On derivations of lattice implication algebras

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Abstract

In this paper, we introduced the notion of derivation in lattice implication algebra, and considered the properties of derivations in lattice implication algebras. We give an equivalent condition to be derivation of a lattice implication algebra. Also, we characterized the fixed set $Fix_d(L)$ and $Kerd$ by derivations. Moreover, we prove that if d is a derivation of a lattice implication algebra, every filter F is a d-invariant.

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1. Introduction

In order to research a logical system whose propositional value is given in a lattice. Y. Xu [10] proposed the concept of lattice implication algebras, and some researchers have studied their properties and the corresponding logic systems. Also, in [11], Y. Xu and K. Y. Qin discussed the properties lattice H implication algebras, and gave some equivalent conditions about lattice H implication algebras. Y. Xu and K. Y. Qin [12] introduced the notion of filters in a lattice implication, and investigated their properties. In this paper, we introduced the notion of derivation, and considered the properties of derivations of lattice implication algebras. We give an equivalent condition to be derivation in a lattice implication algebra. Also, we characterized the fixed set $Fix_d(L)$ and $Kerd$ by derivations. Moreover, we prove that if d is a derivation of a lattice implication algebra, every filter F is a a d-invariant.

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2. Preliminary

A *lattice implication algebra* is an algebra $(L; \wedge, \vee, ', \rightarrow, 0, 1)$ of type $(2,2,1,2,0,0)$, where $(L; \wedge, \vee, 0, 1)$ is a bounded lattice, " $'$ " is an order-reversing involution and " $\rightarrow$ " is a binary operation, satisfying the following axioms:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(I2) $x \rightarrow x = 1$,

(I3) $x \rightarrow y = y' \rightarrow x'$,

(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,

(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,

(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,

(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$,

for all $x, y, z \in L$. If L satisfies conditions (I1) – (I5), we say that L is a *quasi lattice implication algebra*. A lattice implication algebra L is called a *lattice H implication algebra* if it satisfies $x \vee y \vee ((x \wedge y) \rightarrow z) = 1$ for all $x, y, z \in L$.

In the sequel the binary operation " $\rightarrow$ " will be denoted by juxtaposition. We can define a partial ordering " $\leq$ " on a lattice implication algebra L by $x \leq y$ if and only if $x \rightarrow y = 1$.

In a lattice implication algebra L, the following hold (see [10]):

(u1) $0 \rightarrow x = 1$, $1 \rightarrow x = x$ and $x \rightarrow 1 = 1$.

(u2) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.

(u3) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$.

(u4) $x' = x \rightarrow 0$.

(u5) $x \vee y = (x \rightarrow y) \rightarrow y$.

(u6) $((y \rightarrow x) \rightarrow y')' = x \wedge y = ((x \rightarrow y) \rightarrow x')'$.

(u7) $x \leq (x \rightarrow y) \rightarrow y$.

In a lattice H implication algebra L, the following hold:

(u8) $x \rightarrow (x \rightarrow y) = x \rightarrow y$.

(u9) $x \to (y \to z) = (x \to y) \to (x \to z)$.

A subset F of a lattice implication algebra L is called a *filter* of L it it satisfies:

(F1) $1 \in F$,

(F2) $x \in F$ and $x \to y \in F$ imply $y \in F$, for all $x, y \in L$.

3. Derivations of lattice implication algebras

In what follows, let L denote a lattice implication algebra unless otherwise specified.

Definition 3.1. Let L be a lattice implication algebra. A map $d : L \to L$ is a *derivation* of L if

$$d(x \to y) = (x \to d(y)) \vee (d(x) \to y)$$

for all $x, y \in L$.

Example 3.2. Let $L := \{0, a, b, c, 1\}$. Define the partial order relation on L as $0 < a < b < c < 1$, and define

$$x \wedge y := \min\{x, y\},\ x \vee y := \max\{x, y\}$$

for all $x, y \in L$ and "$'$" and "$\to$" as follows:

x	x'
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

Then $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra. Define a map $d : L \to L$ by

$$d(x) = \begin{cases} 1 & \text{if } x = c, 1 \\ b & \text{if } x = a \\ a & \text{if } x = 0 \\ c & \text{if } x = b \end{cases}$$

Then it is easy to check that d is a derivation of lattice implication algebra L.

Example 3.3. Let $L := \{0, a, b, 1\}$ be a set with the Cayley table.

x	x'
0	1
a	b
b	a
1	0

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

For any $x \in L$, we have $x' = x \rightarrow 0$. The operations $\wedge$ and $\vee$ on L are defined as follows:

$$x \vee y = (x \rightarrow y) \rightarrow y, \qquad x \wedge y = ((x' \rightarrow y') \rightarrow y')'.$$

Then $(L, \vee, \wedge, ', \rightarrow)$ is a lattice implication algebra. Define a map $d : L \rightarrow L$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 0, 1, b \\ b & \text{if } x = a \end{cases}$$

Then it is easy to check that d is a derivation of lattice implication algebra L.

Proposition 3.4. Let d be a derivation of L. Then we have $d(1) = 1$.

Proof. Let d be a derivation of L. Then we have

$$\begin{aligned} d(1) &= d(1 \rightarrow 1) = (1 \rightarrow d(1))) \vee (d(1) \rightarrow 1) \\ &= d(1) \vee 1 = (d(1) \rightarrow 1) \rightarrow 1 = 1 \rightarrow 1 = 1. \end{aligned}$$

Proposition 3.5. Let L be a lattice implication algebra and let d be a derivation on L. Then $d(x) = d(x) \vee x$ for all $x \in L$.

Proof. Let $x \in L$. Then we have

$$d(x) = d(1 \rightarrow x) = (1 \rightarrow d(x)) \vee (d(1) \rightarrow x) = (1 \rightarrow d(x)) \vee (1 \rightarrow x) = d(x) \vee x.$$

Corollary 3.6. Let L be a lattice implication algebra and let d be a derivation on L. Then $x \leq d(x)$ for all $x \in L$.

Proof. Let d be a derivation on L. Then by Proposition 3.5, we have

$$\begin{aligned} x \rightarrow d(x) &= x \rightarrow (d(x) \vee x) = x \rightarrow ((d(x) \rightarrow x) \rightarrow x) \\ &= (d(x) \rightarrow x) \rightarrow (x \rightarrow x) = (d(x) \rightarrow x) \rightarrow 1 = 1, \end{aligned}$$

which implies $x \leq d(x)$.

Proposition 3.7. Let f be an expansive map on a lattice implication algebra L, i.e., $x \leq f(x)$ for all $x \in L$. Then $f(x) \to y \leq x \to f(y)$ for all $x, y \in L$.

Proof. Suppose that f is an expansive map on L and $x, y \in L$. Then $x \leq f(x)$ and $y \leq f(y)$. Hence $f(x) \to y \leq x \to y$ and $x \to y \leq x \to f(y)$ by (u3). It follows that $f(x) \to y \leq x \to f(y)$.

Theorem 3.8. Let d be a map on a lattice implication algebra L. Then the following identities are equivalent:

(i) d is a derivation of L :

(ii) $d(x \to y) = x \to d(y)$ for all $x, y \in L$.

Proof. Suppose that d is a derivation of L. Then Since $x \leq d(x)$, we have $d(x) \to y \leq x \to d(y)$ by Proposition 3.7. Hence by (I5)

$$\begin{aligned} d(x \to y) &= (x \to d(y)) \vee (d(x) \to y) \\ &= ((x \to d(y)) \to (d(x) \to y)) \to (d(x) \to y) \\ &= ((d(x) \to y) \to (x \to d(y))) \to (x \to d(y)) \\ &= 1 \to (x \to d(y)) = x \to d(y) \end{aligned}$$

for all $x, y \in L$. Conversely, suppose that d is a map satisfying $d(x \to y) = x \to d(y)$ for all $x, y \in L$. Then $d(1) = d(d(1) \to 1) = d(1) \to d(1) = 1$, hence we have $1 = d(1) = d(x \to x) = x \to d(x)$ for all $x \in L$. This implies that $x \leq d(x)$ for all $x \in L$, and so $d(x) \to y \leq x \to d(y)$ by Proposition 3.7. Hence by (I5),

$$d(x \to y) = x \to d(y) = (x \to d(y)) \vee (d(x) \to y)$$

for all $x, y \in L$.

Proposition 3.9. Let L be a lattice implication algebra. If $d_1, d_2, d_3, ..., d_n$ are derivations of L, $d_1 \circ d_2 \circ d_3 \cdots \circ d_n$ is a derivation of L.

Proof. It is sufficient to prove the theorem in case of two derivations. Let d_1 and d_2 be derivations of L and $x, y \in L$. Then we have

$$\begin{aligned} (d_1 \circ d_2)(x \to y) &= d_1(d_2(x \to y)) \\ &= d_1(x \to d_2(y)) = (x \to d_1(d_2(y))) \\ &= x \to (d_1 \circ d_2)(y), \end{aligned}$$

from Theorem 3.8. This completes the proof.

Let d be a derivation of L. Define a set $Fix_d(L)$ by

$$Fix_d(L) := \{x \in L \mid d(x) = x\}$$

for all $x \in L$.

Proposition 3.10. Let d be a derivation of L. If $x \in Fix_d(L)$, then we have

$$(\overbrace{d \circ d \circ d \cdots \circ d}^{n})(x) = x.$$

Proof. By definition of $Fix_d(L)$, the proof is straightforward.

Proposition 3.11 Let L be a lattice implication algebra and let d be a derivation on L. Then we have the following properties.

(i) If $x \in L$ and $y \in Fix_d(L)$, we have $x \to y \in Fix_d(L)$,

(ii) If $y \in Fix_d(L)$, $x \vee y \in Fix_d(L)$ for all $x \in L$.

Proof. (i) Let $x \in L$ and $y \in Fix_d(L)$. Then we have $d(y) = y$. Thus we get

$$\begin{aligned} d(x \to y) &= x \to d(y) \\ &= x \to y \end{aligned}$$

from Theorem 3.8.

(ii) Let $x \in L$ and $y \in Fix_d(L)$. Then we get

$$\begin{aligned} d(x \vee y) &= d((x \to y) \to y) \\ &= (x \to y) \to d(y) \\ &= (x \to y) \to y) = x \vee y \end{aligned}$$

from Theorem 3.8.

Proposition 3.12. Let L be a lattice implication algebra and let d be a derivation. If $x \leq y$ and $x \in Fix_d(L)$, we have $y \in Fix_d(L)$.

Proof. Let $x \leq y$ and $x \in Fix_d(L)$. Then we have $x \to y = 1$ and $d(x) = x$. Thus we get

$$d(y) = d((1 \to y) = d((x \to y) \to y) = d(x \vee y) = x \vee y = y$$

from Proposition 3.11(ii).

Definition 3.13. Let L be a lattice implication algebra and d be a derivation of L. If $x \leq y$ implies $d(x) \leq d(y)$ for all $x, y \in L$, d is called an *isotone derivation.*

Example 3.14. In Example 3.3, d is an isotone derivation of L.

Proposition 3.15. Let L be a lattice implication algebra and let d be a derivation. If d is an endomorphism on L, d is an isotone derivation.

Proof. Let $x \leq y$. Then $x \to y = 1$, and so

$$d(x) \to d(y) = d(x \to y) = d(1) = 1.$$

This imply $d(x) \leq d(y)$.

Proposition 3.16. Let L be a lattice implication algebra and d be a derivation. Then $d : L \to L$ is an identity map if it satisfies $x \to d(y) = d(x) \to y$ for all $x, y \in L$.

Proof. Let $x, y \in L$ be such that $x \to d(y) = d(x) \to y$. Now $d(x) = d(1 \to x) = 1 \to d(x) = d(1) \to x = 1 \to x = x$ by Theorem3.8. Thus d is an identity map.

Theorem 3.17. Let L be a lattice implication algebra and d a derivation. Then d is one to one if and only if d is an identity derivation.

Proof. Sufficiency is obvious. Suppose that d is one to one. For every $x \in L$, we have

$$d(d(x) \to x) = d(x) \to d(x) = 1 = d(1)$$

and so $d(x) \to x = 1$, i.e., $d(x) \leq x$. Since $x \leq d(x)$ for all $x \in L$ from Corollary 3.6, it follows that $d(x) = x$, which implies that d is the identity derivation.

Let L be a lattice implication algebra. Then, for each $a \in L$, we define a map $d_a : L \to L$ by

$$d_a(x) = a \to x$$

for all $x \in L$.

Theorem 3.18. For each $a \in L$, the map d_a is a derivation of L.

Proof. Suppose that d_a is a map defined by $d_a(x) = a \to x$ for each $x \in L$. Then for any $x, y \in L$, we have

$$d_a(x \to y) = a \to (x \to y) = x \to (a \to y) = x \to d_a(y).$$

Hence d_a is a derivation of L by Theorem 3.8.

We call the derivation d_a of Theorem 3.18 as *simple derivation.*

Proposition 3.19. Let L be a lattice H implication algebra. For every $a \in L$, the simple derivation d_a is an endomorphism.

Proof. Let $x, y \in L$. From (u9), we have

$$d_a(x \to y) = a \to (x \to y) = (a \to x) \to (a \to y) = d_a(x) \to d_a(y).$$

Hence d_a is an endomorphism.

Definition 3.20. Let L be a lattice implication algebra and let d be a derivation of L. Define a $Kerd$ by

$$Kerd = \{x \in L \mid d(x) = 1\}.$$

Proposition 3.21. Let d be a derivation of a lattice implication algebra L. If d is an endomorphism on L, $Kerd$ is a filter of L.

Proof. Clearly, $1 \in Kerd$. Let $x, x \to y \in Kerd$. Then $d(x) = 1$ and $d(x \to y) = 1$. Hence we have

$$1 = d(x \to y) = d(x) \to d(y) = 1 \to d(y) = d(y),$$

which implies $y \in Kerd$.

Proposition 3.22. Let L be a lattice implication algebra and let d be a derivation. If $y \in Kerd$, then we have $x \vee y \in Kerd$ for all $x \in L$.

Proof. Let d be a derivation and $y \in Kerd$. Then we get $d(y) = 1$, and so

$$d(x \vee y) = d((x \to y) \to y) = (x \to y) \to d(y) = (x \to y) \to 1 = 1$$

from Theorem 3.8. Hence we have $x \vee y \in Kerd$.

Proposition 3.23. Let L be a lattice implication algebra and d be a derivation. If $x \leq y$ and $x \in Kerd$, then $y \in Kerd$.

Proof. Let $x \leq y$ and $x \in Kerd$. Then we get $x \to y = 1$ and $d(x) = 1$, and so

$$\begin{aligned} d(y) &= d(1 \to y) = d((x \to y) \to y) \\ &= d((y \to x) \to x) = (y \to x) \to d(x) \\ &= (y \to x) \to 1 = 1 \end{aligned}$$

from Theorem 3.8. Hence we have $y \in Kerd$.

Proposition 3.24. Let L be a lattice implication algebra and let d be a derivation of L. If $y \in Kerd$, we have $x \to y \in Kerd$ for all $x \in L$.

Proof. Let $y \in Kerd$. Then $d(y) = 1$. Thus we have

$$d(x \to y) = x \to d(y) = x \to 1 = 1$$

from Theorem 3.8. Hence we get $x \to y \in Kerd$.

Definition 3.25. Let L be a lattice implication algebra. A nonempty subset F of L is said to be a *d-invariant* if $d(F) \subseteq F$ where $d(F) = \{d(x) \mid x \in F\}$.

Theorem 3.26. Let L be a lattice implication algebra and let d be a derivation. Then every filter F is a d-invariant.

Proof. Let F be a filter of L. Let $y \in d(F)$. Then $y = d(x)$ for some $x \in F$. It follows that $x \to y = x \to d(x) = 1 \in F$, which implies $y \in F$. Thus $d(F) \subseteq F$. Hence F is a d-invariant.

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