graph on 4 vertices consisting of two disjoint edges.

A *k-cycle* C_k, $k \geq 3$ is the graph on vertices $a_1, a_2, \ldots, a_k$ with edges $\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_k, a_1\}$. A C_k will be denoted by any cyclic shift of $(a_1, a_2, \cdots, a_k)$ or $(a_k, a_{k-1}, \ldots, a_1)$. In particular, the triangle K_3 with edges $\{a, b\}, \{a, c\}, \{c, b\}$ will be denoted by (a, b, c).

A $K_3 + e$, or a *kite*, is a simple graph on 4 vertices consisting of a triangle and a single edge (*tail*) sharing one common vertex (see Figure 1). We denote by $(a, b, c) - d$ or $(b, a, c) - d$ the kite having *base* $\{a, b\}$ and tail $\{c, d\}$.

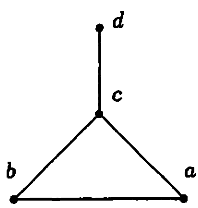

Figure 1: the kite (a,b,c)-d

Definition. *Let $(X, \mathcal{B})$ be a λ-fold G-decomposition of λH. Let G_i, $i = 1, \ldots, \mu$, be non isomorphic proper subgraphs of G, each without isolated vertices. Put $\mathcal{B}_i = \{B_i \mid B \in \mathcal{B}\}$, where B_i is a subgraph of B isomorphic to G_i. A $\{G_1, G_2, \ldots, G_\mu\}$-***metamorphosis*** *of $(X, \mathcal{B})$ is a rearrangement, for each $i = 1, \ldots, \mu$, of the edges of $\bigcup_{B \in \mathcal{B}} (E(B) \setminus E(B_i))$ into a family $\mathcal{B}'_i$ of copies of G_i with leave L_i, such that $(X, \mathcal{B}_i \cup \mathcal{B}'_i, L_i)$ is a maximum packing of λH with copies of G_i.*

Above definition has been introduced in [1] as *simultaneous metamorphosis.* A G_1-metamorphosis is also well-known as metamorphosis into a maximum packing with copies of G_1. The existence problem of $S_\lambda(2, 4, n)$ having a metamorphosis has been studied in many papers (for example [7, 8]).

In this paper, we become to study the simultaneous metamorphosis of an $S_\lambda(2, 4, n)$ when the subgraphs $G_i, i = 1, \ldots, \mu$, $\mu \geq 2$ are obtained by removing from K_4 a fixed number $t \in \{1,2,\dots,5\}$ of edges. In our definition we require that G_i is not isomorphic to G_j when $i \neq j$, so the first case to study is $t = 2$. It is $G = K_4, G_1 = C_4, G_2 = K_3 + e$. In the following we always denote the sets $\mathcal{B}'_1$, $\mathcal{B}'_2$, L_1, L_2 by $\mathcal{C}$, $\mathcal{K}$, $L_\mathcal{C}$ and $L_\mathcal{K}$, respectively.

It is well-known that, for $n \geq 4$: an $S_\lambda(2,4,n)$ exists if and only if $\lambda n(n-1) \equiv 0 \pmod{12}$ and $\lambda(n-1) \equiv 0 \pmod 3$; a λ-fold C_4-system of order n exists if and only if $\lambda n(n-1) \equiv 0 \pmod 8$ and $\lambda(n-1) \equiv 0 \pmod 2$; a λ-fold kite-system of order n if and only if $\lambda n(n-1) \equiv 0 \pmod 8$.

Necessary and sufficient conditions for the existence of an $S_\lambda(2,4,n)$ having a metamorphosis into a maximum packing of λK_n with 4-cycles (with kites) are given in [6] ([5]). See the following table, where $\emptyset$ denotes the empty graph.

$\lambda \pmod{12}$	$n \geq 4$	$L_\mathcal{C}$	$L_\mathcal{K}$
1,5,7,11	$1 \pmod{24}$	$\emptyset$	$\emptyset$
	$4 \pmod{24}$	1-factor	P_3 or, if $n > 4$, E_2
	$13 \pmod{24}$	C_6 or 2 K_3s	P_3 or E_2
	$16 \pmod{24}$	1-factor	$\emptyset$
2,10	$1,4 \pmod{12}$	$\emptyset$	$\emptyset$
	$7,10 \pmod{12}$	$2P_2$	P_3 or $2P_2$ or E_2
3,9	$1 \pmod 8$	$\emptyset$	$\emptyset$
	$0 \pmod 8$	1-factor	$\emptyset$
	$4 \pmod 8$	1-factor	P_3 or $2P_2$ or E_2
	$5 \pmod 8$	$2P_2$	P_3 or $2P_2$ or E_2
4,8	$1 \pmod 3$	$\emptyset$	$\emptyset$
6	$0,1 \pmod 4$	$\emptyset$	$\emptyset$
	$2,3 \pmod 4$	$2P_2$	P_3 or $2P_2$ or E_2
0	$\forall n \geq 4$	$\emptyset$	$\emptyset$

Pairing [5] and [6] it is easy to check that in some cases C_4-metamorphoses and (K_3+e)-metamorphoses follow from a *same* starting $S_\lambda(2,4,n)$. Collecting these results we get our first result.

Theorem 1.1. *[5, 6] If $\lambda = 1$ and $n \equiv 4, 13 \pmod{24}$, $\lambda = 2$ and $n = 7, 10, 19$, $\lambda = 3$ and $n \equiv 4, 5 \pmod 8$, $\lambda = 6$ and $n \equiv 2, 3 \pmod 4$, then there exists an $S_\lambda(2,4,n)$ having a $\{C_4, K_3 + e\}$-metamorphosis.*

Theorem 1.2. *[Weighting construction] . Suppose there exist:*

1. *an $\{r,s\}$-GDD of type $g_1^{u_1} g_2^{u_2} \dots g_h^{u_h}$;*
2. *an $S_\lambda(2,4,\alpha + wg_i), i = 1,\dots,h$, with $\alpha = 0,1$, having a $\{C_4, K_3 + e\}$-metamorphosis;*

3. a 4-GDD of index λ and type w^r, having a $\{C_4, K_3+e\}$-metamorphosis;

4. a 4-GDD of index λ and type w^s, having a $\{C_4, K_3+e\}$-metamorphosis.

Then there is an $S_\lambda(2,4,w(g_1u_1+\ldots+g_hu_h)+\alpha)$ having a $\{C_4, K_3+e\}$-metamorphosis.

Proof The proof follows easily from the well-known Wilson fundamental construction [3]. □

2 $\lambda = 1$

Lemma 2.1. *There exists a 4-GDD of type $(2t)^4$, with $t \geq 2, t \neq 3$, having a $\{C_4, K_3+e\}$-metamorphosis.*

Proof For $t \geq 2, t \neq 3$, let $X = \mathbb{Z}_{2t} \times \mathbb{Z}_4$, $\mathcal{G} = \{\mathbb{Z}_{2t} \times \{k\}, k \in \mathbb{Z}_4\}$ and $\mathcal{B} = \{\{(i,1),(j,2),(i \circ_1 j,3),(i \circ_2 j,0)\} \mid i,j \in \mathbb{Z}_{2t}\}$, where $(\mathbb{Z}_{2t}, \circ_1)$ and $(\mathbb{Z}_{2t}, \circ_2)$ are two orthogonal quasigroups of order $2t$ [2]. Then $\Gamma = (X, \mathcal{G}, \mathcal{B})$ is the 4-GDD of type $(2t)^4$.

Remove from each block the edges $\{(i,1),(j,2)\}$, $\{(i \circ_1 j,3),(i \circ_2 j,0)\}$. These edges cover two complete bipartite graphs $K_{2t,2t}$, then we can rearrange them into the set $\mathcal{C}$ of 4-cycles [10].

For each $0 \leq i \leq 2t-1$ and for each $0 \leq j \leq t-1$, remove the edges $\{(i,1),(j,2)\}$, $\{(j,2),(i \circ_1 j,3)\}$, $\{(i,1),(i \circ_1 (j+t),3)\}$, $\{(i \circ_1 (j+t),3),(i \circ_2 (j+t),0)\}$. Since $\{(j,2),(i \circ_1 j,3) \mid 0 \leq i \leq 2t-1, 0 \leq j \leq t-1\} = \{(j,2),(i \circ_1 (j+t),3) \mid 0 \leq i \leq 2t-1, 0 \leq j \leq t-1\}$, the removed edges can be assembled into the set $\mathcal{K} = \{((i,1),(j,2),(i \circ_1 (j+t),3)) - (i \circ_2 (j+t),0) \mid 0 \leq i \leq 2t-1, 0 \leq j \leq t-1\}$. □

In order to give a $\{G_1, G_2, \ldots, G_\mu\}$-metamorphosis, it is sufficient, for $\lambda = 1$, to indicate, for each i, L_i and $\mathcal{B}'_i$, being straightforward the blocks in $\mathcal{B}_i$.

Lemma 2.2. *For $n = 25, 49, 73$ there is an $S(2,4,n)$ $(X, \mathcal{B})$, having a $\{C_4, K_3+e\}$-metamorphosis with empty leaves.*

Proof **n=25:** $X = \mathbb{Z}_{25}$, $\mathcal{B} = \{\{1,5,12,0\}, \{1,6,13,2\}, \{3,7,14,2\}, \{8,4,3,10\}, \{4,9,11,0\}, \{5,10,17,6\}, \{7,11,18,6\}, \{7,12,19,8\}, \{9,15,13,8\}, \{14,5,16,9\}, \{10,15,22,11\}, \{12,16,23,11\}, \{12,24,17,13\}, \{13,18,20,14\}, \{10,14,21,19\}, \{15,2,20,16\}, \{16,21,3,17\}, \{17,22,4,18\}, \{0,23,19,18\}, \{19,24,1,15\}, \{21,20,7,0\}, \{21,8,1,22\}, \{2,22,9,23\}, \{23,5,3,24\}, \{6,20,24,4\}, \{2,0,24,10\}, \{3,20,11,1\}, \{4,2,21,12\}, \{3,0,13,22\}, \{4,14,23,1\}, \{7,5,4,15\}, \{6,8,16,0\}, \{7,9,17,1\}, \{2,8,5,18\}, \{19,3,9,6\}, \{9,20,12,10\}, \{5,21,13,11\}, \{6,14,22,12\}, \{7,23,10,13\}, \{14,8,24,11\}, \{15,17,0,14\}, \{10,18,1,16\}, \{17,19,2,11\}, \{15,18,12,3\}, \{16,19,13,4\}, \{22,20,19,5\}, \{15,21,23,6\}, \{24,16,22,7\}, \{20,17,23,8\}, \{9,21,24,18\}\}$; $\mathcal{C} = \{(2,3,1,0), (7,5,3,0), (14,13,4,0), (23,7,16,0), (10,11,2,1), (8,6,4,1), (24,10,17,1), (18,15,4,2), (20,11,9,2), (21,22,4,3),$

$(19,5,12,3)$, $(9,7,6,5)$, $(21,24,8,5)$, $(22,20,9,6)$, $(15,22,13,6)$, $(14,10,8,7)$, $(17,18,9,8)$, $(13,11,12,10)$, $(18,16,14,11)$, $(19,15,13,12)$, $(21,23,14,12)$, $(17,19,16,15)$, $(23,24,17,16)$, $(20,21,19,18)$, $(24,22,23,20)\}$
$\mathcal{K} = \{(4,0,1) - 20$, $(5,9,0) - 22$, $(1,3,2) - 12$, $(6,2,7) - 1$, $(10,8,9) - 1$, $(6,10,4) - 12$, $(11,5,6) - 3$, $(15,8,7) - 16$, $(11,8,12) - 6$, $(18,9,14) - 1$, $(11,16,15) - 6$, $(19,10,11) - 14$, $(14,12,13) - 7$, $(23,13,24) - 7$, $(15,14,19) - 16$, $(10,2,16) - 4$, $(11,21,17)-8$, $(18,16,17)-14$, $(21,18,22)-3$, $(23,2,18)-15$, $(8,0,18)-3$, $(24,15,5)-22$, $(10,0,20) - 4$, $(21,0,6) - 19$, $(20,8,22) - 23\}$
n=49: $X = \mathbb{Z}_{49}$. The starters blocks of $\mathcal{B}$ are $\{0,8,3,1\}$, $\{0,29,4,18\}$, $\{6,33,21,0\}$, $\{32,19,9,0\}$. The starters blocks of $\mathcal{C}$ are $(0,5,4,22)$ and $(0,9,34,13)$. The starters blocks of $\mathcal{K}$ are $(0,1,19) - 12$, $(6,17,0) - 16$.
n=73: $X = \mathbb{Z}_{73}$. The starters blocks of $\mathcal{B}$ are $\{1,4,6,0\}$, $\{7,28,0,20\}$, $\{9,33,44,0\}$, $\{0,25,47,15\}$, $\{46,12,30,0\}$, $\{0,31,14,50\}$. The starters blocks of $\mathcal{C}$ are $(0,1,3,13)$, $(0,26,54,24)$ and $(0,29,65,31)$. The starters blocks of $\mathcal{K}$ are $(10,1,0)-4$, $(40,27,0)-12$, $(0,23,8) - 22$. □

Lemma 2.3. *For $n \equiv 1 \pmod{24}$, there exists an $S(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis.*

Proof For $n = 25, 49, 73$, the result follows from Lemma 2.2. Let Γ be the 4-GDD in Lemma 2.1 with $t = 12$. Add an infinite point to each group $G_i = \mathbb{Z}_{24} \times \{i\}$, $i = 0,1,2,3$, and place on it a copy of the $S(2,4,25)$ given in Lemma 2.2. The result is an $S(2,4,97)$ having a $\{C_4, K_3 + e\}$-metamorphosis. Now let $n = 24u + 1$, with $u \geq 5$. Add an infinite point to the vertex set of a 4-GDD of type 6^u [3] and apply to it the weighting construction with $r = s = 4$, $\alpha = 1$ and $w = 4$. This completes the proof. □

Lemma 2.4. *There exist an $S(2,4,16)$ and an $S(2,4,40)$ having a $\{C_4, K_3+e\}$-metamorphosis where $L_{\mathcal{C}}$ is an 1-factor and $L_{\mathcal{K}}$ is the empty graph.*

Proof **n=16:** $X = \mathbb{Z}_{16}$, $\mathcal{B} = \{\{1,2,0,3\}$, $\{4,6,0,5\}$, $\{0,7,8,9\}$, $\{11,13,0,12\}$, $\{15,0,10,14\}$, $\{4,1,7,11\}$, $\{1,12,14,5\}$, $\{1,8,15,6\}$, $\{9,13,10,1\}$, $\{2,13,15,4\}$, $\{2,10,5,7\}$, $\{2,9,12,6\}$, $\{8,11,14,2\}$, $\{3,9,14,4\}$, $\{3,5,8,13\}$, $\{3,11,10,6\}$, $\{3,7,12,15\}$, $\{8,10,4,12\}$, $\{9,15,5,11\}$, $\{7,14,6,13\}\}$;
$\mathcal{C} = \{(1,2,9,8)$, $(11,13,9,3)$, $(0,3,5,7)$, $(11,7,14,5)$, $(13,2,10,8)$, $(4,15,12,1)$, $(6,10,14,4)$, $(0,15,6,12)\}$;
$L_{\mathcal{C}} = \{(0,5), (1,10), (2,14), (3,7), (4,12), (6,13), (8,11), (9,15)\}$;
$\mathcal{K} = \{(4,1,0)-6$, $(10,0,7)-1$, $(13,14,12)-7$, $(2,6,8)-13$, $(6,3,1)-12$, $(3,9,13)-15$, $(14,7,8) - 10$, $(11,12,15) - 0$, $(13,2,10) - 4$, $(11,6,9) - 14\}$.
n=40: $X = \mathbb{Z}_{40}$. $\mathcal{B} = \{\{i, 1+i, 4+i, 13+i\}$, $\{i, 2+i, 7+i, 24+i\}$, $\{i, 6+i, 14+i, 25+i\}$,$\{j, 10+j, 20+j, 30+j\} \mid 0 \leq i \leq 39,\ 0 \leq j \leq 9\}$;
$\mathcal{C} = \{(i, 4+i, 20+i, 24+i)$, $(i, 5+i, 20+i, 25+i)$, $(i, 8+i, 20+i, 28+i) \mid 0 \leq i \leq 19\}$;
$L_{\mathcal{C}} = \{j, 20+j), (10+j, 30+j) \mid 0 \leq j \leq 9\}$;
$\mathcal{K} = \{(6,21,15) - 25$, $(7,22,16) - 26$, $(7,22,16) - 26$, $(8,23,17) - 27$, $(9,24,18) - 28$, $(10,25,19) - 29$, $(11,26,20) - 30$, $(12,27,21) - 31$, $(13,28,22) - 32$, $(14,29,23) - 33$, $(15,30,24) - 34$, $(16,31,25) - 30$, $(17,32,26) - 31$, $(18,33,27) - 32$, $(19,34,28) - 33$,

(20, 35, 29) − 34, (21, 36, 30) − 35, (22, 37, 31) − 36, (23, 38, 32) − 37, (24, 39, 33) − 38, (25, 0, 34) − 39, (26, 1, 35) − 0, (27, 2, 36) − 1, (28, 3, 37) − 2, (29, 4, 38) − 3, (30, 5, 39) − 4, (31, 6, 0) − 17, (32, 7, 1) − 18, (33, 8, 2) − 19, (34, 9, 3) − 20, (35, 10, 4) − 21, (36, 11, 5) − 22, (37, 12, 6) − 23, (38, 13, 7) − 24, (39, 14, 8) − 25, (0, 15, 9) − 26, (1, 16, 10) − 27, (2, 17, 11) − 28, (3, 18, 12) − 29, (4, 19, 13) − 30, (5, 20, 14) − 31, (0, 5, 17) − 29, (1, 6, 18) − 30, (2, 7, 19) − 31, (3, 8, 20) − 32, (4, 9, 21) − 33, (5, 10, 22) − 34, (6, 11, 23) − 35, (7, 12, 24) − 36, (8, 13, 25) − 37, (9, 14, 26) − 38, (10, 15, 27) − 39, (11, 16, 28) − 0, (12, 17, 29) − 1, (13, 18, 30) − 2, (14, 19, 31) − 3, (15, 20, 32) − 2, (16, 21, 33) − 3, (17, 22, 34) − 4, (18, 23, 35) − 5, (19, 24, 36) − 6, (20, 25, 37) − 7, (21, 26, 38) − 8, (22, 27, 39) − 9, (23, 28, 0) − 10, (24, 29, 1) − 11}. □

Remark 2.1. In the $S(2,4,16)$ given in Lemma 2.4, it is possible to choose a path of lenght 2 from each $B \in \mathcal{B}\setminus\{0,1,2,3\}$ so that the edges belonging to these paths can be reassembled into the set of (K_3+e)s $\{(13,14,2)-5,\ (12,8,7)-13,\ (2,8,6)-15,\ (6,3,5)-14,\ (3,13,9)-14,\ (11,12,15)-10,\ (13,10,12)-5,\ (9,6,11)-4,\ (4,5,7)-9\}$ and into the edges $\{0,15\},\{2,4\}$.

Remark 2.2. In the $S(2,4,16)$ given in Lemma 2.4, it is possible to choose a path of lenght 2 from each $B \in \mathcal{B}\setminus\{0,1,2,3\}$ so that the edges belonging to these paths can be reassembled into the set of (K_3+e)s $\{(12,8,7)-11,\ (6,2,8)-15,\ (3,6,5)-12,\ (3,13,9)-14,\ (11,12,15)-13,\ (13,12,10)-15,\ (9,11,6)-14,\ (4,7,5)-14\}$ and into the triangles $(0,7,10),(2,13,14)$.

The $6t+4$ Construction[6]. Let $n = 6t+4$, where t is even and $t \geq 10$. Let $X = \{1,2,\ldots,t\}$ and let R be a skew room frame of type $2^{t/2}$ with holes $H = \{h_1,h_2,\ldots,h_{t/2}\}$ of size 2. For the definition of a skew room frame and for results on its existence see [4].

1. For the hole $h_1 \in H$, let $(X_{h_1},\mathcal{B}_1)$ be a copy of the $S(2,4,16)$ in Lemma 2.4 on $X_{h_1} = \{a,b,c,d\} \cup (h_1 \times \mathbb{Z}_6)$.
2. For each hole $h_i \in H \setminus \{h_1\}$, let $(X_{h_i},\mathcal{B}_i)$ be a copy of the $S(2,4,16)$ in Lemma 2.4 on $X_{h_i} = \{a,b,c,d\} \cup (h_i \times \mathbb{Z}_6)$ such that $\{a,b,c,d\} \in \mathcal{B}_i$.
3. If x and y belong to different holes in H, then there exists only one cell (r,c) in R containing the pair $\{x,y\}$. Let $\mathcal{D} = \{\{(x,i),(y,i),(r,i+1),(c,i+4)\} \mid i \in \mathbb{Z}_6\}$.

Let $X = \bigcup_{h_i\in H} X_{h_i}$ and $\mathcal{B} = (\bigcup_{h_i\in H\setminus\{h_1\}} \mathcal{B}_i \setminus \{\{a,b,c,d\}\}) \cup \mathcal{B}_1 \cup \mathcal{D}$. It is straightforward to see that $(X,\mathcal{B})$ is an $S(2,4,n)$. For $i,j \in \mathbb{Z}_6$, the vertices $(x,i) \in X$ will be called "of level i" and the edge $\{(x,i),(y,j)\}$ will be called "between levels i and j".

Lemma 2.5. *For $n \equiv 16 \pmod{24}$, there exists an $S(2,4,n)$ having $\{C_4, K_3+e\}$-metamorphosis.*

Proof Let $n = 16 + 24k$. By Lemma 2.4 we can assume $k \geq 2$. Let $(X, \mathcal{B})$ the $S(2, 4, n)$ given by the $6t + 4$ Construction with $t = 4k + 2$. It is proved in [6](Lemma 2.5) that $(X, \mathcal{B})$ has a C_4-metamorphosis with leave a 1-factor. So we have only to prove the $(K_3 + e)$-metamorphosis of $(X, \mathcal{B})$.

- Take a $(K_3 + e)$-metamorphosis of $(X_{h_1}, \mathcal{B}_1)$ as in Lemma 2.4.
- For each hole h_{2i}, $1 \leq i \leq k$, delete the edges from type 2 blocks and reassemble them as in Remark 2.1, where we put a, b, c, d instead of $0, 1, 2, 3$.
- For each hole h_{2i+1}, $1 \leq i \leq k$, delete the edges from type 2 blocks and reassemble them as in Remark 2.2, where we put a, b, c, d instead of $0, 1, 2, 3$. Note that the edges from Remark 2.1 and the triangles from 2.2 can be reassembled into $(K_3 + e)$s.
- Delete the paths $[(x, 2), (c, 0), (y, 2)]$, $[(x, 3), (c, 1), (y, 3)]$ and $[(x, 4), (r, 5), (y, 4)]$ from all blocks in $\mathcal{D}$ of the form $\{(x, 2), (y, 2), (c, 0), (r, 3)\}$, $\{(x, 3), (y, 3), (c, 1), (r, 4)\}$ and $\{(x, 4), (y, 4), (r, 5), (c, 2)\}$. Delete the paths $[(y, 0), (x, 0), (r, 1)]$, $[(y, 1), (x, 1), (r, 2)]$, $[(y, 5), (x, 5), (r, 0)]$ from all blocks in $\mathcal{D}$ of the form $\{(x, 0), (y, 0), (r, 1), (c, 4)\}$, $\{(x, 1), (y, 1), (r, 2), (c, 5)\}$ and $\{(x, 5), (y, 5), (r, 0), (c, 3)\}$, respectively.

The deleted edges don't belong to the same hole and we can split them into the following classes:

1. edges between levels 0 and 2;
2. edges between levels 1 and 3;
3. edges between levels 4 and 5;
4. edges on level 0;
5. edges on level 1;
6. edges on level 5;
7. edges between levels 0 and 1;
8. edges between levels 1 and 2;
9. edges between levels 0 and 5.

Reassemble the edges of type 1, 4, 7 into the (K_3+e)s $((c, 2), (y, 0), (x, 0)) - (r, 1)$, the edges of type 2, 5, 8 into the (K_3+e)s $((c, 3), (y, 1), (x, 1)) - (r, 2)$, the edges of type 3, 6, 9 into the $(K_3 + e)$s $((c, 4), (y, 5), (x, 5)) - (r, 0)$. Note that, for example, $\{\{(x, 2), (c, 0)\}, \{(y, 2), (c, 0)\}\} = \{\{(c, 2), (y, 0)\}, \{(c, 2), (x, 0)\}\} = \{\{(a, 2), (1, 0)\}, \{(a, 2), (2, 0)\}, \{(b, 2), (3, 0)\}, \{(b, 2), (4, 0)\}, \ldots \mid a \neq 1, 2, b \neq 3, 4, \ldots\} = \{\{(1, 2), (a, 0)\}, \{(2, 2), (a, 0)\}, \{(3, 2), (b, 0)\}, \{(4, 2), (b, 0)\}, \ldots \mid a \neq 1, 2, b \neq 3, 4, \ldots\}$. Therefore we obtain a $(K_3 + e)$-design of order n. □

Theorem 2.6. *For $n \equiv 1, 4 \pmod{12}$, there exists an $S(2,4,n)$ having a $\{C_4, K_3 + e\}$-metamorphosis.*

Proof The result follows from Theorem 1.1 and Lemmas 2.3 and 2.5. □

3 $\lambda = 3$

Lemma 3.1. *There exist $\{4,5\}$-GDDs of type $2^1 4^5$, $3^1 5^4$, $6^1(6u+4)^4$, $u \geq 2$.*

Proof Let $(S, \mathcal{G}, \mathcal{B})$ be a 5-GDD of type 5^5 [3], where the groups are $G_i = \mathbb{Z}_5 \times \{i\}$, $i = 1, \ldots, 5$. Let $B_1, \ldots, B_5$ be the blocks of $\mathcal{B}$ meeting $(0,1)$. Remove the vertices $(0,1), (1,1), (2,1)$ and form a new GDD of type $2^1 4^5$ having $G_1 \setminus \{(0,1),(1,1),(2,1)\}$ and $B_i \setminus \{(0,1)\}$, $i = 1, \ldots, 5$ as groups and $G_i, i = 2,3,4,5$ and $B \setminus \{(1,1),(2,1)\}$, for every $B \in \mathcal{B} \setminus \{B_1, B_2, \ldots, B_5\}$, as blocks. Note that the blocks of size 5 of this new GDD are those meeting $(3,1)$ or $(4,1)$. The remaining blocks are of size 4.
Now delete $(0,1),(1,1)$ in $(S, \mathcal{G}, \mathcal{B})$. We get a $\{4,5\}$-GDD of type $3^1 5^4$. The blocks of the new GDD have size 5 if they contain one of the points $(2,1)$, $(3,1)$, $(4,1)$, otherwise have size 4.

Let $(S, \mathcal{G}, \mathcal{B})$ be a 5-GDD of type $(6u+4)^5$ $u \geq 2$ [3], where the groups are $G_i = \mathbb{Z}_{6u+4} \times \{i\}$, for $1 \leq i \leq 5$. By deleting the points $(0,1), (1,1), \ldots, (6u-3,1)$, we obtain a $\{4,5\}$-GDD of type $6^1(6u+4)^4$. The blocks of the new GDD have size 4 or 5. The blocks of size 5 are those containing $(x,1)$, for some $6u-2 \leq x \leq 6u+3$. □

Lemma 3.2. *For $t \geq 2, t \neq 3$, there exist 4-GDDs of index 3 and type $(2t)^4$ or $(2t)^5$ having a $\{C_4, K_3 + e\}$-metamorphosis.*

Proof Take the 4-GDD of type $(2t)^4$ constructed in Lemma 2.1 and repeat three times its blocks. The result is a 4-GDD of type $(2t)^4$ and index $\lambda = 3$. Now let $(X, \mathcal{B})$ be an $S_3(2,4,5)$. Place in each block $\{x_1, x_2, x_3, x_4\} \in \mathcal{B}$ a 4-GDD of type $(2t)^4$ with groups $G_i = \{x_i\} \times \mathbb{Z}_{2t}$ having a $\{C_4, K_3 + e\}$-metamorphosis. The result is the required 4-GDD of index 3 and type $(2t)^5$ having a $\{C_4, K_3 + e\}$-metamorphosis. □

Lemma 3.3. *For $n \equiv 1 \pmod 8$, $n \geq 9$, there exists an $S_3(2,4,n)$ having a $\{C_4, K_3 + e\}$-metamorphosis.*

Proof
n = 9. $X = \mathbb{Z}_9$. The starters blocks of $\mathcal{B}$ are $\{2,0,4,1\}$, $\{1,6,0,4\}$. If we delete the edges $\{a,b\}$, $\{c,d\}$ from each block $\{a,b,c,d\}$, we can reassemble these edges into a set $\mathcal{C} =$ with starter block $(0,4,8,2)$. If we delete the paths with starters $[4,2,1]$, $[1,0,6]$, we can reassemble these edges into a set $\mathcal{K}$ with starter block $(0,1,3) - 4$.

n = 17. $X = \mathbb{Z}_{17}$. The starters blocks of $\mathcal{B}$ are $\{6,4,1,0\}$, $\{2,12,8,0\}$, $\{16,7,4,0\}$, $\{15,8,14,0\}$. If we delete the edges $\{a,b\}$, $\{c,d\}$ from each block $\{a,b,c,d\}$, we can reassemble these edges into a set $\mathcal{C}$ with starter blocks $(0,8,16,3)$, $(0,1,3,10)$. If we delete the paths with starters $[1,4,0]$, $[8,0,12]$, $[16,4,7]$, $[0,15,14]$, we can reassemble these edges into a set $\mathcal{K}$ with starter blocks $(0,1,4)-9$, $(0,5,8)-10$.
n = 24u + 1, $u \geq 1$. Take 3 copies of the $S(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis given in Lemma 2.3.
n = 33. Take the 4-GDD of index 3 and type 8^4 constructed in Lemma 3.2. Add an infinite point to each group $G_i, i = 0,1,2,3$, and place on it a copy of the $S_3(2,4,9)$ above constructed. We obtain an $S_3(2,4,33)$ having a $\{C_4, K_3+e\}$-metamorphosis.
n = 24u + 9, $u \geq 2$ or $n = 48u + 17$, $u \geq 1$. Add an infinite point to the vertex set of a 4-GDD of type 2^{3u+1} (4^{3u+1})[3] and apply Theorem 1.2 with $r = s = 4$ and $w = 4$. The result is an $S_3(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis.
n = 96u + 41, $u \geq 0$. Blow up by 8 an $S_3(2,4,12u+5)$ $(\mathbb{Z}_{12u+5}, \mathcal{B})$ and place in each expanded block a 4-GDD of type 8^4 having a $\{C_4, K_3+e\}$-metamorphosis (see Lemma 2.1). To complete the proof add an infinite point to each expanded vertex of $\mathbb{Z}_{12u+5}$ and place on it an $S_3(2,4,9)$ having a $\{C_4, K_3+e\}$-metamorphosis.
n = 96u + 89, $u \geq 0$. Apply Theorem 1.2 with $\lambda = 3$, $\alpha = 1$, $r = 4$, $s = 5$ (Lemma 3.2) and the following ingredients given in Lemma 3.1:

- if $u = 0$: $w = 4$, a $\{4,5\}$-GDD of type $2^1 4^5$;
- if $u = 1$: $w = 8$, a $\{4,5\}$-GDD of type $3^1 5^4$;
- if $u \geq 2$: $w = 4$, a $\{4,5\}$-GDD of type $6^1(6u+4)^4$.

□

Lemma 3.4. *For $n = 8, 24$ there exist an $S_3(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis.*

Proof
n=8: $X = \mathbb{Z}_8$, $\mathcal{B} = \{\{0,1,3,7\}, \{1,2,4,7\}, \{2,3,5,7\}, \{3,4,6,7\}, \{4,5,0,7\}, \{5,6,1,7\}, \{0,6,2,7\}, \{2,4,5,6\}, \{3,5,6,0\}, \{4,6,0,1\}, \{5,1,0,2\}, \{6,3,1,2\}, \{0,3,2,4\}, \{1,3,4,5\}\}$. Delete the edges (a,b),(c,d) from each block $\{a,b,c,d\} \in \mathcal{B}$ and reassemble them into $\mathcal{C} = \{(0,1,2,7), (6,5,1,7), (5,4,3,7), (2,3,5,4), (6,0,2,4), (0,6,3,1)\}$ and $L_C = \{(1,2),(3,0),(4,7),(5,6)\}$. Delete from the blocks in $\mathcal{B}$ the paths $[1,0,3]$, $[1,4,7]$, $[4,6,7]$, $[0,5,7]$, $[5,1,7]$, $[2,0,7]$, $[5,4,6]$, $[3,6,5]$, $[6,0,4]$, $[1,0,5]$, $[2,1,3]$, $[0,3,4]$, $[1,3,5]$ and reassemble their edges into $\mathcal{K} = \{(2,1,0)-4, (3,5,0)-1, (3,7,1)-4, (6,7,4)-3, (0,6,3)-1, (0,7,5)-1, (4,6,5)-3\}$.

n=24: $X = \mathbb{Z}_{12} \times \{1,2\}$. $\mathcal{B} = \{\{(i,1),(11+i,2),\ (1+i,1),(2+i,2)\},\ \{(i,1),(i,2),(3+i,1),(5+i,1)\},\ \{(i,1),(9+i,2),\ (4+i,1),(6+i,1)\},\ \{(i,1),(7+i,2),\ (3+i,1),(5+i,1)\},\ \{(i,1),(6+i,2),\ (4+i,1),(5+i,1)\},\ \{(i,1),(8+i,2),\ (3+i,1),(4+i,1)\},\ \{(i,1),(6+i,2),\ (10+i,2),(11+i,2)\},\ \{(i,1),(4+i,2),\ (8+i,2),(9+i,2)\},\ \{(i,1),(11+i,2),(8+i,2),(10+i,2)\},\ \{(i,1),(i,2),\ (3+i,2),(5+i,2)\},\ \{(i,1),(7+i,2),\ (1+i,2),(3+i,2)\},\ \{(j,1),(j,2),\ (6+j,1),(6+j,2)\} \mid i \in \mathbb{Z}_{12}, j \in \mathbb{Z}_6\}$. Delete the edges $\{a,b\}$, $\{c,d\}$ from each block $\{a,b,c,d\}$ and reassemble them into $\mathcal{C} = \{((i,1),(2+i,1),(1+i,2),(11+i,2)),\ ((i,1),(2+i,1),(2+i,2),(1+i,2)),\ ((i,1),(1+i,1),(10+i,2),(8+i,2)),\ ((j,1),(6+j,1),(j,2),(6+j,2)) \mid i \in \mathbb{Z}_{12}, j \in \mathbb{Z}_6\}$ and $L_{\mathcal{C}} = \{\{(j,1),(j,2)\},\{(6+j,1),(6+j,2)\} \mid j \in \mathbb{Z}_6\}$.
$\mathcal{K} = \{((i,2),(5+i,1),(i,1)) - (2+i,2),\ ((9+i,2),(6+i,1),(i,1)) - (5+i,1),\ ((3+i,2),(1+i,2),(i,1)) - (4+i,2),\ ((11+i,2),(8+i,2),(i,1)) - (i,2) \mid i \in \mathbb{Z}_{12}\} \cup \{((10,2),(0,1),(6,2)) - (0,2),\ ((11,2),(1,1),(7,2)) - (1,2),\ ((12,2),(2,1),(8,2)) - (2,2),\ ((13,2),(3,1),(9,2)) - (3,2),\ ((14,2),(4,1),(10,2)) - (4,2),$
$((3,2),(5,1),(11,2)) - (7,2),\ ((4,2),(6,1),(12,2)) - (8,2),\ ((4,1),(0,1),(3,1)) - (3,2),\ ((5,1),(1,1),(4,1)) - (4,2),\ ((6,1),(2,1),(5,1)) - (5,2),\ ((7,1),(3,1),(6,1)) - (6,2),\ ((8,1),(4,1),(7,1)) - (7,2),\ ((8,1),(5,1),(9,1)) - (3,2),\ ((9,1),(6,1),(10,1)) - (4,2),\ ((10,1),(7,1),(11,1)) - (5,2),\ ((0,1),(1,1),(9,1)) - (7,2),$
$((1,1),(2,1),(10,1)) - (8,2),\ ((2,1),(3,1),(11,1)) - (9,2),\ ((1,2),(7,1),(5,2)) - (11,2),\ ((2,2),(8,1),(6,2)) - (10,2),\ ((0,1),(11,1),(8,1)) - (8,2)\}$. □

The $4t$ Construction. [6] Let $n = 4t$, where $t \geq 4$ and $t \neq 6$. Let $S = \{1,2,\ldots,t\}$ and let $(S,\circ)$ be an idempotent self-orthogonal quasigroup of order t [2]. Set $X = S \times \mathbb{Z}_4$ and define a collection of blocks $\mathcal{B}$ as follows:

1. For each $x \in S$, place in $\mathcal{B}$ three copies of the block $\{(x,0),(x,1),(x,2),(x,3)\}$.

2. For each pair $x,y \in S, x < y$, place in $\mathcal{B}$ the blocks $\{(x,i),(y,i),(x \circ y, i+1),(y \circ x, i+1)\}$, where $i \in \mathbb{Z}_4$ and the second coordinates are reduced modulo 4.

3. For each pair $x,y \in S, x < y$, place in $\mathcal{B}$ the blocks $\{(x,i),(y,i),(x \circ y, i+2),(y \circ x, i+2)\}$, where $i = 0,1$ and the second coordinates are reduced modulo 4.

4. For each pair $x,y \in S, x \neq y$, place in $\mathcal{B}$ the block $\{(x,0),(y,1),(x \circ y,2),(y \circ x,3)\}$.

Then $(X,\mathcal{B})$ is an $S_3(2,4,n)$. For $i \in \mathbb{Z}_4$, the vertices $(x,i) \in X$ will be called "of level i" and the edge $\{(x,i),(x,j)\}$ will be called "belonging to the same column".

Lemma 3.5. *For $n \equiv 0 \pmod 8$, there exists an $S_3(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis.*

Proof Let $n = 8k$. For $k = 1, 3$, the result follows from Lemma 3.4. Now let $k \neq 1, 3$ and let $(X, \mathcal{B})$ be the $S_3(2, 4, 8k)$ given in the $4t$ Construction with $t = 2k$. Lemma 4.4 in [6] proves that $(X, \mathcal{B})$ has a C_4-metamorphosis.

Now we prove that $(X, \mathcal{B})$ has a $(K_3 + e)$-metamorphosis:

- For each odd $x \in S$, delete the paths $2[(x,1), (x,0), (x,2)]$ and $[(x,1),(x,2),(x,3)]$ from type 1 blocks; for each even $x \in S$, delete the paths $2[(x,0),(x,1),(x,2)]$ and $[(x,0),(x,2),(x,3)]$ from type 1 blocks. Reassemble these paths into $(K_3 + e)s$ with leave $[(x,1),(x,0),(x,2)]$ for x odd and
$[(x,0),(x,1),(x,2)]$ for x even.

- From each type 2 block delete the path $[(x,i),(x \circ y, i+1),(y,i)]$. The deleted edges don't belong to the same column and we can split them into the following classes: (I) edges between levels 0 and 1, (II) edges between levels 1 and 2, (III) edges between levels 2 and 3, (IV) edges between levels 0 and 3.

- From each type 3 block delete the path $[(y,i),(x,i),(y \circ x, i+2)]$ if $x = 2j-1$ and $y \circ x = 2j$, $j = 1, \ldots, k$, otherwise delete the path $[(x,i),(y,i),(y \circ x, i+2)]$. The deleted edges don't belong to the same column and we can split them into the following classes: (V) edges on level 0, (VI) edges on level 1, (VII) edges between levels 0 and 2, (VIII) edges between levels 1 and 3.

- From each type 4 block delete the path $[(y,1),(x,0),(x \circ y, 2)]$. The deleted edges don't belong to the same column and we can split them into the following classes: (IX) edges between levels 0 and 2, (X) edges between levels 0 and 1.

Reassemble the deleted edges (I), (V) and (VII) into the (K_3+e)s $((y,0),(x \circ y,1),(x,0)) - (y \circ x, 2)$ if $x = 2j-1$ and $y \circ x = 2j$, $j = 1, \ldots, k$; otherwise, into the (K_3+e)s $((x,0),(x \circ y,1),(y,0)) - (y \circ x, 2)$.

Reassemble the deleted edges (II), (VI), (VIII) into the (K_3+e)s $((y,1),(x \circ y,2),(x,1)) - (y \circ x, 3)$ if $x = 2j-1$ and $y \circ x = 2j$, $j = 1, \ldots, k$; otherwise, into the (K_3+e)s $((x,1),(x \circ y,2),(y,1)) - (y \circ x, 3)$.

Reassemble the deleted edges (III), (IV), (IX) and (X) into the (K_3+e)s $((y \circ x,3),(x \circ y,2),(x,0)) - (y,1)$.

Next we need to rearrange these (K_3+e)s to use the paths obtained from type 1 blocks, $[(x,1),(x,0),(x,2)]$, for x odd, and $[(x,0),(x,1),(x,2)]$, for x even. For each $j = 1, \ldots, k$, replace the (K_3+e) $((y,0),(x \circ y,1),(2j-1,0)) - (2j,2)$, obtained by rearranging the deleted edges (I), (V) and (VII), by $((y,0),(x \circ y,1),(2j-1,0)) - (2j-1,2)$. Replace the (K_3+e) $((y,3),(x \circ y,2),(2j-1,0)) - (2j,1)$, obtained by rearranging the deleted edges (III), (IV), (IX) and (X), by $((y,3),(x \circ y,2),(2j-1,0)) - (2j-1,1)$.

Next arrange the remaining edges $\{(2j,0),(2j,1)\}$, $\{(2j,1),(2j,2)\}$, $\{(2j-1,0),(2j,1)\}$ and $\{(2j-1,0),(2j,2)\}$, $j=1,\dots,k$, into the (K_3+e)s $((2j-1,0),(2j,2),(2j,1))-(2j,0)$, $j=1,\dots,k$.

We obtain a 3-fold (K_3+e)-design of order n and so an $S_3(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis. □

Theorem 3.6. *For $n \equiv 0,1 \pmod 4$, there exists an $S_3(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis.*

Proof For $n \equiv 4,5 \pmod 8$, the result follows from Theorem 1.1. For $n \equiv 0 \pmod 8$ and for $n \equiv 1 \pmod 8$, the result follows from Lemmas 3.5 and 3.3, respectively. □

4 Summary

Lemma 4.1. *For $\lambda = 2$ with $n \equiv 1,4 \pmod{12}, n \geq 4$, $\lambda = 6$ with $n \equiv 0,1 \pmod 4, n \geq 4$, $\lambda = 4,8$ with $n \equiv 1 \pmod 3, n \geq 4$ and $\lambda = 12$, with $n \geq 4$, there exists an $S_\lambda(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis.*

Proof For the values of λ and n as in hypothesis, there exists an $S_{\lambda/2}(2,4,n)$, $(X,\mathcal{B})$. By repeating two times each block of $(X,\mathcal{B})$, we obtain an $S_\lambda(2,4,n)$. For each $B_1, B_2 \in \mathcal{B}$ such that $B_1 = B_2 = \{x,y,z,t\}$, remove the edges $\{x,y\}$, $\{z,t\}$ ($\{x,y\}$ and $\{x,t\}$) from B_1 and the edges $\{x,t\}$, $\{y,z\}$ ($\{y,t\}$, $\{z,t\}$) from B_2. Rearrange the removed edges into the 4-cycle (x,y,z,t) (into the K_3+e $(x,y,t)-z$). This completes the proof. □

Theorem 4.2. *There exists an $S_\lambda(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis if and only if $n \geq 4$, $\lambda n(n-1) \equiv 0 \pmod{12}$ and $\lambda(n-1) \equiv 0 \pmod 3$.*

Proof The necessity is trivial. For $\lambda = 1,3$ the result follows from Theorems 2.6, 3.6. For $\lambda = 2$ with $n \equiv 1,4 \pmod{12}$, $\lambda = 6$ with $n \equiv 0,1 \pmod 4$, $\lambda = 4,8,12$, the result follows from Lemma 4.1. For $\lambda = 2$, $n = 7,10,19$, the result follows from Theorem 1.1. For $\lambda = 2$, $n \equiv 7,10 \pmod{12}$, $n \geq 22$, take a $PBD(n)$ with one block of size 7 and others of size 4 [9] and place an $S_2(2,4,4)$ or an $S_2(2,4,7)$ having a $\{C_4, K_3+e\}$-metamorphosis on each block. For $\lambda = 6$ and $n \equiv 2,3 \pmod 4$, the result follows from Theorem 1.1. For $\lambda = 5,7,9,10,11$ combine a $S_\nu(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis with a $S_\mu(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis, with $(\lambda,\nu,\mu) = (5,4,1)$, $(7,6,1)$, $(9,6,3)$, $(10,8,2)$, $(11,6,5)$, respectively. For $\lambda = 12k+h$, with $0 \leq h \leq 11$, combine k $S_{12}(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis with an $S_h(2,4,n)$ having a $\{C_4, K_3+e\}$-metamorphosis. □

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