Unimodal rays of the generalized Pascal's triangle *

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Abstract

In this paper, we present two criteria for a sequence lying along a ray of a combinatorial triangle to be unimodal, and give a correct proof for the result of Belbachir and Szalay on unimodal rays of the generalized Pascal's triangle.

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1 Introduction

A sequence $\{a_i\}_{i\geq 0}$ of nonnegative numbers is called *unimodal* if $a_0 \leq a_1 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots$ and *log-concave* if for all $i \geq 1$, $a_{i-1}a_{i+1} \leq a_i^2$. It is well known that a log-concave sequence $\{a_i\}_{i\geq 0}$ of positive numbers is unimodal (see [6]), and a sequence of positive numbers $\{a_i\}_{i\geq 0}$ is log-concave if and only if $a_{i-1}a_{j+1} \leq a_i a_j$ for $j \geq i \geq 1$ (e.g., see [3, Proposition 2.5.1]). Unimodal and log-concave sequences often arise in combinatorics and other branches of mathematics. See survey articles [3, 4, 6].

Recently the binomial sequences in the form of $\{\binom{n_0+ia}{k_0+ib}\}_{i\geq 0}$ have been shown to share various unimodality properties (see [1, 2, 8, 10]). Arrange a binomial coefficient $\binom{n}{k}$ as a lattice point in two-dimensional plane. Clearly, the sequence $\{\binom{n_0+ia}{k_0+ib}\}_{i\geq 0}$ locates along a ray of Pascal's triangle and the

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corresponding ray starts with $\binom{n_0}{k_0}$ and its slope is determined by two parameters a and b. When $a = 0$ and $b = 0$, we get the rows and the columns of Pascal's triangle respectively.

In this paper, we present two criteria for a sequence lying along a ray of a combinatorial triangle to be unimodal. As an application, we give a correct proof for the results of Belbachir and Szalay [2] on the unimodal rays of the generalized Pascal's triangle.

2 Two criteria for unimodal rays

Many combinatorial numbers form a doubly indexed array $\{T(n,k)\}_{n,k\geq 0}$ of positive numbers, such as $\{\binom{n}{k}\}_{n,k\geq 0}$ (the ordinary Pascal's triangle). We call this kind of arrays a combinatorial triangle, in which the indices $n = 0, 1, 2, \ldots$, and the indices k form an arithmetic progression with the first term 0. The generalized Pascal's triangle in next section is such a triangle that its column-indices form an arithmetic progression.

Given a combinatorial triangle $\{T(n,k)\}_{n,k\geq 0}$, it is not difficult to see that a sequence lying along certain a ray of the triangle has the form of $\{T(n_0 - ia, k_0 + ib)\}_{i\geq 0}$ with $a, b > 0$, or $\{T(n_0 + ia, k_0 + ib)\}_{i\geq 0}$ with $b \geq a \geq 0$. In this section, we will show two criteria for these two kinds of sequences to be unimodal.

Theorem 1. *Suppose a combinatorial triangle $\{T(n,k)\}_{n,k\geq 0}$ satisfies:*

(i) *every row is log-concave;*

(ii) *every column is log-concave;*

(iii) $T(n-1,k)T(n,k-1) \leq T(n-1,k-1)T(n,k)$.

Then the sequence $\{T(n_0 - ia, k_0 + ib)\}_{i\geq 0}$ is log-concave, where $a, b \geq 0$ and $n_0 \geq k_0 \geq 0$.

Proof. By the definition of log-concavity, it suffices to prove that

$$T(n-a,k+b)T(n+a,k-b) \leq T(n,k)^2 \quad \text{for } a, b \geq 0.$$

Let n, k and l be fixed nonnegative integers and $k \leq l \leq n$. We first show that for all $a \geq 0$,

$$T(n-a,l+1)T(n+a,k-1) \leq T(n-a,l)T(n+a,k). \tag{1}$$

We proceed by induction on a. The case that $a = 0$ follows from the condition (i). Now suppose (1) holds for $a = j$, i.e.,

$$T(n-j,l+1)T(n+j,k-1) \leq T(n-j,l)T(n+j,k).$$

By condition (iii), we have

$$T(n-j,l)T(n-j-1,l+1) \le T(n-j-1,l)T(n-j,l+1),$$

$$T(n+j,k)T(n+j+1,k-1) \le T(n+j,k-1)T(n+j+1,k).$$

Multiplying these two inequalities together and cancelling, we have

$$T(n-j-1,l+1)T(n+j+1,k-1) \le T(n-j-1,l)T(n+j+1,k).$$

Thus (1) holds for all a.

Then (1) yields a series of inequalities as follows,

$$\begin{aligned} T(n-a,k+b)T(n+a,k-b) &\le T(n-a,k+b-1)T(n+a,k-b+1) \\ \le \cdots &\le T(n-a,k+1)T(n+a,k-1) \le T(n-a,k)T(n+a,k). \end{aligned}$$

By condition (ii), we have

$$T(n-a,k)T(n+a,k) \le T(n-a+1,k)T(n+a-1,k) \le \cdots \le T(n,k)^2.$$

Hence

$$T(n-a,k+b)T(n+a,k-b) \le T(n-a,k)T(n+a,k) \le T(n,k)^2.$$

as desired. □

Theorem 2. *Suppose a combinatorial triangle* $\{T(n,k)\}_{n,k\ge 0}$ *satisfies:*

(i) *every row is log-concave;*

(ii) *the sequence* $\{T(n+i,k+i)\}_{i\ge 0}$ *is log-concave for fixed* $n,k \ge 0$*;*

(iii) $T(n-1,k-1)T(n,k+1) \le T(n-1,k)T(n,k)$.

Then the sequence $\{T(n_0+ia,k_0+ib)\}_{i\ge 0}$ *is log-concave, where* $b \ge a \ge 0$ *and* $n_0 \ge k_0$.

Proof. It suffices to show that

$$T(n-a,k-b)T(n+a,k+b) \le T(n,k)^2 \quad \text{for } b \ge a \ge 0.$$

Next we will prove that for fixed $a \ge 0$ and all $k \le l$,

$$T(n-a,k-a-1)T(n+a,l+a+1) \le T(n-a,k-a)T(n+a,l+a). \quad (2)$$

The case that $a = 0$ follows from condition (i); Suppose the above inequality holds for $a = j$, i.e.,

$$T(n-j,k-j-1)T(n+j,l+j+1) \le T(n-j,k-j)T(n+j,l+j).$$

By condition (iii),

$$\begin{aligned}&T(n-j-1,k-j-2)T(n-j,k-j)\\ \le\ &T(n-j-1,k-j-1)T(n-j,k-j-1),\\ &T(n+j,l+j)T(n+j+1,l+j+2)\\ \le\ &T(n+j,l+j+1)T(n+j+1,l+j+1).\end{aligned}$$

Multiplying the previous two inequalities and cancelling, we have

$$T(n-j-1,k-j-2)T(n+j+1,l+j+2) \le T(n-j-1,k-j-1)T(n+j+1,l+j+1).$$

Thus (2) holds for fixed $a \ge 0$ and all $k \le l$.

Then (2) and condition (ii) yield a series of inequalities as follows,

$$\begin{aligned}&T(n-a,k-b)T(n+a,k+b) \le T(n-a,k-b+1)T(n+a,k+b-1)\\ &\le \cdots \le T(n-a,k-a)T(n+a,k+a)\\ &\le T(n-a+1,k-a+1)T(n+a-1,k+a-1)\\ &\le \cdots \le T(n-1,k-1)T(n+1,k+1) \le T(n,k)^2.\end{aligned}$$

□

3 The correct proof for the unimodal rays of the generalized Pascal's triangle

The generalized binomial coefficient $\binom{n,s}{k}$, also known as polynomial coefficient (see [5]), is given by

$$(1+x+x^2+\cdots+x^{s-1})^n = \sum_{k=0}^{ns-n} \binom{n,s}{k} x^k.$$

It satisfies the following recurrence relation

$$\binom{n,s}{k} = \sum_{j=0}^{s-1} \binom{n-1,s}{k-j}$$

and has the symmetry property

$$\binom{n,s}{k} = \binom{n,s}{ns-n-k}.$$

When $s = 2$, it is the ordinary binomial coefficient. When $s = 3$, we get the triangle $\{\binom{n,3}{k}\}_{k=0}^{2n}$ of trinomial coefficients:

$$\begin{array}{ccccccc} 1 & & & & & & \\ 1 & 1 & 1 & & & & \\ 1 & 2 & 3 & 2 & 1 & & \\ 1 & 3 & 6 & 7 & 6 & 3 & 1 \\ & & & \cdots & & & \end{array}$$

Clearly, the column indices of the generalized Pascal's triangle $\{\binom{n,s}{k}\}_{n,k\geq 0}$ form an arithmetic progression with the common difference $s-1$.

The following propsition was proved by Belbachir and Szalay, which is equivalent to the original result in [2] by the symmetry of the generalized binomial coefficient.

Proposition 1. *The sequence* $\left\{\binom{n_0-ia,s}{k_0+ib}\right\}_{i\geq 0}$ *is log-concave, with* $a, b \geq 0$.

Nevertheless, their proof for Proposition 1 is wrong. Taking the ordinary Pascal's triangle for instance, Belbachir and Szalay [2] prove Proposition 1 in the following way. They notice that: (a) The ordinary convolution preserves log-concavity, that is, if two positive sequences $\{a_i\}$ and $\{b_i\}$ are both log-concave, then so is the sequence c_i with $c_i = \sum_{j=0}^{i} a_j b_{i-j}$ (e.g., see [9]); (b) By Vandermonde's convolution formula,

$$\binom{n_0 - ia}{k_0 + ib} = \sum_{j=0}^{i} \binom{i}{j} \binom{n_0 - i(a+1)}{k_0 + ib - j}. \tag{3}$$

Then they claim that $\{\binom{n_0-ia}{k_0+ib}\}_{i\geq 0}$ is the convolution of $\{\binom{i}{j}\}_{j=0}^{i}$ and $\{\binom{n_0-i(a+1)}{k_0+ib-j}\}_{j=0}^{i}$, and is therefore log-concave by the log-concavity of the latter two sequences. However, the equation (3) is not the ordinary convolution since it belongs to the following type of equations

$$C_i^{(i)} = \sum_{j=0}^{i} A_j^{(i)} B_{i-j}^{(i)}.$$

Next we will give a correct proof for Proposition 1. The following lemma is needed.

Lemma 1. *If a sequence* $x_0, x_1, \cdots, x_{2s-2}$ *of positive numbers is log-concave, then*

$$\left(\sum_{j=0}^{s-1} x_j\right)^2 \geq x_0 \sum_{j=0}^{2s-2} \binom{2,s}{j} x_{2s-2-j}. \tag{4}$$

Proof. Note that

$$(1+x+\cdots+x^{s-1})^2 = \sum_{j=0}^{2s-2} \binom{2,s}{j} x^{2s-2-j}.$$

By setting $x_i \equiv x^i$ $(i = 0, 1, \cdots, 2s-2)$, we see that the numbers of the terms in the expansions of

$$\left(\sum_{j=0}^{s-1} x_j\right)^2 \quad \text{and} \quad x_0 \sum_{j=0}^{2s-2} \binom{2,s}{j} x_{2s-2-j}$$

are equal. Then there is a bijection from the terms of $\left(\sum_{j=0}^{s-1} x_j\right)^2$ to the terms of $x_0 \sum_{j=0}^{2s-2} \binom{2,s}{j} x_{2s-2-j}$ under the rule

$$x_i x_j \longmapsto x_0 x_{i+j}.$$

By the log-concavity of the sequence $\{x_i\}_{i=0}^{2s-2}$, we have $x_0 x_{i+j} \le x_i x_j$, no matter $i \ge j$ or $i \le j$. Then the required inequality (4) follows.

□

Now we are in a position to prove Proposition 1 by Theorem 1.

The proof of Proposition 1. It suffices to verify three conditions in Theorem 1 for the generalized Pascal's triangle respectively.

Condition (i): From the polynomial $(1+x+x^2+\cdots+x^{s-1})^n$, the n-th row is the n-convolution of the sequence $1, 1, \ldots, 1$(the number of 1's is s). Since the ordinary convolution preserves log-concavity, every row of the generalized Pascal's triangle is log-concave.

Condition (ii): Note that

$$\begin{aligned}&\binom{n,s}{k}^2 - \binom{n-1,s}{k}\binom{n+1,s}{k}\\ &= \left(\sum_{j=0}^{s-1}\binom{n-1,s}{k-j}\right)^2 - \binom{n-1,s}{k}\left(\sum_{j=0}^{2s-2}\binom{2,s}{j}\binom{n-1,s}{k-j}\right).\end{aligned}$$

Let $x_j = \binom{n-1,s}{k-j}$, $j = 0, 1, \cdots, 2s-2$. Clearly, the sequence $\{x_j\}_{j=0}^{2s-2}$ is log-concave. By Lemma 1, we claim that the difference $\binom{n,s}{k}^2 - \binom{n-1,s}{k}\binom{n+1,s}{k}$ is nonnegative. Thus every column of the generalized Pascal's triangle is log-concave.

Condition (iii): We have

$$\begin{aligned}
&\binom{n-1,s}{k-1}\binom{n,s}{k}-\binom{n-1,s}{k}\binom{n,s}{k-1}\\
&=\binom{n-1,s}{k-1}\left(\sum_{j=0}^{s-1}\binom{n-1,s}{k-j}\right)-\binom{n-1,s}{k}\left(\sum_{j=0}^{s-1}\binom{n-1,s}{k-1-j}\right)\\
&=\sum_{j=0}^{s-1}\left(\binom{n-1,s}{k-1}\binom{n-1,s}{k-j}-\binom{n-1,s}{k}\binom{n-1,s}{k-1-j}\right)\\
&\geq 0,
\end{aligned}$$

where the last inequality follows from the log-concavity of the $(n-1)$-th row in the generalized Pascal's triangle. □

Remark 1. Proposition 1 also can be derived from Theorem 2. Moreover, many other combinatorial triangles can be shown to have unimodal rays by Theorem 1 and Theorem 2, such as the Stirling's triangles of two kinds.

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