MORE IDENTITIES AND TRIDIAGONAL MATRICES ABOUT FIBONACCI AND LUCAS NUMBERS

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ABSTRACT. In this paper, we derive a family of identities on the arbitrary subscripted Fibonacci and Lucas numbers. Furthermore, we construct the tridiagonal and symmetric tridiagonal family of matrices whose determinants form any linear subsequence of the Fibonacci numbers and Lucas numbers. Thus, we give a generalization of the presented in Nalli and Civciv [A. Nalli, H. Civciv, A generalization of tridiagonal matrix determinants, Fibonacci and Lucas numbers, Chaos, Solitons and Fractals 2009;40 (1): 355 61] and Cahill and Narayan [N. D. Cahill, D. A. Narayan, Fibonacci and Lucas numbers as tridiagonal matrix determinants, The Fibonacci Quarterly, 2004;42 (1): 216-221].

1. Introduction

The Fibonacci sequence $\{F_n\}$, is defined by the recurrence relation, for $n \geq 1$

$$F_{n+1} = F_n + F_{n-1} \tag{1.1}$$

with $F_0 = 0, F_1 = 1$. The first few Fibonacci numbers are:
$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, \cdots$

The Lucas sequence $\{L_n\}$, is defined by the recurrence relation, for $n \geq 1$

$$L_{n+1} = L_n + L_{n-1} \tag{1.2}$$

with $L_0 = 2, L_1 = 1$. The first few Lucas numbers are:
$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, 5778, \cdots$

There are many known connections between determinants of tridiagonal matrices and the Fibonacci and Lucas numbers. For example, Strang [5][6] presents a family of tridiagonal matrices given by:

$$M(k) = \begin{bmatrix} 3 & 1 & & & \\ 1 & 3 & 1 & & \\ & 1 & 3 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 3 \end{bmatrix} \tag{1.3}$$

where $M(k)$ is $k \times k$. It is easy to show by induction that the determinants $|M(k)|$ are the Fibonacci numbers F_{2k+2}. Using the method of Laplace expansion to evaluate the determinant of tridiagonal matrix:

$$A(k) = \begin{bmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & a_{32} & a_{33} & \ddots & \\ & & \ddots & \ddots & a_{k-1,k} \\ & & & a_{k,k-1} & a_{kk} \end{bmatrix}, \tag{1.4}$$

Cahill and Narayan [1] provides symmetric tridiagonal matrices as follows:

$$M_{\alpha,\beta}(k) = \begin{bmatrix} F_{\alpha+\beta} & m_{1,2} & & & & \\ m_{2,1} & \left\lceil \frac{F_{2\alpha+\beta}}{F_{\alpha+\beta}} \right\rceil & \sqrt{(-1)^{\alpha}} & & & \\ & \sqrt{(-1)^{\alpha}} & L_{\alpha} & \sqrt{(-1)^{\alpha}} & & \\ & & \sqrt{(-1)^{\alpha}} & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & L_{\alpha} & \sqrt{(-1)^{\alpha}} \\ & & & & & \sqrt{(-1)^{\alpha}} & L_{\alpha} \end{bmatrix} \tag{1.5}$$

where $m_{1,2} = m_{2,1} = \sqrt{\left\lceil \frac{F_{2\alpha+\beta}}{F_{\alpha+\beta}} \right\rceil F_{\alpha+\beta} - F_{2\alpha+\beta}}, \alpha \in Z^{+}$ and $\beta \in N$. Using lemma: $F_{k+n} = L_n F_k + (-1)^{n+1} F_{k-n}$ (for $n \geq 1$), the author proves that: $|M_{\alpha,\beta}(k)| = F_{\alpha k+\beta}$. Nalli and Civciv [4] generalizes Cahill's result, gives the following matrix:

$$M_{-\alpha,-\beta}(k) = \begin{bmatrix} F_{-\alpha-\beta} & m_{1,2} & & & \\ m_{2,1} & \left\lceil \frac{F_{-2\alpha-\beta}}{F_{-\alpha-\beta}} \right\rceil & \sqrt{(-1)^{-\alpha}} & & \\ & \sqrt{(-1)^{-\alpha}} & L_{-\alpha} & & \\ & & & \ddots & \\ & & & \ddots & L_{-\alpha} & \sqrt{(-1)^{-\alpha}} \\ & & & & \sqrt{(-1)^{-\alpha}} & L_{-\alpha} \end{bmatrix} \tag{1.6}$$

where $m_{1,2} = m_{2,1} = \sqrt{\left\lceil \frac{F_{-2\alpha-\beta}}{F_{-\alpha-\beta}} \right\rceil F_{-\alpha-\beta} - F_{-2\alpha-\beta}}, \alpha \in Z^{+}$ and $\beta \in N$, using relation $F_{-\alpha-\beta} = -F_{\alpha+\beta}$, the authors prove that the determinant $|M_{-\alpha,-\beta}(k)|$ equals to $F_{\alpha k+\beta}$ or $-F_{\alpha k+\beta}$(see Ref. [4]). The purpose of this paper is to get more identities, and present another method to construct tridiagonal matrices or symmetric tridiagonal matrices which have connections between determinants of tridiagonal matrices and the arbitrary subscripted Fibonacci and Lucas numbers.

2. More recurrence relations (identities)

Suppose we want an identity of the form (w, h, k and $n \geq 1$ are arbitrary positive integers)

$$F_{w(n+h)+k} = x_1 F_{wn} + x_2 F_{w(n-1)}, \tag{2.1}$$

we write an augmented matrix A_n^*:

$$A_n^* = \begin{bmatrix} F_{wn} & F_{w(n-1)} & F_{w(n+h)+k} \\ F_{w(n+1)} & F_{wn} & F_{w(n+h+1)+k} \end{bmatrix}. \tag{2.2}$$

Then take a convenient value for n, say $n = 1$, and use elementary row operations on the augmented matrix A_1^* [3],

$$A_1^* = \begin{bmatrix} F_w & F_0 & F_{w(1+h)+k} \\ F_{2w} & F_w & F_{w(2+h)+k} \end{bmatrix} \tag{2.3}$$

to obtain a classes of identities (for w, h and k various integers):

$$F_{2(n+4)} = 21F_{2(n+1)} - 8F_{2n}; \tag{2.4}$$

$$F_{2(n+5)} = 55F_{2(n+1)} - 21F_{2n}; \tag{2.5}$$

$$F_{2(n+3)+5} = 89F_{2(n+1)} - 34F_{2n}; \tag{2.6}$$

$$F_{3(n+4)} = 72F_{3(n+1)} + 17F_{3n}; \tag{2.7}$$

$$F_{3(n+4)+2} = \frac{377}{2}F_{3(n+1)} + \frac{89}{2}F_{3n}; \tag{2.8}$$

$$F_{4(n+4)} = 329F_{4(n+1)} - 48F_{4n}; \tag{2.9}$$

$$F_{4(n+4)+3} = \frac{4181}{3}F_{4(n+1)} - \frac{610}{3}F_{4n}; \tag{2.10}$$

$$F_{5(n+5)} = 15005F_{5(n+1)} + 1353F_{5n}; \tag{2.11}$$

$$F_{5(n+5)+4} = \frac{514229}{5}F_{5(n+1)} + \frac{46368}{5}F_{5n}; \tag{2.12}$$

$$F_{7(n+8)} = 17373187209F_{7(n+1)} + 598364773F_{7n}; \tag{2.13}$$

$$F_{7(n+8)+5} = \frac{2504730781961}{13}F_{7(n+1)} + \frac{86267571272}{13}F_{7n}; \tag{2.14}$$

$$\cdots$$

Note: For (2.4-2.14), using the principle of induction, we can get the general identities:

$$F_{2(n+r)} = F_{2r}F_{2(n+1)} - F_{2(r-1)}F_{2n}; \tag{2.15}$$

$$F_{3(n+h)+k} = \frac{F_{3h+k}}{2}F_{3(n+1)} + \frac{F_{3(h-1)+k}}{2}F_{3n}; \tag{2.16}$$

$$F_{4(n+h)+k} = \frac{F_{4h+k}}{3}F_{4(n+1)} - \frac{F_{4(h-1)+k}}{3}F_{4n}; \tag{2.17}$$

$$F_{5(n+h)+k} = \frac{F_{5h+k}}{5} F_{5(n+1)} + \frac{F_{5(h-1)+k}}{5} F_{5n}; \tag{2.18}$$

$$F_{7(n+h)+k} = \frac{F_{7h+k}}{13} F_{7(n+1)} + \frac{F_{7(h-1)+k}}{13} F_{7n}; \tag{2.19}$$

Suppose we want an identity of the form (w, h, k and $n \geq 1$ are arbitrary positive integers)

$$L_{w(n+h)+k} = x_1 L_{wn} + x_2 L_{w(n-1)}, \tag{2.20}$$

we write an augmented matrix B_n^*:

$$B_n^* = \begin{bmatrix} L_{wn} & L_{w(n-1)} & L_{w(n+h)+k} \\ L_{w(n+1)} & L_{wn} & L_{w(n+h+1)+k} \end{bmatrix}, \tag{2.21}$$

Then take a convenient value for n, say $n = 1$, and use elementary row operations on the augmented matrix B_1^*,

$$B_1^* = \begin{bmatrix} L_w & L_0 & L_{w(1+h)+k} \\ L_{2w} & L_w & L_{w(2+h)+k} \end{bmatrix} \tag{2.22}$$

to obtain a classes of identities (for w, h and k various integers):

$$L_{2(n+4)} = 21L_{2(n+1)} - 8L_{2n}; \tag{2.23}$$

$$L_{2(n+5)} = 55L_{2(n+1)} - 21L_{2n}; \tag{2.24}$$

$$L_{2(n+3)+5} = 89L_{2(n+1)} - 34L_{2n}; \tag{2.25}$$

$$L_{3(n+4)} = 72L_{3(n+1)} + 17L_{3n}; \tag{2.26}$$

$$L_{3(n+4)+2} = \frac{377}{2} L_{3(n+1)} + \frac{89}{2} L_{3n}; \tag{2.27}$$

$$L_{4(n+4)} = 329F_{4(n+1)} - 48L_{4n}; \tag{2.28}$$

$$L_{4(n+4)+3} = \frac{4181}{3} L_{4(n+1)} - \frac{610}{3} L_{4n}; \tag{2.29}$$

$$L_{5(n+5)} = 15005L_{5(n+1)} + 1353L_{5n}; \tag{2.30}$$

$$L_{5(n+5)+4} = \frac{514229}{5} L_{5(n+1)} + \frac{46368}{5} L_{5n}; \tag{2.31}$$

$$L_{7(n+8)} = 17373187209L_{7(n+1)} + 598364773L_{7n}; \tag{2.32}$$

$$L_{7(n+8)+5} = \frac{2504730781961}{13} L_{7(n+1)} + \frac{86267571272}{13} L_{7n}; \tag{2.33}$$

$$\cdots$$

Note: For (2.23-2.33), using the principle of induction, we can get the general identities:

$$L_{2(n+r)} = F_{2r} L_{2(n+1)} - F_{2(r-1)} L_{2n}; \tag{2.34}$$

$$L_{3(n+h)+k} = \frac{F_{3h+k}}{2} L_{3(n+1)} + \frac{F_{3(h-1)+k}}{2} L_{3n}; \tag{2.35}$$

$$L_{4(n+h)+k} = \frac{F_{4h+k}}{3} L_{4(n+1)} - \frac{F_{4(h-1)+k}}{3} L_{4n}; \tag{2.36}$$

$$L_{5(n+h)+k} = \frac{F_{5h+k}}{5} L_{5(n+1)} + \frac{F_{5(h-1)+k}}{5} L_{5n}; \tag{2.37}$$

$$L_{7(n+h)+k} = \frac{F_{7h+k}}{13} L_{7(n+1)} + \frac{F_{7(h-1)+k}}{13} L_{7n}; \tag{2.38}$$

3. More tridiagonal matrices

Suppose we want an identity of the form (w, k and $n \geq 1$ are arbitrary positive integers)

$$F_{w(n+1)+k} = x_1 F_{wn+k} + x_2 F_{w(n-1)+k}. \tag{3.1}$$

Using the generating matrix A [2],

$$A = \begin{bmatrix} 0 & 1 \\ x_2 & x_1 \end{bmatrix} \tag{3.2}$$

for augmented matrix A_n^*:

$$A_n^* = \begin{bmatrix} F_{wn+k} & F_{w(n-1)+k} & F_{w(n+1)+k} \\ F_{w(n+1)+k} & F_{wn+k} & F_{w(n+2)+k} \end{bmatrix}, \tag{3.3}$$

using direct matrix computation and the principle of induction, we have

$$A_n^* = AA_{n-1}^* = A^2 A_{n-2}^* = \cdots = A^{n-1} A_1^*. \tag{3.4}$$

Thus, we use elementary row operations on the augmented matrix A_1^*,

$$A_1^* = \begin{bmatrix} F_{w+k} & F_k & F_{2w+k} \\ F_{2w+k} & F_{w+k} & F_{3w+k} \end{bmatrix} \tag{3.5}$$

to obtain x_1 and x_2. We can construct the $n \times n$ tridiagonal matrix:

$$M_F^{(1)}(n) = \begin{bmatrix} F_{tw+k} & m & & & & \\ 1 & \left\lceil \frac{F_{(t+1)w+k}}{F_{tw+k}} \right\rceil & -x_2 & & & \\ & 1 & x_1 & -x_2 & & \\ & & 1 & x_1 & -x_2 & \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & -x_2 \\ & & & & & 1 & x_1 \end{bmatrix} \tag{3.6}$$

or $n \times n$ symmetric tridiagonal matrix:

$$M_F^{(2)}(n) = \begin{bmatrix} F_{tw+k} & \sqrt{m} & & & & & \\ \sqrt{m} & \left\lceil \frac{F_{(t+1)w+k}}{F_{tw+k}} \right\rceil & \sqrt{-x_2} & & & & \\ & \sqrt{-x_2} & x_1 & \sqrt{-x_2} & & & \\ & & \sqrt{-x_2} & x_1 & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & \sqrt{-x_2} \\ & & & & & \sqrt{-x_2} & x_1 \end{bmatrix} \tag{3.7}$$

where $m = \left\lceil \frac{F_{(t+1)w+k}}{F_{tw+k}} \right\rceil \cdot F_{tw+k} - F_{(t+1)w+k}$, $M_F^{(1)}(t)$ and $M_F^{(2)}(t)$ are $n \times n$, $t \in N$, $i^2 = -1$, when $m < 0$, $\sqrt{m} = i\sqrt{|m|}$, when $-x_2 < 0$, $\sqrt{-x_2} = i\sqrt{|x_2|}$. It is easy to show by induction that the determinants $\left|M_F^{(1)}(t)\right|$ and $\left|M_F^{(2)}(t)\right|$ are the Fibonacci numbers $F_{w(n+t-1)+k}$, that is $\left|M_F^{(1)}(t)\right| = \left|M_F^{(2)}(t)\right| = F_{w(n+t-1)+k}$.

For the Lucas numbers, by the above method, we construct the similar $n \times n$ tridiagonal matrix:

$$M_L^{(1)}(n) = \begin{bmatrix} L_{tw+k} & r & & & & & \\ 1 & \left\lceil \frac{L_{(t+1)w+k}}{L_{tw+k}} \right\rceil & -x_2 & & & & \\ & 1 & x_1 & -x_2 & & & \\ & & 1 & x_1 & -x_2 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & -x_2 \\ & & & & & 1 & x_1 \end{bmatrix} \tag{3.8}$$

or $n \times n$ symmetric tridiagonal matrix:

$$M_L^{(2)}(n) = \begin{bmatrix} L_{tw+k} & \sqrt{r} & & & & & \\ \sqrt{r} & \left\lceil \frac{L_{(t+1)w+k}}{L_{tw+k}} \right\rceil & \sqrt{-x_2} & & & & \\ & \sqrt{-x_2} & x_1 & \sqrt{-x_2} & & & \\ & & \sqrt{-x_2} & x_1 & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & \sqrt{-x_2} \\ & & & & & \sqrt{-x_2} & x_1 \end{bmatrix} \tag{3.9}$$

where $r = \left\lceil \frac{L_{(t+1)w+k}}{L_{tw+k}} \right\rceil \cdot L_{tw+k} - L_{(t+1)w+k}$, $M_{L1}(t)$ and $M_{L2}(t)$ are $n \times n$, $t \in N$, $i^2 = -1$, when $r < 0$, $\sqrt{r} = i\sqrt{|r|}$, when $-x_2 < 0$, $\sqrt{-x_2} = i\sqrt{|x_2|}$. It is easy to show by induction that the determinants $\left|M_L^{(1)}(n)\right|$ and $\left|M_L^{(2)}(n)\right|$ are the Lucas numbers $L_{w(n+t-1)+k}$, that is $\left|M_L^{(1)}(n)\right| = \left|M_L^{(2)}(n)\right| = L_{w(n+t-1)+k}$.

Example 3.1. *If $w = 2, k = 5$, (3.1) becomes*

$$F_{2(n+1)+5} = x_1 F_{2n+5} + x_2 F_{2(n-1)+5}, \tag{3.10}$$

the augmented matrix A_1^* can be transformed to

$$A_1^* = \begin{bmatrix} F_7 & F_5 & F_9 \\ F_9 & F_7 & F_{11} \end{bmatrix} = \begin{bmatrix} 13 & 5 & 34 \\ 34 & 13 & 89 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \end{bmatrix}. \tag{3.11}$$

Thus, we have identity

$$F_{2(n+1)+5} = 3F_{2n+5} - F_{2(n-1)+5}. \tag{3.12}$$

If we set $t = 2$ in (3.6, 3.7), the determinants of

$$\begin{bmatrix} 34 & 13 & & & \\ 1 & 3 & 1 & & \\ & 1 & 3 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 3 \end{bmatrix}, \begin{bmatrix} 34 & \sqrt{13} & & & \\ \sqrt{13} & 3 & 1 & & \\ & 1 & 3 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 3 \end{bmatrix}$$

are the Fibonacci numbers $F_{2(n+1)+5}$.
If we set $t = 1$ in (3.6, 3.7), the determinants of

$$\begin{bmatrix} 13 & 5 & & & \\ 1 & 3 & 1 & & \\ & 1 & 3 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 3 \end{bmatrix}, \begin{bmatrix} 13 & \sqrt{5} & & & \\ \sqrt{5} & 3 & 1 & & \\ & 1 & 3 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 3 \end{bmatrix}$$

are the Fibonacci numbers F_{2n+5}.

If we set $t = 0$ in (3.6, 3.7), the determinants of

$$\begin{bmatrix} 5 & 2 & & & \\ 1 & 3 & 1 & & \\ & 1 & 3 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 3 \end{bmatrix}, \begin{bmatrix} 5 & \sqrt{2} & & & \\ \sqrt{2} & 3 & 1 & & \\ & 1 & 3 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 3 \end{bmatrix}$$

are the Fibonacci numbers $F_{2(n-1)+5}$.

Example 3.2. *Suppose we want an identity of the form* (w, k *and* $n \geq 1$ *are arbitrary positive integers)*

$$L_{w(n+1)+k} = x_1 L_{wn+k} + x_2 L_{w(n-1)+k}. \tag{3.13}$$

If $w = -3$, $k = -4$, *(3.13) becomes*

$$L_{-3(n+1)-4} = x_1 L_{-3n-4} + x_2 L_{-3(n-1)-4}, \tag{3.14}$$

the augmented matrix B_1^* *can be transformed to*

$$B_1^* = \begin{bmatrix} L_{-7} & L_{-4} & L_{-10} \\ L_{-10} & L_{-7} & L_{-13} \end{bmatrix} = \begin{bmatrix} -29 & 7 & 123 \\ 123 & -29 & -521 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -4 \\ 0 & 1 & 1 \end{bmatrix}. \tag{3.15}$$

Thus, we have identity

$$L_{-3(n+1)-4} = -4L_{-3n-4} + L_{-3(n-1)-4}. \tag{3.16}$$

If we set $t = 2$ *in (3.8, 3.9), the determinants of*

$$\begin{bmatrix} 123 & 29 & & & \\ 1 & -4 & -1 & & \\ & 1 & -4 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -4 \end{bmatrix}, \begin{bmatrix} 123 & \sqrt{29} & & & \\ \sqrt{29} & -4 & i & & \\ & i & -4 & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & -4 \end{bmatrix}$$

are the Lucas numbers $L_{-3(n+1)-4}$.
If we set $t = 1$ *in (3.8, 3.9), the determinants of*

$$\begin{bmatrix} -29 & -7 & & & \\ 1 & -4 & -1 & & \\ & 1 & -4 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -4 \end{bmatrix}, \begin{bmatrix} -29 & i\sqrt{7} & & & \\ i\sqrt{7} & -4 & i & & \\ & i & -4 & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & -4 \end{bmatrix}$$

are the Lucas numbers L_{-3n-4}.

If we set $t = 0$ *in (3.8, 3.9), the determinants of*

$$\begin{bmatrix} 7 & 1 & & & \\ 1 & -4 & -1 & & \\ & 1 & -4 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & -4 \end{bmatrix}, \begin{bmatrix} 7 & 1 & & & \\ 1 & -4 & i & & \\ & i & -4 & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & -4 \end{bmatrix}$$

are the Lucas numbers $L_{-3(n-1)-4}$.

4. Conclusion

In this paper, we derive family relations on the arbitrary subscripted Fibonacci and Lucas numbers. Furthermore, we construct the tridiagonal and symmetric tridiagonal family of matrices which determinants form any linear subsequence of the Fibonacci numbers and Lucas numbers. Thus, we give a generalization of the presented in Nalli and Civciv [4] and Cahill and Narayan [1].

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