Quasi-tree graphs with the second largest number of maximal independent sets

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Abstract

A maximal independent set is an independent set that is not a proper subset of any other independent set. A connected graph (respectively, graph) G with vertex set $V(G)$ is called a quasi-tree graph (respectively, quasi-forest graph), if there exists a vertex $x \in V(G)$ such that $G - x$ is a tree (respectively, forest). In this paper, we determine the second largest numbers of maximal independent sets among all quasi-tree graphs and quasi-forest graphs. We also characterize those extremal graphs achieving these values.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph. An *independent set* is a subset S of V such that no two vertices in S are adjacent. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. The set of all maximal independent sets of a graph G is denoted by $\mathrm{MI}(G)$ and its cardinality by $mi(G)$.

The problem of determining the largest value of $mi(G)$ in a general graph of order n and those graphs achieving the largest number was proposed by Erdös and Moser, and solved by Moon and Moser [8]. It was then studied for various families of graphs, including trees, forests, (connected) graphs with at most one cycle, (connected) triangle-free graphs, (k-)connected graphs, bipartite graphs; for a survey see [4]. Jin and Li [1] investigated the second largest number of $mi(G)$ among all graphs of order n; Jou and Lin [5] further explored the same problem for trees and forests.

A connected graph (respectively, graph) G with vertex set $V(G)$ is called a *quasi-tree graph* (respectively, *quasi-forest graph*), if there exists a vertex $x \in V(G)$ such that $G - x$ is a tree (respectively, forest). The concept of quasi-tree graphs was mentioned by Liu and Lu in [7]. Recently, the problem of determining the largest numbers of $mi(G)$ among all quasi-tree graphs and quasi-forest graphs of order n was solved by Lin [6].

The purpose of this paper is to determine the second largest numbers of maximal independent sets among all quasi-tree graphs and quasi-forest graphs of order n. Additionally, extremal graphs achieving these values are also given.

2 Preliminary

In this section, we present some notations and preliminary results, which will be helpful to the proof of our main result in next section. For a vertex $x \in V(G)$, let $\mathrm{MI}_{-x}(G) = \{I \in \mathrm{MI}(G) : x \notin I\}$ and $\mathrm{MI}_{+x}(G) = \{I \in \mathrm{MI}(G) : x \in I\}$. The *neighborhood* $N_G(x)$ of a vertex x is the set of vertices adjacent to x in G and the *closed neighborhood* $N_G[x]$ is $\{x\} \cup N_G(x)$. The *degree* of x is the cardinality of $N_G(x)$, denoted by $\deg_G(x)$. Let $\Delta(G) = \max\{\deg_G(x) : x \in V(G)\}$. A vertex x is called a *leaf* if $\deg_G x = 1$. For a set $A \subseteq V(G)$, the *deletion* of A from G is the graph $G-A$ obtained from G by removing all vertices in A and their incident edges. Two graphs G_1 and G_2 are *disjoint* if $V(G_1) \cap V(G_2) = \emptyset$. The *union* of two disjoint graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. nG is the short notation for the union of n copies of disjoint graphs isomorphic to G. Denote by C_n a *cycle* with n vertices and P_n a *path* with n vertices.

Throughout this paper, for simplicity, let $r = \sqrt{2}$.

Lemma 2.1. ([9]) *For any vertex x in a graph G, $mi(G) \leq mi(G-x) + mi(G - N_G[x])$.*

Lemma 2.2. ([6]) *Let x be the vertex in a graph G such that $mi(G) = mi(G-x) + mi(G - N_G[x])$, the following hold.*
(1) $mi(G-x) = |\mathrm{MI}_{-x}(G)|$.
(2) For a maximal independent set $I \in \mathrm{MI}(G-x)$, $I \cap N_G(x) \neq \emptyset$.

Lemma 2.3. ([2]) *If G is the union of two disjoint graphs G_1 and G_2, then $mi(G) = mi(G_1)mi(G_2)$.*

The results of the largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.4, 2.5, respectively.

Theorem 2.4. ([2], [3]) *If T is a tree with $n \geq 1$ vertices, then $mi(T) \leq t(n)$, where*

$$t(n) = \begin{cases} r^{n-2}+1, & \text{if } n \text{ is even;} \\ r^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(T) = t(n)$ if and only if $T = T(n)$, where

$$T(n) = \begin{cases} B(2, \frac{n-2}{2}) \text{ or } B(4, \frac{n-4}{2}), & \text{if } n \text{ is even;} \\ B(1, \frac{n-1}{2}), & \text{if } n \text{ is odd.} \end{cases}$$

where $B(i,j)$ *is the set of batons, which are the graphs obtained from the basic path* P *of* $i \geq 1$ *vertices by attaching* $j \geq 0$ *paths of length two to the endpoints of* P *in all possible ways (see Figure* 1*).*

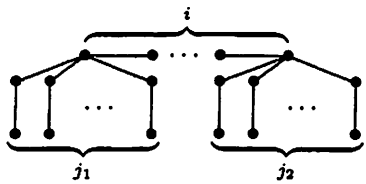


Figure 1: The baton $B(i,j)$ with $j = j_1 + j_2$

Theorem 2.5. ([2], [3]) *If* F *is a forest with* $n \geq 1$ *vertices, then* $mi(F) \leq f(n)$, *where*

$$f(n) = \begin{cases} r^n, & \text{if } n \text{ is even;} \\ r^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(F) = f(n)$ *if and only if* $F = F(n)$, *where*

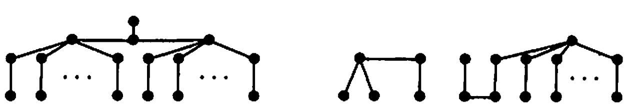
$$F(n) = \begin{cases} \frac{n}{2}P_2, & \text{if } n \text{ is even;} \\ B(1, \frac{n-1-2s}{2}) \cup sP_2 \text{ for some } s \text{ with } 0 \leq s \leq \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

The results of the second largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.6 and 2.7, respectively.

Theorem 2.6. ([5]) *If* T *is a tree with* $n \geq 4$ *vertices having* $T \neq T(n)$, *then* $mi(T) \leq t'(n)$, *where*

$$t'(n) = \begin{cases} r^{n-2}, & \text{if } n \text{ is even;} \\ 3, & \text{if } n = 5; \\ 3r^{n-5} + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(T) = t'(n)$ *if and only if* $T = T_1^*(8), T_2^*(8), P_{10}$ *or* $T = T'(n)$, *where* $T'(n)$ *and* $T_1^*(8)$, $T_2^*(8)$ *are shown in Figures* 2 *and* 3, *respectively.*



$T'_e(n)$, $n \geq 4$ is even $\quad$ $T'(5)$ $\quad$ $T'_o(n)$, $n \geq 7$ is odd

Figure 2: The graph $T'(n)$

Figure 3: The graphs $T_1^*(8)$ and $T_2^*(8)$

Theorem 2.7. ([5]) *If F is a forest with $n \geq 4$ vertices having $F \neq F(n)$, then $mi(F) \leq f'(n)$, where*

$$f'(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even;} \\ 3, & \text{if } n = 5; \\ 7r^{n-7}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(F) = f'(n)$ if and only if $F = F'(n)$, where

$$F'(n) = \begin{cases} P_4 \cup \frac{n-4}{2} P_2, & \text{if } n \geq 4 \text{ is even;} \\ T'(5) \text{ or } P_1 \cup P_4, & \text{if } n = 5; \\ P_7 \cup \frac{n-7}{2} P_2, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

The results of the largest numbers of maximal independent sets among all quasi-tree graphs and quasi-forest graphs are described in Theorems 2.8 and 2.9, respectively.

Theorem 2.8. ([6])*If Q is a quasi-tree graph with $n \geq 5$ vertices, then $mi(Q) \leq q(n)$, where*

$$q(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even;} \\ r^{n-1} + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(Q) = q(n)$ if and only if $Q = Q(n)$ or $Q = C_5$, where $Q(n)$ is shown in Figure 4.

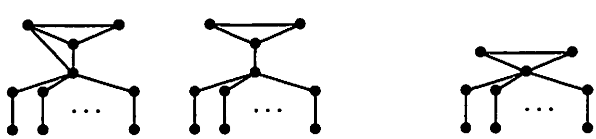


Figure 4: The graph $Q(n)$

Theorem 2.9. ([6])*If Q is a quasi-forest graph with $n \geq 2$ vertices, then $mi(Q) \leq \overline{q}(n)$, where*

$$\overline{q}(n) = \begin{cases} r^n, & \text{if } n \text{ is even;} \\ 3r^{n-3}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $mi(Q) = \overline{q}(n)$ *if and only if* $Q = \overline{Q}(n)$*, where*

$$\overline{Q}(n) = \begin{cases} \frac{n}{2}P_2, & \text{if } n \text{ is even;} \\ C_3 \cup \frac{n-3}{2}P_2, & \text{if } n \text{ is odd.} \end{cases}$$

3 Main results

In this section, we determine the second largest values of $mi(G)$ among all quasi-tree graphs and quasi-forest graphs of order $n \geq 4$, respectively. Moreover, the extremal graphs achieving these values are also determined.

For even $n \geq 6$, $Q'_e(n)$ is the graph obtained from $B(1, \frac{n-4}{2})$ by adding a C_3 and a new edge joining a vertex of C_3 and a leaf of $B(1, \frac{n-4}{2})$; $Q_e^{*\prime}(n)$ is the graph obtained from $Q'_e(n)$ by adding a new edge joining a vertex with degree 2 of induced C_3 of $Q'_e(n)$ and the only vertex in the basic path of $B(1, \frac{n-4}{2})$, see Figure 5.

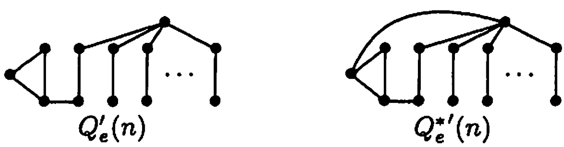


Figure 5: The graphs $Q'_e(n)$ and $Q_e^{*\prime}(n)$ for even $n \geq 6$

Theorem 3.1. *If Q is a quasi-tree graph of even order $n \geq 8$ having $Q \neq Q(n)$, then $mi(Q) \leq 5r^{n-6} + 1$. Furthermore, the equality holds if and only if $Q = Q'_e(n)$ or $Q = Q_e^{*\prime}(n)$.*

Proof. It is straightforward to check that $mi(Q'_e(n)) = mi(Q_e^{*\prime}(n)) = 5r^{n-6} + 1$. Let Q be a quasi-tree graph of even order $n \geq 8$ having $Q \neq Q(n)$ such that $mi(Q)$ is as large as possible. If Q is a tree, then $5r^{n-6} + 1 \leq mi(Q) \leq t(n) = r^{n-2} + 1 < 5r^{n-6} + 1$. This is a contradiction, so Q contains at least one cycle. Let x be a vertex such that $Q - x$ is a tree. Then x is on some cycle of Q, it follows that $\deg_Q(x) \geq 2$. In addition, by Theorem 2.4, $mi(Q - x) \leq t(n-1)$.

First, suppose that $Q - x = T(n-1) = B(1, \frac{n-2}{2})$. By Lemma 2.1, we have $mi(Q - N_Q[x]) \geq mi(Q) - mi(Q-x) \geq (5r^{n-6}+1) - r^{n-2} = r^{n-6}+1$. If $\deg_Q(x) \geq 4$ then $Q - N_Q[x]$ is a forest with at most $n-5$ vertices, by Theorem 2.5, $r^{n-6} + 1 \leq mi(Q - N_Q[x]) \leq f(n-5) = r^{n-6}$. This is a contradiction. So we assume that $2 \leq \deg_Q(x) \leq 3$. We consider the following cases:

- The vertices in $N_Q(x)$ are on only one P_2 of $B(1, \frac{n-2}{2})$. Since $Q \neq Q(n)$, there are two possibilities for graph Q. See Figure 6. By simple

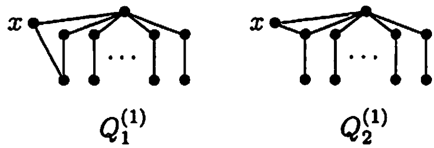


Figure 6: The graphs $Q_i^{(1)}$ $(i = 1, 2)$

calculation, we have $mi(Q_i^{(1)}) \leq 4r^{n-6}+1$ for $i = 1, 2$, a contradiction to $mi(Q) \geq 5r^{n-6} + 1$.

- The vertices in $N_Q(x)$ are on exactly two P_2's of $B(1, \frac{n-2}{2})$. Suppose that $N_Q(x)$ contains the only vertex in the basic path of $B(1, \frac{n-2}{2})$, then $mi(Q - N_Q[x]) = r^{n-6}$, a contradiction to $mi(Q - N_Q[x]) \geq r^{n-6} + 1$. Hence there are five possibilities for graph Q. See Figure 7. Note that $Q_5^{(2)} = Q_e^{*\prime}(n)$. On the other hand, by simple calculation, we have $mi(Q_i^{(2)}) \leq 5r^{n-6}$ for $i = 1, 2, 3, 4$, a contradiction to $mi(Q) \geq 5r^{n-6} + 1$.

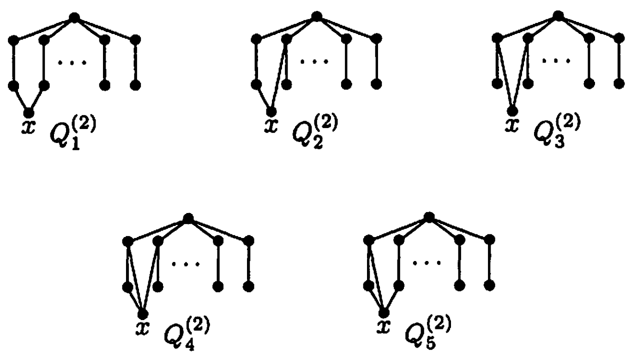
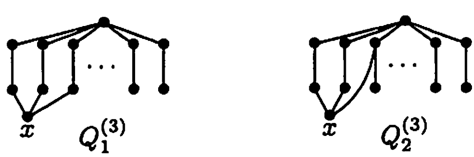


Figure 7: The graphs $Q_i^{(2)}$ $(i = 1, 2, 3, 4, 5)$

- The vertices in $N_Q(x)$ are on three P_2's of $B(1, \frac{n-2}{2})$. There are four possibilities for graph Q. See Figure 8. By simple calculation, $mi(Q - N_Q[x]) \leq r^{n-8} + 1$, a contradiction to $mi(Q - N_Q[x]) \geq r^{n-6} + 1$.



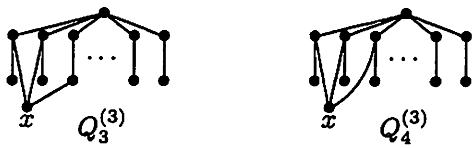


Figure 8: The graphs $Q_i^{(3)}$ $(i = 1, 2, 3, 4)$

Now we assume that $Q - x \neq T(n-1)$. By Theorem 2.6, we assume that $mi(Q - x) \leq t'(n-1)$. By Lemma 2.1 and $\deg_Q(x) \geq 2$, we have

$$\begin{aligned} 5r^{n-6} + 1 \leq mi(Q) &\leq mi(Q - x) + mi(Q - N_Q[x]) \\ &\leq t'(n-1) + f(n-3) \\ &= (3r^{n-6} + 1) + r^{n-4} \\ &= 5r^{n-6} + 1. \end{aligned}$$

Furthermore, the equalities holding imply that $|\mathrm{MI}_{-x}(Q)| = mi(Q - x) = t'(n-1)$ and $|\mathrm{MI}_{+x}(Q)| = mi(Q - N_Q[x]) = f(n-3)$.

Since $|\mathrm{MI}_{-x}(Q)| = mi(Q - x) = t'(n-1)$, by Theorem 2.6, we have that $Q - x = T_o'(n-1)$. On the other hand, $|\mathrm{MI}_{+x}(Q)| = mi(Q - N_Q[x]) = f(n-3)$, by Theorem 2.5, we have that $Q - N_Q[x] = F(n-4)$ or $Q - N_Q[x] = F(n-3)$. We consider two following cases.

Case 1. $\deg_Q(x) = 3$. By Theorem 2.5, we have that $Q - N_Q[x] = F(n-4) = \frac{n-4}{2}P_2$. Hence we obtain that $Q = Q_e^{*\prime}(n)$.

Case 2. $\deg_Q(x) = 2$. Since $Q - x = T_o'(n-1)$ and by Theorem 2.5, we have that $Q - N_Q[x] = F(n-3) = B(1, \frac{n-4-2s}{2}) \cup sP_2$ for some s with $0 \leq s \leq \frac{n-4}{2}$. Hence there are seven possibilities for graph Q meeting the requirements. See Figure 9.

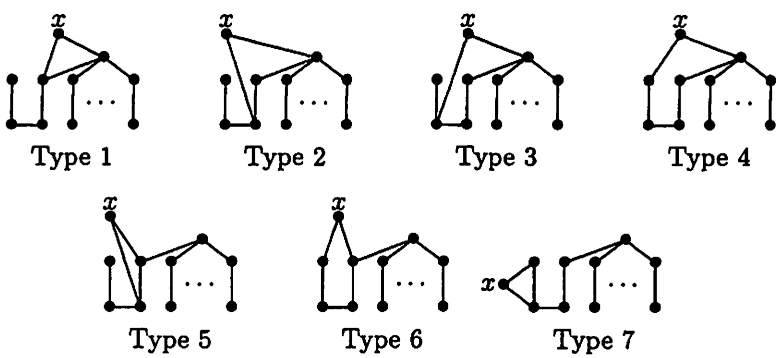


Figure 9: The seven possibilities for graph Q

Moreover, among these only that of Type 7 satisfies Lemma 2.2 (2), hence we obtain that $Q = Q_e'(n)$. □

Theorem 3.2. *If Q is a quasi-forest graph of even order $n \geq 4$ having $Q \neq \overline{Q}(n)$, then $mi(Q) \leq 3r^{n-4}$. Furthermore, the equality holds if and only if*

$$Q = \overline{Q}'_e(n) = \begin{cases} P_4 \cup \frac{n-4}{2}P_2 \\ Q_e(n-2s) \cup sP_2 \\ Q_e^*(n-2s) \cup sP_2 \\ Q_e^{*\prime}(6) \cup \frac{n-6}{2}P_2 \\ C_3 \cup B(1, \frac{n-4-2s}{2}) \cup sP_2 \end{cases}$$

for some s with $0 \leq s \leq \frac{n-4}{2}$.

Proof. It is straightforward to check that $mi(\overline{Q}'_e(n)) = 3r^{n-4}$. Let Q be a quasi-forest graph of even order $n \geq 4$ having $Q \neq \overline{Q}(n)$ such that $mi(Q)$ is as large as possible. Then $mi(Q) \geq mi(\overline{Q}'_e(n)) = 3r^{n-4}$. If Q is a forest and $Q \neq \overline{Q}(n)$, by Theorem 2.7, then $3r^{n-4} \leq mi(Q) \leq f'(n) = 3r^{n-4}$. Thus $Q = P_4 \cup \frac{n-4}{2}P_2$. Now we assume that Q is a quasi-forest graph with at least one cycle. Let x be a vertex such that $Q - x$ is a forest and $\deg_Q(x)$ is as large as possible . Then x is on some cycle of Q, it follows that $\deg_Q(x) \geq 2$. By Theorem 2.5, $mi(Q - x) \leq f(n-1)$. On the other hand, $Q - N_Q[x]$ is a forest with at most $n-3$ vertices, by Theorem 2.5 again, $mi(Q - N_Q[x]) \leq f(n-3)$. Thus, by Lemma 2.1, we have

$$\begin{aligned} 3r^{n-4} \leq mi(Q) &\leq mi(Q-x) + mi(Q - N_Q[x]) \\ &\leq f(n-1) + f(n-3) \\ &= r^{n-2} + r^{n-4} \\ &= 3r^{n-4}. \end{aligned}$$

Furthermore, the equalities holding imply that $|\mathrm{MI}_{-x}(Q)| = mi(Q-x) = f(n-1)$ and $|\mathrm{MI}_{+x}(Q)| = mi(Q - N_Q[x]) = f(n-3)$. By Theorem 2.5, we have that $Q - x = F(n-1)$. Note that $F(n-1)$ is the union of a baton and some P_2's. In addition, $Q - N_Q[x] = F(n-4)$ or $Q - N_Q[x] = F(n-3)$. Let s be an integer with $0 \leq s \leq \frac{n-4}{2}$. We consider two following cases.

Case 1. $\deg_Q(x) = 3$. Then $Q - N_Q[x] = F(n-4) = \frac{n-4}{2}P_2$. Hence we obtain that $Q = Q_e(n-2s) \cup sP_2$, or $Q_e^*(n-2s) \cup sP_2$, or $Q_e^{*\prime}(6) \cup \frac{n-6}{2}P_2$.

Case 2. $\deg_Q(x) = 2$. Then $Q - N_Q[x] = F(n-3) = B(1, \frac{n-4-2s}{2}) \cup sP_2$. On the other hand, $\deg_Q(x)$ is as large as possible, hence we obtain that $Q = C_3 \cup B(1, \frac{n-4-2s}{2}) \cup sP_2$. □

Theorem 3.3. *If Q is a quasi-forest graph of odd order $n \geq 5$ having $Q \neq \overline{Q}(n)$, then $mi(Q) \leq 5r^{n-5}$. Furthermore, the equality holds if and* *only if*

$$Q = \overline{Q}'_o(n) = \begin{cases} Q_o(5) \cup \frac{n-5}{2}P_2 \\ W \cup \frac{n-5}{2}P_2 \\ C_5 \cup \frac{n-5}{2}P_2 \end{cases}$$

where W is a bow, that is, two triangles C_3 having one common vertex.

Proof. It is straightforward to check that $mi(\overline{Q}'_o(n)) = 5r^{n-5}$. Let Q be a quasi-forest graph of odd order $n \geq 5$ having $Q \neq \overline{Q}(n)$ such that $mi(Q)$ is as large as possible. Then $mi(Q) \geq mi(\overline{Q}'_o(n)) = 5r^{n-5}$. If Q is a forest, then $5r^{n-5} \leq mi(Q) \leq f(n) = r^{n-1} < 5r^{n-5}$. This is a contradiction, so Q contains at least one cycle. Let x be a vertex such that $Q - x$ is a forest and $\deg_Q(x)$ is as large as possible. Then x is on some cycle of Q, it follows that $\deg_Q(x) \geq 2$. Thus $Q - N_Q[x]$ is a forest with at most $n-3$ vertices, by Theorem 2.5, $mi(Q - N_Q[x]) \leq f(n-3)$. By Lemma 2.1, we obtain that $mi(Q-x) \geq mi(Q) - mi(Q - N_Q[x]) \geq 5r^{n-5} - r^{n-3} = 3r^{n-5} = f'(n-1)$. By Theorem 2.7, we have $mi(Q-x) = f(n-1)$ or $mi(Q-x) = f'(n-1)$. Hence we consider two following cases.

Case 1. $mi(Q-x) = f(n-1)$. Then $Q - x = F(n-1) = \frac{n-1}{2}P_2$. Suppose that $\deg_Q x = 2$, then $Q = \overline{Q}(n)$. This is a contradiction. Hence $\deg_Q x \geq 3$, that is, $Q - N_Q[x]$ is a forest with at most $n-4$ vertices. By Lemma 2.1 and Theorem 2.5, we have $r^{n-5} = f(n-4) \geq mi(Q - N_Q[x]) \geq mi(Q) - mi(Q-x) \geq 5r^{n-5} - r^{n-1} = r^{n-5} = f(n-4) = f(n-5)$, it follows that $Q - N_Q[x] = F(n-4)$ or $F(n-5)$. For the case of $Q - N_Q[x] = F(n-4) = P_1 \cup \frac{n-5}{2}P_2$, then $Q = Q_o(5) \cup \frac{n-5}{2}P_2$. For the case of $Q - N_Q[x] = F(n-5) = \frac{n-5}{2}P_2$, then $Q = W \cup \frac{n-5}{2}P_2$, where W is a bow.

Case 2. $mi(Q-x) = f'(n-1)$. Then $Q - x = F'(n-1) = P_4 \cup \frac{n-5}{2}P_2$. Since $Q - N_Q[x]$ is a forest with at most $n-3$ vertices, by Lemma 2.1 and Theorem 2.5, we have $r^{n-3} = f(n-3) \geq mi(Q - N_Q[x]) \geq mi(Q) - mi(Q-x) \geq 5r^{n-5} - 3r^{n-5} = r^{n-3}$. It follows that $\deg_Q x = 2$. Then $Q - x = P_4 \cup \frac{n-5}{2}P_2$ and $Q - N_Q[x] = \frac{n-3}{2}P_2$. In addition, $\deg_Q(x)$ is as large as possible, hence we obtain that $Q = C_5 \cup \frac{n-5}{2}P_2$. □

Theorem 3.4. *If Q is a quasi-tree graph of odd order $n \geq 7$ having $Q \neq Q(n)$, then $mi(Q) \leq r^{n-1}$. Furthermore, the equality holds if and only if $Q = Q_1^*(7), Q_2^*(7), Q_3^*(7), Q_4^*(7)$ or $B(1, \frac{n-1}{2})$, where $Q_1^*(7), Q_2^*(7), Q_3^*(7)$ and $Q_4^*(7)$ are shown in Figure 10.*

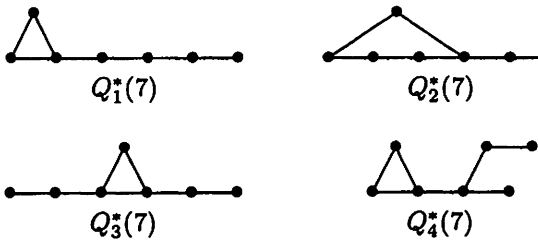


Figure 10: The graphs $Q_1^*(7), Q_2^*(7), Q_3^*(7)$ and $Q_4^*(7)$

Proof. Let Q be a quasi-tree graph of odd order $n \geq 7$ having $Q \neq Q(n)$ such that $mi(Q)$ is as large as possible. Since $Q \neq Q(n)$ and $mi(B(1, \frac{n-1}{2}))$ $= r^{n-1}$, then $r^{n-1} \leq mi(Q) \leq mi(Q(n)) - 1 = (r^{n-1} + 1) - 1 = r^{n-1}$. This implies that $mi(Q) = r^{r-1}$. If Q is a tree, by Theorem 2.4, $r^{n-1} = mi(Q) \leq t(n) = r^{n-1}$. This follows that $Q = B(1, \frac{n-1}{2})$.

Now we assume that Q contains at least one cycle. We claim that $\Delta(Q) = 3$. Let v be a vertex of Q such that $\deg_Q(v) = \Delta(Q)$. If $Q - v = \frac{n-1}{2}P_2$, then $Q = Q_o(n)$. This is a contradiction, so $Q - v \neq \frac{n-1}{2}P_2$. Note that $Q - v$ is a quasi-forest graph of even order $n-1$. By Theorem 3.2, we have $mi(Q - v) \leq 3r^{n-5}$. Hence $mi(Q - N_Q[v]) \geq mi(Q) - mi(Q - v) \geq r^{n-1} - 3r^{n-5} = r^{n-5}$. If $\deg_Q(v) \geq 5$, by Theorem 2.9, $r^{n-5} \leq mi(Q - N_Q[v]) \leq \overline{q}(n-6) = 3r^{n-9}$, this is a contradiction. If $\deg_Q(v) = 4$, by Theorem 2.9, then $r^{n-5} \leq mi(Q - N_Q[v]) \leq \overline{q}(n-5) = r^{n-5}$, hence we obtain that $Q - N_Q[v] = \frac{n-5}{2}P_2$. It is not difficult to see that there does not exist a quasi-tree graph Q such that $Q - v = \overline{Q}'_e(n-1)$ and $Q - N_Q[v] = \frac{n-5}{2}P_2$. On the other hand, it is obvious that $mi(C_n) < r^{n-1}$, hence we obtain that $\deg_Q(v) = 3$. Since Q is a quasi-tree graph and $\Delta(Q) = 3$, there exists a vertex $x \in V(Q)$ such that x is on some cycle in Q and $\deg_Q(x) = 3$. It follows that $Q - x$ is a forest of even order $n-1$ and $Q - x$ contains at most two components. Since $Q - N_Q[x]$ is a forest of odd order $n-4$, by Lemma 2.1, Theorems 2.5 and 2.7, we have

$$
\begin{aligned}
r^{n-1} = mi(Q) &\leq mi(Q - x) + mi(Q - N_Q[x]) \\
&\leq f'(n-1) + f(n-4) \\
&\leq 3r^{n-5} + r^{n-5} \\
&= r^{n-1}.
\end{aligned}
$$

The equalities holding imply that $Q - x = P_4 \cup \frac{n-5}{2}P_2$. Since $n \geq 7$, $Q - x$ contains exactly two components, these imply $n = 7$ and $mi(Q - N_Q[x]) = r^{n-5} = 2$. Hence we obtain that $Q = Q_1^*(7), Q_2^*(7), Q_3^*(7)$ or $Q_4^*(7)$. □

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