Eulerian Zero-Divisor Graphs

Emad Abu Osba
University of Jordan, Faculty of Science, Math. Department,
Amman 11942, Jordan. E-mail: eabuosba@ju.edu.jo
Hasan Al-Ezeh
University of Jordan, Faculty of Science, Math. Department,
Amman 11942, Jordan. E-mail: alezehh@ju.edu.jo

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Abstract

In this article, we characterize for which finite commutative ring R, the zero-divisor graph $\Gamma(R)$, the line graph $L(\Gamma(R))$, the complement graph $\overline{\Gamma(R)}$, and the line graph for the complement graph $L(\overline{\Gamma(R)})$ are Eulerian.

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1 Introduction

Let R be a finite commutative ring with 1. $Z(R)$ is the set of zero-divisors of R, and $Z^*(R) = Z(R)\backslash\{0\}$. The zero-divisor graph of R, $\Gamma(Z^*(R))$, usually written $\Gamma(R)$, is the graph in which each element of $Z^*(R)$ is a vertex, i.e., $V(\Gamma(R)) = Z^*(R)$, and two distinct vertices x and y are adjacent if $xy = 0$. The complement graph $\overline{\Gamma(R)}$ is defined on the same vertex set, but two distinct vertices x and y are adjacent if $xy \neq 0$. For more properties of this graph, see [4] and [5]. The line graph of G, denoted by $L(G)$, is a graph whose vertices are the edges of G, and two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are, see [7]. It is clear that if G is connected, then so is $L(G)$, while if $L(G)$ is connected and G has no isolated vertices, then G is also connected. If v, w are vertices in G and vw is an edge joining them in G, then the degree of vw in $L(G)$ is $\deg(vw) = \deg(v) + \deg(w) - 2$. For distinct vertices x and y of a graph G, let $d(x, y)$ be the length of a shortest path from x to y and let $N(x)$ denote the set of all neighbors of x in G. In a ring R, $Ann(X) = \{a \in R : ax = 0$ for all $x \in X\}$ and $U(R)$ is the set of all units in R.

For any undefined terminology the reader may consult [8].

In this article, we characterize the cases in which the graphs $\Gamma(R)$ and $\overline{\Gamma(R)}$ are Eulerian, where R is a finite commutative ring with 1. This problem was partially solved in [6], where the authors dealt with the ring $\mathbb{Z}_n$, and in [1] and [2], where the authors dealt with the ring $\mathbb{Z}_n[i]$. In [3], the authors dealt with non-commutative finite rings and directed graphs, but our results and techniques are easier than theirs. In [7], the author characterized when $L(\Gamma(R))$ is Eulerian, where R is a non-local ring and gave a partial result when R is local.

2 When is $\Gamma(R)$ Eulerian?

Definition 1 *A graph Γ is called **Eulerian** if there exists a closed trail containing every edge of Γ.*

In this article, we will use another criterion for a graph to be Eulerian, which was proved by Euler.

Proposition 2 (Euler) *A connected graph Γ is Eulerian if and only if the degree of each vertex of Γ is even.*

We begin with some preliminaries needed for the coming work.

Recall that if a ring R has non-zero nilpotent elements, then it has a non-zero element x such that $x^2 = 0$. If R is a local ring, then $|R| = p^n$ for some prime integer p, and so $Char(R) = p^m$, being the order of 1 in the additive group $(R, +)$.

One may wonder if in a local ring R with unique maximal ideal M, the conditions $M^2 = \{0\}$ and $x^2 = 0$ for all $x \in M$ are equivalent. The following example shows that this need not be true.

Example 3 *Let $R = \mathbb{Z}_2[X,Y]/(X^2,Y^2) = \mathbb{Z}_2[x,y]$, where $x = X + (X^2,Y^2)$ and $y = Y + (X^2,Y^2)$. Then R is a finite local ring with $Char(R) = 2$. The maximal ideal in R is $M = (x,y)$ in which $z^2 = 0$ for each $z \in M$, but $xy \neq 0$.*

Example 4 *Let $R = \mathbb{Z}_4[X,Y]/(X^2,XY,Y^2) = \mathbb{Z}_4[x,y]$, where $x = X + (X^2,XY,Y^2)$ and $y = Y + (X^2,XY,Y^2)$. Then R is a finite local ring with $Char(R) = 4$. The maximal ideal in R is $M = (2,x,y)$ in which $z^2 = 0$ for each $z \in M$, but $2x \neq 0$.*

Lemma 5 *In a local ring R with maximal ideal M, if $Char(R)$ is neither 2 nor 4, then $M^2 = \{0\}$ if and only if $x^2 = 0$ for all $x \in M$.*

Proof. If $M^2 = \{0\}$, then $x^2 = 0$ for all $x \in M$.

Conversely, assume $x^2 = 0$ for all $x \in M$. If $Char(R) = 2^n$ with $n > 2$, then $0 \neq 4 = 2^2$. So assume that $Char(R)$ is an odd integer. Let $x, y \in M$. Then $0 = (x+y)^2 = 2xy$, and since $2 \in U(R)$, we must have $xy = 0$. Thus $M^2 = \{0\}$. ∎

Theorem 6 *For a finite local ring R with a maximal ideal M, the graph $\Gamma(R)$ is Eulerian if and only if $|R|$ is even and $x^2 = 0$ for each $x \in M$.*

Proof. Assume R is a local ring with maximal ideal M. In this case, $|R| = p^n$ for some prime integer p.

Suppose that $y^2 \neq 0$, for some $y \in M$, and let $x \in M$ such that $x^2 = 0$. Then $\deg(x) = |Ann(x)| - 2$, while $\deg(y) = |Ann(y)| - 1$. But these two integers cannot both be even, hence $\Gamma(R)$ could not be Eulerian.

Conversely, if $x^2 = 0$ for each $x \in M$, then $\deg(x) = |Ann(x)| - 2$ for each $x \in M - \{0\}$. Since $|Ann(x)| = p^m$ for some $m \leq n$, we have $\Gamma(R)$ is Eulerian if R has even order. ∎

Lemma 7 *Let R be a finite non-local ring such that $|R|$ is even. Then $\Gamma(R)$ cannot be a Eulerian graph.*

Proof. Assume $R = R_1 \times R_2$. Since $|R|$ is even, then $|R_1|$ is even or $|R_2|$ is even. Assume $|R_1|$ is even, then $N((0,1)) = R_1 \times \{0\} \setminus \{(0,0)\}$. So, $\deg((0,1)) = |R_1| - 1$ which is an odd number. Thus $\Gamma(R)$ cannot be a Eulerian graph. ∎

Theorem 8 *For a finite non-local ring R, the graph $\Gamma(R)$ is Eulerian if and only if R has no non-zero nilpotent elements and $|R|$ is odd.*

Proof. Assume that $\Gamma(R)$ is Eulerian. Then by Lemma 7, $|R|$ is odd. If x is a non-zero nilpotent element in R such that $x^2 = 0$, then $N(x) = Ann(x) \setminus \{0, x\}$, and so $\deg(x) = |Ann(x)| - 2$, which is an odd integer. Hence $\Gamma(R)$ cannot be Eulerian.

Conversely, assume that R has no non-zero nilpotent elements and $|R|$ is odd. Then for each $x \in Z^*(R)$, $\deg(x) = |Ann(x)| - 1$, which is an even integer. Hence $\Gamma(R)$ is Eulerian. ∎

Corollary 9 *For a finite non-local ring R, the graph $\Gamma(R)$ is Eulerian if and only if R is a direct product of fields, each of which is of odd order.*

Now, one can easily deduce Proposition 1 in [3], Theorem 3.1 in [6], and Theorem 29 in [2].

3 When is $L(\Gamma(R))$ Eulerian?

For a ring R, the graph $\Gamma(R)$ is connected, see [4], and so is $L(\Gamma(R))$. Hence $L(\Gamma(R))$ is Eulerian if and only if $\deg(vw)$ is even for each vertex vw in $L(\Gamma(R))$, if and only if $\deg(v)$ is even for each vertex v in $\Gamma(R)$ or $\deg(v)$ is odd for each vertex v in $\Gamma(R)$. So if a ring R has non-zero nilpotent elements x, y such that $x^2 = 0$ and $y^n = 0$ with $n \neq 2$, then $\deg(x)$ and $\deg(y)$ are not both even and are not both odd; so $L(\Gamma(R))$ is not Eulerian. Thus we have the following theorem, see also [7].

Theorem 10 *(1) Let R be a finite local ring with maximal ideal M. Then $L(\Gamma(R))$ is Eulerian if and only if $x^2 = 0$ for each $x \in M$.*

(2) Let R be a finite non-local ring. Then $L(\Gamma(R))$ is Eulerian if and only if R is a product of fields, each of which is of even order or each of which is of odd order.

4 When is $\overline{\Gamma(R)}$ Eulerian?

Recall that a connected graph Γ is Eulerian if and only if the degree of each vertex of Γ is even. If Γ is disconnected, then it is Eulerian if and only if it is a disjoint union of a discrete subgraph and a Eulerian subgraph. So a disconnected graph is Eulerian if and only if it is a disjoint union of a discrete subgraph and a connected graph in which each vertex has even degree.

If R is a finite local ring with maximal ideal M, then there exists a positive integer n such that $M^n = \{0\}$ and $M^{n-1} \neq \{0\}$. If $a \in M^{n-1} \setminus \{0\}$, then $ax = 0$ for each $x \in M$ and so a is an isolated vertex in $\overline{\Gamma(R)}$; hence $\overline{\Gamma(R)}$ cannot be connected.

Lemma 11 *If R is a finite local ring with maximal ideal M such that $M^2 \neq \{0\}$, then the subgraph of $\overline{\Gamma(R)}$ consisting of the vertices $M \setminus Ann(M)$ is connected with diameter at most 2.*

Proof. Since $M^2 \neq \{0\}$, $Ann(M)$ is a proper subset of M. Let $a, b \in M \setminus Ann(M)$. If $ab \neq 0$, then $a - b$ is a path in $\overline{\Gamma(R)}$. If $ab = 0$, then there exists $z_1, z_2 \in M \setminus Ann(M)$ such that $az_1 \neq 0$ and $bz_2 \neq 0$. We have many cases:

Case (I): If $z_1 = z_2$, then $a - z_1 - b$ is a path in $\overline{\Gamma(R)}$.

Case (II): If $z_1 \neq z_2$, and $bz_1 \neq 0$, then $a - z_1 - b$ is a path in $\overline{\Gamma(R)}$. If $az_2 \neq 0$, then $a - z_2 - b$ is a path in $\overline{\Gamma(R)}$.

Case (III): If $z_1 \neq z_2, az_2 = 0$ and $bz_1 = 0$, then $a - (z_1 + z_2) - b$ is a path in $\overline{\Gamma(R)}$. Note that in this case, $z_1 + z_2$ is not equal to $0, a$ nor b.

Thus the subgraph is connected and has diameter less than or equal to 2. Note that in this case, the vertices of $Ann(M) \setminus \{0\}$ are isolated. ■

If R is a finite local ring with maximal ideal M and $M^2 = \{0\}$, then $\overline{\Gamma(R)} = |M^*| K_1$, and so it is Eulerian. If $M^2 \neq \{0\}$, but $x^2 = 0$ for all $x \in M \setminus \{0\}$, then $N(x) = M \setminus Ann(x)$ and $\deg(x) = |M| - |Ann(x)|$ which is always even, and so $\overline{\Gamma(R)}$ is Eulerian. If there exists $x \in M$ such that $x^2 \neq 0$, then $N(x) = M \setminus (Ann(x) \cup \{x\})$ and $\deg(x) = |M| - |Ann(x)| - 1$, which is always odd. Thus $\overline{\Gamma(R)}$ is cannot be Eulerian. The following theorem summarizes the above work.

Theorem 12 *Let R be a finite local ring with maximal ideal M. Then $\overline{\Gamma(R)}$ is Eulerian if and only if $x^2 = 0$ for all $x \in M$.*

The following theorem was proved in [1].

Theorem 13 *Let R be a finite ring that is a product of two rings with at least one of them not an integral domain. Then $\overline{\Gamma(R)}$ is connected with $diam(\overline{\Gamma(R)}) \leq 3$.*

Now, we will discuss the cases in which $\overline{\Gamma(R)}$ is not connected when R is a finite non-local ring. Assume $R = \mathbb{Z}_2 \times R_2$ with R_2 an integral domain. Then in $\overline{\Gamma(R)}$, $(1,0)$ is isolated, while the elements $\{(0,a) : a \in R_2^*\}$ form the subgraph $K_{|R_2^*|}$. So $\overline{\Gamma(R)}$ is Eulerian if and only if $|R_2|$ is even. If $R = R_1 \times R_2$ with R_1 and R_2 integral domains and $|R_i| > 2$ for $i = 1, 2$, then $\overline{\Gamma(R)}$ is $K_{|R_1^*|} \cup K_{|R_2^*|}$, which cannot be Eulerian.

Now, we discuss the cases in which $\overline{\Gamma(R)}$ is connected.

Theorem 14 *If R is a finite non-local ring with odd order, then $\overline{\Gamma(R)}$ is not Eulerian.*

Proof. Assume $R = R_1 \times R_2$ with $|R|$ odd and R_1 not an integral domain. Then $\overline{\Gamma(R)}$ is connected. If R has a non-zero nilpotent element (x, y), then $N((x,y) = Z(R) \setminus Ann((x,y))$ and $N((x,1)) = (Z(R) \setminus Ann((x,1))) \setminus \{(x,1)\}$, which implies that $\deg((x,y))$ and $\deg((x,1))$ cannot both be even, and so $\overline{\Gamma(R)}$ is not Eulerian. If R has no non-zero nilpotents, then R is a product of fields, and to get a connected graph, R must be a product of at least 3 fields, say $R = \prod_{k=1}^{m} R_k$ with $m > 2$. For each k, $U(R_k) = R_k^*$ has an even number of elements, which implies that $Z(R) = R \setminus (U(R)) = R \setminus \prod_{k=1}^{m} U(R_k)$, which has an odd number of elements. Now for each $(x_k) \in Z^*(R)$, $N((x_k)) = (Z(R) \setminus Ann((x_k))) \setminus \{(x_k)\}$ which always has an odd number of elements; so $\overline{\Gamma(R)}$ is not Eulerian. ■

Theorem 15 *If R is a finite non-local ring with even order, then $\overline{\Gamma(R)}$ is Eulerian if and only if $R = \mathbb{Z}_2 \times R_2$ with R_2 an integral domain of even order or R is a product of at least 3 fields, each of which has even order.*

Proof. The case of a non-connected graph was proved in the paragraph after Theorem 13. Assume $R = R_1 \times R_2$ with $|R|$ even and R_1 not an integral domain. Assume first that $|R_1|$ is even and $|R_2|$ is odd. Then $Ann((1,0)) = \{0\} \times R_2$ and $Ann((0,1)) = R_1 \times \{0\}$; so $\deg((1,0)) = |Z(R)| - |R_2| - 1$ and $\deg((0,1)) = |Z(R)| - |R_1| - 1$, which cannot both be even, hence $\overline{\Gamma(R)}$ is not Eulerian. Now we can assume that R is a product of local rings, each of which has order 2^{n_k} with $n_k > 1$ for each k. If R has a non-zero nilpotent element (x, y), then $N((x,y) = Z(R) \setminus Ann((x,y))$ and $N((x,1)) = (Z(R) \setminus Ann((x,1))) \setminus \{(x,1)\}$, which implies that $\deg((x,y))$ and $\deg((x,1))$ cannot both be even, and hence $\overline{\Gamma(R)}$ is not Eulerian. So assume that R is a product of at least 3 fields, each of order 2^{n_k}. For each k, $U(R_k) = R_k^*$ has an odd number of elements, which implies that $Z(R) = R \setminus (U(R)) = R \setminus \prod_{k=1}^{m} U(R_k)$ has an odd number of elements. Now for each $(x_k) \in Z^*(R)$, $N((x_k)) = (Z(R) \setminus Ann((x_k))) \setminus \{(x_k)\}$ always has an even number of elements; so $\overline{\Gamma(R)}$ is Eulerian. ∎

Now we can deduce Theorem 8.2 in [6] that $\overline{\Gamma(\mathbb{Z}_n)}$ is Eulerian if and only if $n = p^2$ for some prime p.

5 When is $L(\overline{\Gamma(R)})$ Eulerian?

If R is a finite local ring with maximal ideal M, then $L(\overline{\Gamma(R)})$ is connected, since $\Gamma(R)$ is the union of the discrete subgraph $Ann(M)$ and the connected subgraph $\overline{M \setminus Ann(M)}$. Clearly if M has an element y such that $y^2 \neq 0$, then $L(\overline{\Gamma(R)})$ cannot be Eulerian; hence $L(\overline{\Gamma(R)})$ is Eulerian if and only if $x^2 = 0$ for all $x \in M$.

If R is a product of local rings and has non-zero nilpotent elements, then $L(\overline{\Gamma(R)})$ cannot be Eulerian. So R must be a product of fields. If R is a product of two fields with neither of them is $\mathbb{Z}_2$, then $L(\overline{\Gamma(R)})$ cannot be Eulerian. If these fields have even and odd orders, then the degrees of the vertices of $L(\overline{\Gamma(R)})$ are odd and even, and so $L(\overline{\Gamma(R)})$ cannot be Eulerian. So, assume that R is a product of fields with all of their orders even or all odd. Hence we have the following result.

Theorem 16 *(1) Let R be a finite local ring with maximal ideal M. Then $L(\overline{\Gamma(R)})$ is Eulerian if and only if $x^2 = 0$ for each $x \in M$.*

(2) Let R be a finite non-local ring. Then $L(\overline{\Gamma(R)})$ is Eulerian if and only if $R = \mathbb{Z}_2 \times F$, where F is a field or R is a product of at least 3 fields, each of which is of even order or each of which is of odd order.

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