On strict-double-bound numbers of spiders and ladders

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Abstract

For a poset $P = (X, \leq_P)$, the *strict-double-bound graph* (*sDB-graph* sDB(P)) is the graph on X for which vertices u and v of sDB(P) are adjacent if and only if $u \neq v$ and there exist x and y in X distinct from u and v such that $x \leq_P u \leq_P y$ and $x \leq_P v \leq_P y$. The *strict-double-bound number* $\zeta(G)$ of a graph G is defined as $\min\{n; G \cup \overline{K}_n$ is a strict-double-bound graph $\}$.

We obtain that for a spider $S_{n,m}$ $(n, m \geq 3)$ and a ladder L_n $(n \geq 4)$, $\lceil 2\sqrt{nm} \rceil \leq \zeta(S_{n,m}) \leq n + m$, $\zeta(S_{n,n}) = 2n$, and $\lceil 2\sqrt{3n+2} \rceil \leq \zeta(L_n) \leq 2n$.

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1 Introduction

In this paper we consider finite undirected simple graphs. For a graph G, $\overline{G}$ is the complement of G. A *clique* in a graph G is the vertex set of a maximal complete subgraph of G. A family $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_n\}$ is an *edge clique cover* of G if each Q_i is a clique of G and for each $\{u, v\} \in E(G)$, there exists $Q_i \in \mathcal{Q}$ such that $u, v \in Q_i$. For a graph G and $S \subseteq V(G)$, $\langle S \rangle_V$ is the induced subgraph of S. For a graph G and $v \in V(G)$, $N_G(v) = \{u\ ; \{u, v\} \in E(G)\}$.

For a poset $P = (X, \leq_P)$ and an element $x \in X$ of P, we put $U_P(x) = \{y \in X \ ; \ x \leq_P y\}$ and $L_P(x) = \{y \in X \ ; \ y \leq_P x\}$. For a poset P, let $\mathrm{Max}(P)$ be the set of all maximal elements of P and let $\mathrm{Min}(P)$ be the set of all minimal elements of P.

McMorris and Zaslavsky [4] introduced a concept of double bound graphs. Diny [1] characterized double bound graphs. We consider strict-double-bound graphs and strict-double-bound numbers. For a poset $P = (X, \leq_P)$, the *strict-double-bound graph* (*sDB-graph*) of $P = (X, \leq_P)$ is the graph $\mathrm{sDB}(P)$ on X for which vertices u and v of $\mathrm{sDB}(P)$ are adjacent if and only if $u \neq v$ and there exist $x \in X$ and $y \in X$ distinct from u and v such that $x \leq_P u \leq_P y$ and $x \leq_P v \leq_p y$. We say that a graph G is a *strict-double-bound graph* if there exists a poset whose strict-double-bound graph is isomorphic to G.

Maximal elements and minimal elements of posets are isolated vertices of strict-double-bound graphs. So a connected graph with $p \geq 2$ vertices is not a strict-double-bound graph. Era, Tsuchiya [2] and Scott [6] dealt with strict-double-bound graphs. Scott [6] gave the following result.

Proposition 1.1 (Scott [6]) *Any graph that is the disjoint union of a non-trivial component and enough number of isolated vertices is a strict-double-bound graph.*

We introduce the strict-double-bound number of a graph. The *strict-double-bound number* $\zeta(G)$ is defined as $\min\{n \ ; \ G \cup \overline{K}_n$ is a strict-double-bound graph $\}$. In this paper, we consider properties of strict-double-bound numbers.

Scott [6] obtains the following result, using a concept of transitive double competition numbers.

Theorem 1.2 (Scott [6]) *For a non-trivial connected graph G and a minimal edge clique cover $\mathcal{Q}$ of G,* $\left\lceil 2\sqrt{|\mathcal{Q}|} \right\rceil \leq \zeta(G) \leq |\mathcal{Q}| + 1$.

We already know $\zeta(K_n) = 2$ for $n \geq 2$ by Theorem 1.2. Konishi, Ogawa, Tagusari, Tsuchiya [3] and Ogawa, Tagusari, Tsuchiya [5] obtained that $\zeta(K_{1,n}) = \left\lceil 2\sqrt{n} \right\rceil$ $(n \geq 1)$, $\zeta(P_n) = \left\lceil 2\sqrt{n-1} \right\rceil$ $(n \geq 2)$, $\zeta(C_n) = \left\lceil 2\sqrt{n} \right\rceil$ $(n \geq 4)$, and $\zeta(W_n) = \left\lceil 2\sqrt{n-1} \right\rceil$ $(n \geq 5)$. In [5] Ogawa, Tagusari, Tsuchiya also gave that $\lceil 2\sqrt{n-1} \rceil \leq \zeta(T) \leq \sum_{v \in \mathrm{IN}(T)} \left\lceil 2\sqrt{\deg_T(v)} \right\rceil - 2(|\mathrm{IN}(T)| - 1)$, where T is a non-trivial tree with $n \geq 2$ vertices, and $\mathrm{IN}(T)$

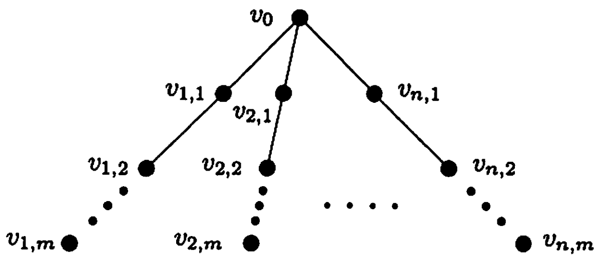

Figure 1: A spider $S_{n,m}$

is the vertex set of non-leaves of T. In this paper we deal with strict-double-bound numbers of other graphs, that is, spiders and ladders.

2 On spiders

A *spider* $S_{n,m}$ $(n \geq 3)$ is a graph as follows:

(1) $V(S_{n,m}) = \{v_0\} \cup \{v_{i,j}\ ;\ i = 1, 2, ..., n, j = 1, 2, ..., m\}$,

(2) $E(S_{n,m}) = \{\{v_0, v_{i,1}\}\ ;\ 1 \leq i \leq n\} \cup \{\{v_{i,j}, v_{i,j+1}\}\ ;\ 1 \leq i \leq n, 1 \leq j \leq m-1\}$.

We have the following result on strict-double-bound numbers of spiders.

Proposition 2.1 *For a spider* $S_{n,m}$ $(n, m \geq 3)$, $\lceil 2\sqrt{nm} \rceil \leq \zeta(S_{n,m}) \leq n+m$,

Proof. We construct a poset P for $S_{n,m}$ as follows:

(1) $V(P) = V(S_{n,m}) \cup \{x_1, x_2, ..., x_n, y_1, y_2, ..., y_m\}$,

(2) $V(S_{n,m})$, $\{x_1, x_2, ..., x_n\}$ and $\{y_1, y_2, ..., y_m\}$ are antichains of P,

$$(3)\begin{cases} v_0 \leq_P x_i & (1 \leq i \leq n), \\ v_{i,j} \leq_P x_i & (1 \leq i \leq n, 1 \leq j \leq m), \\ y_1 \leq_P v_0, & \\ y_j \leq_P v_{i,j-1} & (1 \leq i \leq n, 2 \leq j \leq m), \\ y_j \leq_P v_{i,j} & (1 \leq i \leq n, 1 \leq j \leq m). \end{cases}$$

We show that $\mathrm{sDB}(P) \cong S_{n,m} \cup \overline{K}_{n+m}$.

(1) For $v_0, v_{i,1}$ $(1 \leq i \leq n)$, $L_P(v_0) \cap L_P(v_{i,1}) = \{y_1\} \neq \emptyset$ and $U_P(v_0) \cap U_P(v_{i,1}) = \{x_i\} \neq \emptyset$. So there exist edges $\{v_0, v_{i,1}\} (1 \leq i \leq n)$ in sDB(P).

(2) For $v_{i,j-1}, v_{i,j}$ $(1 \leq i \leq n,\ 2 \leq j \leq m)$, $L_P(v_{i,j-1}) \cap L_P(v_{i,j}) = \{y_j\} \neq \emptyset, U_P(v_{i,j-1}) \cap U_P(v_{i,j}) = \{x_i\} \neq \emptyset$. So there exist edges $\{v_{i,j-1}, v_{i,j}\}$ $(1 \leq i \leq n,\ 2 \leq j \leq m)$ in sDB(P).

(3) For $v_0, v_{i,j}$ $(1 \leq i \leq n,\ 2 \leq j \leq m)$, $L_P(v_0) \cap L_P(v_{i,j}) = \emptyset$. Next we consider adjacency relations of $v_{i,j}$ and $v_{k,l}$ for $1 \leq i, k \leq n$ and $1 \leq j, l \leq m$. In the case $i \neq k$, $U_P(v_{i,j}) \cap U_P(v_{k,l}) = \emptyset$ for $1 \leq i, k \leq n$ and $1 \leq j, l \leq m$. In the case $i = k$ and $|j - l| \geq 2$, $L_P(v_{i,j}) \cap L_P(v_{i,l}) = \emptyset$ for $1 \leq i (= k) \leq n$ and $1 \leq j, l \leq m$.

Thus $\mathrm{sDB}(P) \cong S_{n,m} \cup \overline{K}_{n+m}$ and $\zeta(S_{n,m}) \leq n + m$. Since $E(S_{n,m})$ is a minimal edge clique cover of $S_{n,m}$ and $|E(S_{n,m})| = nm$, $\lceil 2\sqrt{nm} \rceil \leq \zeta(S_{n,m})$ by Theorem 1.2. Therefore $\lceil 2\sqrt{nm} \rceil \leq \zeta(S_{n,m}) \leq n + m$. □

We obtain the following result by Proposition 2.1.

Corollary 2.2 *For a graph $S_{n,n}$ $(n \geq 3)$, $\zeta(S_{n,n}) = 2n$.*

Proof. By Proposition 2.1, $2n = \left\lceil 2\sqrt{n^2} \right\rceil \leq \zeta(S_{n,n}) \leq n + n = 2n$. Thus $\zeta(S_{n,n}) = 2n$. □

3 On ladders

The *ladder* L_n is a graph as follows:

(1) $V(L_n) = \{v_1, v_2, ..., v_{2n+3}, v_{2n+4}\}$,

(2) $E(L_n) = \{\{v_i, v_{i+2}\};\ 1 \leq i \leq 2n + 2\} \cup \{\{v_{2i+1}, v_{2i+2}\}; 1 \leq i \leq n\}$.

We have the following result on strict-double-bound numbers of ladders.

Proposition 3.1 *For a ladder L_n $(n \geq 4)$, $\lceil 2\sqrt{3n + 2} \rceil \leq \zeta(L_n) \leq 2n$.*

Proof. We construct a poset P for L_n as follows:

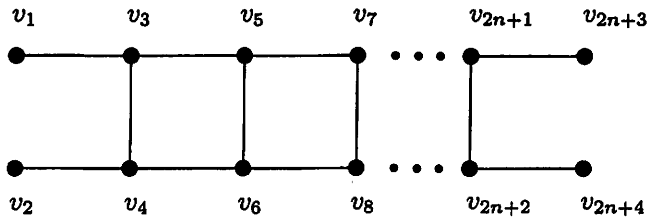

Figure 2: A ladder L_n

(1) $V(P) = V(L_n) \cup \{x_1, x_2, ..., x_n, y_1, y_2, ..., y_n\}$.

(2) $V(L_n), \{x_1, x_2, ..., x_n\}$ and $\{y_1, y_2, ..., y_n\}$ are antichains of P,

(3) In the case n is odd:

(3-1) On relations of x_i and v_i:

$$\begin{cases} (i) & v_{4i}, v_{4i+1}, v_{4i+2}, v_{4i+4} \leq_P x_{2i} & (1 \leq i \leq \frac{n-1}{2}), \\ (ii) & v_{4i-3}, v_{4i-1}, v_{4i}, v_{4i+1} \leq_P x_{2i-1} & (1 \leq i \leq \frac{n+1}{2}), \\ (iii) & v_{2n+2}, v_{2n+4} \leq_P x_2, & \\ (iv) & v_2, v_4 \leq_P x_{n-1}. & \end{cases}$$

(3-2) On relations on y_i and v_i:

$$\begin{cases} (i) & y_{2i} \leq_P v_{4i-1}, v_{4i+1}, v_{4i+2}, v_{4i+3} & (1 \leq i \leq \frac{n-1}{2}), \\ (ii) & y_{2i-1} \leq_P v_{4i-2}, v_{4i-1}, v_{4i}, v_{4i+2} & (1 \leq i \leq \frac{n+1}{2}), \\ (iii) & y_2 \leq_P v_{2n+1}, v_{2n+3}, & \\ (iv) & y_{n-1} \leq_P v_1, v_3. & \end{cases}$$

(4) In the case n is even:

(4-1) On relations of x_i and v_i:

$$\begin{cases} (i) & v_{4i}, v_{4i+1}, v_{4i+2}, v_{4i+4} \leq_P x_{2i} & (1 \leq i \leq \frac{n}{2}), \\ (ii) & v_{4i-3}, v_{4i-1}, v_{4i}, v_{4i+1} \leq_P x_{2i-1} & (1 \leq i \leq \frac{n}{2}), \\ (iii) & v_{2n+1}, v_{2n+3} \leq_P x_1, & \\ (iv) & v_2, v_4 \leq_P x_n. & \end{cases}$$

(4-2) On relations of y_i and v_i:

$$\begin{cases} (i) & y_{2i} \leq_P v_{4i-1}, v_{4i+1}, v_{4i+2}, v_{4i+3} \quad (1 \leq i \leq \frac{n}{2}), \\ (ii) & y_{2i-1} \leq_P v_{4i-2}, v_{4i-1}, v_{4i}, v_{4i+2} \quad (1 \leq i \leq \frac{n}{2}), \\ (iii) & y_2 \leq_P v_{2n+2}, v_{2n+4}, \\ (iv) & y_{n-1} \leq_P v_1, v_3. \end{cases}$$

By the construction of P, we have the following:

(1) $U_P(v_{2l+1}) \cap U_P(v_{2l+2}) = \{x_l\} \neq \emptyset$ and $L_P(v_{2l+1}) \cap L_P(v_{2l+2}) = \{y_l\} \neq \emptyset$ for $l = 1, 2, ..., n$.

(2) In the case n is odd: for $1 \leq i \leq \frac{n-1}{2}$,

(i)	$U_P(v_1) \cap U_P(v_3) = \{x_1\}$,	$L_P(v_1) \cap L_P(v_3) = \{y_{n-1}\}$,
(ii)	$U_P(v_{4i-1}) \cap U_P(v_{4i+1}) = \{x_{2i-1}\}$,	$L_P(v_{4i-1}) \cap L_P(v_{4i+1}) = \{y_{2i}\}$,
(iii)	$U_P(v_{4i+1}) \cap U_P(v_{4i+3}) = \{x_{2i+1}\}$,	$L_P(v_{4i+1}) \cap L_P(v_{4i+3}) = \{y_{2i}\}$,
(iv)	$U_P(v_{2n+1}) \cap U_P(v_{2n+3}) = \{x_n\}$,	$L_P(v_{2n+1}) \cap L_P(v_{2n+3}) = \{y_2\}$,
(v)	$U_P(v_2) \cap U_P(v_4) = \{x_{n-1}\}$,	$L_P(v_2) \cap L_P(v_4) = \{y_1\}$,
(vi)	$U_P(v_{4i}) \cap U_P(v_{4i+2}) = \{x_{2i}\}$,	$L_P(v_{4i}) \cap L_P(v_{4i+2}) = \{y_{2i-1}\}$,
(vii)	$U_P(v_{4i+2}) \cap U_P(v_{4i+4}) = \{x_{2i}\}$,	$L_P(v_{4i+2}) \cap L_P(v_{4i+4}) = \{y_{2i+1}\}$,
$(viii)$	$U_P(v_{2n+2}) \cap U_P(v_{2n+4}) = \{x_2\}$,	$L_P(v_{2n+2}) \cap L_P(v_{2n+4}) = \{y_n\}$.

(3) In the case n is even: for $1 \leq i \leq \frac{n}{2} - 1$,

(i)	$U_P(v_1) \cap U_P(v_3) = \{x_1\}$,	$L_P(v_1) \cap L_P(v_3) = \{y_{n-1}\}$,
(ii)	$U_P(v_{4i-1}) \cap U_P(v_{4i+1}) = \{x_{2i-1}\}$,	$L_P(v_{4i-1}) \cap L_P(v_{4i+1}) = \{y_{2i}\}$,
(iii)	$U_P(v_{4i+1}) \cap U_P(v_{4i+3}) = \{x_{2i+1}\}$,	$L_P(v_{4i+1}) \cap L_P(v_{4i+3}) = \{y_{2i}\}$,
(iv)	$U_P(v_{2n-1}) \cap U_P(v_{2n+1}) = \{x_{n-1}\}$,	$L_P(v_{2n-1}) \cap L_P(v_{2n+1}) = \{y_n,\}$
(v)	$U_P(v_{2n+1}) \cap U_P(v_{2n+3}) = \{x_1\}$,	$L_P(v_{2n+1}) \cap L_P(v_{2n+3}) = \{y_n\}$,
(vi)	$U_P(v_2) \cap U_P(v_4) = \{x_n\}$,	$L_P(v_2) \cap L_P(v_4) = \{y_1\}$,
(vii)	$U_P(v_{4i}) \cap U_P(v_{4i+2}) = \{x_{2i}\}$,	$L_P(v_{4i}) \cap L_P(v_{4i+2}) = \{y_{2i-1}\}$,
$(viii)$	$U_P(v_{4i+2}) \cap U_P(v_{4i+4}) = \{x_{2i}\}$,	$L_P(v_{4i+2}) \cap L_P(v_{4i+4}) = \{y_{2i+1}\}$,
(ix)	$U_P(v_{2n}) \cap U_P(v_{2n+2}) = \{x_n\}$,	$L_P(v_{2n}) \cap L_P(v_{2n+2}) = \{y_{n-1}\}$,
(x)	$U_P(v_{2n+2}) \cap U_P(v_{2n+4}) = \{x_n\}$,	$L_P(v_{2n+2}) \cap L_P(v_{2n+4}) = \{y_2\}$.

So there exist edges $\{v_l, v_{l+2}\}$ $(l = 1, 2, ..., 2n + 2)$ in sDB(P) and edges $\{v_{2l-1}, v_{2l}\}$ $(l = 2, 3, ..., n + 1)$ in sDB(P).

Next we consider non-adjacent vertices v_i and v_j of $V(L_n)$. We easily check by definitions that for $U_P(v_i) \cap U_P(v_j) \neq \emptyset$, $L_P(v_i) \cap L_P(v_j) = \emptyset$.

Thus sDB(P) has the edge set $E(\text{sDB}(P)) = \{\{v_i, v_{i+2}\}; 1 \leq i \leq 2n + 2\} \cup \{\{v_{2i+1}, v_{2i+2}\}; 1 \leq i \leq n\}$. So sDB$(P) = L_n \cup \overline{K}_{2n}$ and $\zeta(L_n) \leq 2n$.

$E(L_n)$ is a minimal edge clique cover of L_n and $|E(L_n)| = 3n + 2$. Therefore $\lceil 2\sqrt{3n+2} \rceil \leq \zeta(L_n)$ by Theorem 1.2. □

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