Weakly split graphs and regular cellulations of the 3-sphere

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ABSTRACT. We conjectured in [3] that every biconnected cyclic graph is the one-dimensional skeleton of a regular cellulation of the 3-sphere and proved it is true for planar and hamiltonian graphs. In this paper we introduce the class of weakly split graphs and prove the conjecture is true for such class. Hamiltonian, split, complete k-partite and matrogenic cyclic graphs are weakly split.

1. Introduction

Let X be a CW-complex [5] on the 3-sphere $S^3 = \{x \in R^4 : |x| = 1\}$ with its standard topology. X is also called a *cellulation* of the 3-sphere. The ascending sequence $X^0 \subset X^1 \subset X^2 \subset X^3 = X$ of closed subspaces of X satisfies the following conditions:

[1] X^0 is a discrete set of points (0-cells)

[2] For $0 < k \leq 3$, $X^k - X^{k-1}$ is the disjoint union of open subspaces, called k-cells, each of which homeomorphic to the open k - dimensional ball $U^k (= \{x \in R^k : |x| < 1\})$.

X^k is the k-dimensional skeleton of X and is a k-dimensional CW-complex for $1 \leq k \leq 3$ on a subspace of the 3-sphere. X is a *regular* CW-complex if the boundary of every k-cell is homeomorphic to the $k-1$ dimensional sphere S^{k-1}, for $1 \leq k \leq 3$. Then, X is called a regular cellulation of S^3. If X is regular, the boundary of every 1-cell is a pair of 0-cells. It follows that the one-dimensional skeleton of a regular CW-complex represents a graph with no loops where the 0-cells correspond to the vertices and the 1-cells correspond to the edges. From now on, we will consider simple graphs (no loops and no multiple edges between two vertices). In particular we are interested in *cyclic* graphs, that is, graphs which contain at least one cycle. Since the graphs are simple, the cycles must be closed paths comprising at least three vertices.

We conjectured in [3] that every biconnected cyclic graph is the one-dimensional skeleton of a regular cellulation of the 3-sphere and proved it is true for planar and hamiltonian graphs. In [3] this conjecture was given for graphs with at least two cycles, because we assumed that two 2-cells could not share the same boundary in order to relate it to the concept of spatiality degree [1, 2, 4].

In this paper we introduce the class of weakly split graphs and prove the conjecture is true for such class. Hamiltonian, split, complete k-partite and matrogenic cyclic graphs are weakly split. Matrogenic graphs include matroidal graphs. Split matrogenic graphs include threshold graphs. Several characterizations of these classes are given in [6].

We define split, matrogenic and weakly split graphs in the next section and show that biconnected graphs which are weakly split verify the conjecture.

2. Split and Weakly Split Graphs

A connected graph $G = (V, E)$ is *weakly split* if V is the union of three disjoint sets I, K and C such that:

- I is empty or a stable set in G;
- K is non-empty and the subgraph induced by K is hamiltonian;
- C is either empty or none of its vertices is adjacent to a vertex in I and C induces a subgraph such that each connected component is a simple path and the terminal vertices of each path are adjacent to at least two vertices in K while the internal ones are adjacent either to at least two vertices in K or to none.

We call the subgraph induced by C the *crown* of G. We call G *split* if the property required on the subgraph induced by K is to be complete and C is empty.

Lemma 1.1. A cyclic split graph $G = (V, E)$ is weakly split.

Proof. Since G is cyclic and split, V is the union of two disjoint sets I and K, with I stable (or empty if $|V| = 3$) and K inducing a complete subgraph with at least three vertices. Therefore, G is weakly split. □

A connected graph $G = (V, E)$ is *matrogenic* if V is the union of three disjoint sets I, K and C such that:

- K is non-empty and the subgraph induced by K is complete;
- C is either empty or each vertex of C is adjacent to every vertex in K and to none in I and the subgraph induced by C is either a chordless simple cycle of five vertices or a matching (a set of disjoint edges) or the complement of a matching (anti-matching);
- I is empty or a stable set in G such that if the neighborouds of two vertices in I (that is, subsets of K) are incomparable then their symmetric difference is equal to 2.

Lemma 1.2. A cyclic matrogenic graph $G = (V, E)$ is weakly split.

Proof. Since G is a cyclic matrogenic graph, V is the union of three disjoint sets I, K and C such that I is stable (or empty if $|V| = 3$) and either the subgraph induced by K is complete with at least three vertices or C is not empty. If C induces a chordless simple cycle of five vertices or an anti-matching then $K \cup C$ induces a hamiltonian subgraph since C induces a hamiltonian subgraph and every vertex of C is connected to every vertex of K. Therefore, G is weakly split with an empty crown. Otherwise, C induces a matching, that is, a crown according to the definition of a weakly split graph. The union of K with two adjacent vertices in C induces a hamiltonian subgraph. Therefore, G is weakly split. □

Lemma 1.3. Let $m_1, m_2 > 1$ if $k = 2$. Then, a complete k-partite graph $K_{m_1,m_2,\cdots,m_k}$ is weakly split.

Proof. Since $m_1, m_2 > 1$ if $k = 2$, $K_{m_1,m_2,\cdots,m_k}$ is always cyclic. Let $I^1 = \{i_1^1 \cdots i_{m_1}^1\}, \cdots, I^k = \{i_1^k \cdots i_{m_k}^k\}$ be the k elements of the partition and $m_1 \leq m_2 \leq \cdots \leq m_k$. Without loss of generality we assume that m_k is greater than 1 (otherwise the graph would be weakly split since it is complete) and $k_1 < \cdots < k_d$ is the subsequence of $1 \cdots k$ such that $k_1 = 1$ and $m_{k_j - 1} < m_{k_j}$ for $2 \leq j \leq d$. We compute a simple cycle with the following procedure. As first step, start the cycle with the simple path $i_1^1, i_1^2, \cdots i_1^k$. As second step, continue the path with the vertex i_2^{k-1} (if there is no such vertex, the graph is weakly split with an empty crown and a stable set of two vertices totally connected to a complete graph). Then, there is either a sequence $i_2^{k-1}, \cdots i_2^1$ or a sequence $i_2^{k-1}, \cdots i_2^{k_2}$ (if $m_1 = 1$) which can continue the path. Generally speaking, the odd steps add sequences of vertices from the leftmost stable set in $I^1, \cdots, I^k$ which has vertices not covered yet by the path to I^k while the even steps add sequences from I_k to such set. For stable sets with even cardinality m_{k_j} and $j < d$, the last vertex is covered by the path at an even step. The successive odd step continues the path with a sequence of vertices from $I^{k_{j+1}}$ to I^k. Finally, only vertices of the sets $I^{k_d}, \cdots, I^k$ are not covered. If $k_d = k$, the graph is weakly split because an edge from the last vertex of I^k covered by the path to i_1^1 can be added to have a cycle which covers all the vertices but a subset of I^k. Otherwise, we add sequences of vertices from I^k to I^{k_d} and viceversa until all of them are covered and then we close the path with an edge from the last covered vertex of I^k to i_1^1. Then, the graph is weakly split because it is hamiltonian. □

We give as a lemma the result shown in [1] that hamiltonian graphs verify the conjecture since it is needed to prove the conjecture for weakly split graphs.

Lemma 1.4. Every hamiltonian graph $G = (V, E)$ is the one-dimensional skeleton of a regular cellulation of S^3.

Proof. We embed V into the 3-sphere. Let $v_1, v_2, ...v_n, v_1$ be the sequence of vertices (0-cells) ordered by a hamiltonian cycle h of G, where $|V| = n$. We embed the edges of h (1-cells) into the 3-sphere so that we have a one-dimensional complex X. Then, we add to X a 2-cell with boundary h. If G is a simple cycle, another 2-cell with boundary h is added to X. At this point, by adding two 3-cells to X we obtain a regular cellulation of the 3-sphere. If G is not a simple cycle, let us consider any edge, say (v_i, v_j), which does not belong to h, with $i < j$. We add to X the edge (v_i, v_j) as a 1-cell and two 2-cells with the cycles $v_1, ..., v_i, v_j, ...v_n, v_1$ and $v_i, v_j, v_{j-1}, ..., v_i$ as boundaries, respectively. These 2-cells are added so that their intersection is the edge (v_i, v_j) to satisfy the property of a CW-complex on the disjointness of cells. Then, we add one 3-cell bounded by these 2-cells and the 2-cell with h as boundary. Since we added only one 3-cell, we can embed the remaining edges of G and, similarly, the corresponding two 2-cells and one 3-cell for each edge. Differently from the first 3-cell we added, the boundaries of these additional 3-cells comprise four 2-cells instead of three. Finally, we add to X one more 3-cell to obtain the regular cellulation of the 3-sphere with G as one-dimensional skeleton. □

Now, we can prove our result.

Theorem 2.1. A biconnected weakly split graph $G = (V, E)$ is the one-dimensional skeleton of a regular cellulation of S^3.

Proof. Since G is weakly split, V is the union of three disjoint sets I, K and C such that I is stable, the subgraph induced by K is hamiltonian and C is the crown. We embed K into the 3-sphere. Let $w_1, w_2, ...w_k, w_1$ be the sequence of vertices ordered by the hamiltonian cycle h of the subgraph induced by K. We embed the edges of h into the 3-sphere so that we have a one-dimesional complex X and we add to X a 2-cell with boundary h. Then, we can apply to X the procedure of Lemma 1.4 to produce a regular cellulation of a proper subspace B_1 of S^3. B_1 is a proper subspace of S^3 because we do not add to X the last 3-cell produced by the procedure of lemma 1.4. Therefore, B_1 is homeomorphic to a closed 3-dimensional ball while the complement B_2 of B_1 in S^3 is an open 3-dimensional ball where we embed the vertices $u_1, u_2, ...u_i$ of I. For each vertex u_j, $1 \leq j \leq i$, first we add the edges connecting u_j to the adjacent vertices in h to X. Since G is biconnected, there are at least two such vertices for each u_j. Then, for each pair of vertices w and w' adjacent to u_j and consecutive in h, we add to X a 2-cell with boundary the cycle defined by u_j, w, w' and the vertices in h between w and w' (which, obviously, are not adjacent to u_j). These 2-cells can be added so that they are disjoint and a 3-cell bounded by these 2-cells and the 2-cells determined by u_{j-1} (if $j = 1$, the 2-cell with boundary h) is added as well. The homeomorphism of such boundary to the 2-sphere follows from the disjointness of the 2-cells. If C is empty, we add to X one more 3-cell to obtain the regular cellulation of the 3-sphere with G as one-dimensional skeleton. Otherwise, the vertices in C are embedded into the portion of B_2 which has not been covered by any cell yet. C induces a graph with p connected components where each connected component is a simple path. Let $C_1, \cdots C_p$ be the partition of C such that each element of the partition induces one of the p connected components. Let $t_1, \cdots t_c$ be the vertices of C_1 in the order induced by the corresponding simple path. Then, for $1 \leq j \leq c$ we add to X the edges (if any) connecting t_j to the adjacent vertices in h and, for each pair of vertices w and w' adjacent to t_j and consecutive in h, we add to X a 2-cell with boundary the cycle defined by t_j, w, w' and the vertices in h between w and w' (which are not adjacent to t_j since w and w' are consecutive in h). As for the vertices in I, these 2-cells can be added so that they are disjoint. Let $j_1 \cdots j_\ell$ be the subsequence of $1 \cdots c$ such that $t_{j_1} \cdots t_{j_\ell}$ are the vertices of C_1 adjacent to at least two vertices in K (obviously, $j_1 = 1$ and $j_\ell = c$). Then, for $1 \leq r \leq \ell$, we add to X the edges of the path from t_{j_r} to $t_{j_{r+1}}$. It follows from the definition of weakly split graph that we can select in h two vertices adjacent to t_{j_r} and two vertices adjacent to $t_{j_{r+1}}$. These selections define a set S of vertices in h of cardinality between two and four, depending on whether two, one or none of the selected vertices adjacent to t_{j_r} coincide with the two selected vertices adjacent to $t_{j_{r+1}}$. Then, we add two 2-cells with boundaries the cycles defined by the vertices of the path from t_{j_r} to $t_{j_{r+1}}$, two vertices of S respectively adjacent to t_{j_r} and $t_{j_{r+1}}$ which are consecutive (unless they coincide) in h with respect to S and the vertices in h (if any) between them (which do not belong to S since the two vertices of S are consecutive). It follows that these two 2-cells can be added to X so that they are disjoint. Therefore, two disjoint 3-cells can be added to X bounded by these two 2-cells and complementary subsets of the 2-cells determined by $t_{j_{r+1}}$

and by t_{j_r}. Moreover, we add one 3-cell bounded by the 2-cells determined by t_{j_1} and the ones determined by u_i. Again, the boundaries of these 3-cells are homeomorphic to the 2-sphere. Such embedding procedure is repeated for each connected component $C_2, \cdots C_p$ of the crown (for each of these components, the last 3-cell added to X is partially bounded by 2-cells of the previous component). Finally, we add to X one more 3-cell to obtain the regular cellulation of the 3-sphere with G as one-dimensional skeleton. □

Corollary 2.1. If a biconnected cyclic graph G is split, matrogenic or complete k-partite then it is the one-dimensional skeleton of a regular cellulation of S^3.

Proof. It follows from lemma 1.1, lemma 1.2, lemma 1.3 and theorem 2.1. □

2. Conclusion

We introduced the class of weakly split graphs and proved that a weakly split graph is the one-dimensional skeleton of a regular cellulation of the 3-sphere. Weakly split graphs include hamiltonian, split, complete k-partite and matrogenic cyclic graphs. Matrogenic graphs include matroidal graphs. Threshold graphs are split and matrogenic. Hamiltonian graphs include complete graphs. Over all the graphs with n vertices, the complete graph is an obvious case where the genus is maximized. On the other hand, when the genus of the graph is 0 the regular cellulation of the 3-sphere is obviously provided by the graph embedding into the 2-sphere. This consideration suggested the conjecture that every biconnected graph is the one-dimensional skeleton of a regular cellulation of the 3-sphere since this property might hold when the graph lies, as far as embeddability into surfaces is concerned, in between a planar one and a complete one. In conclusion, we want to point out that such extremal results were obtained in this paper for k-partite graphs since complete k-partite graphs are weakly split.

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