Maximum Genus Embeddings and Genus Embeddings on Orientable Surfaces [1]

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Abstract: In this paper, the estimations of maximum genus orientable embeddings of graphs are studied, and an exponential lower bound for such numbers are found. Moreover, such two extremal embeddings (*i.e.*, the maximum genus orientable embedding of the current graph and the minimum genus orientable embedding of the complete graph) are sometimes closely related with each other. As applications, we estimate the number of the minimum genus orientable embeddings for complete graph, by estimating the number of maximum genus orientable embedings for current graph.

Key words Maximum genus embedding; Minimum genus embedding; Complete graph; Current graph.

MR(2000) Classification 05C10

I. Introduction

In this paper, all graphs are simple connected. Concepts and terms are standard and follow from[1]. A spanning tree T in a graph G is called *optimal* if the number of odd components, denoted by $\omega(T)$, of $G - T$ is smallest among spanning trees of G. It is well known that $\omega(T)$ is much related to graph embedding, especially in the maximum orientable embedding of graphs (or MOGE in short).

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There are many results on the maximum genus embedding, for example, Liu[2,3] has studied those problem; M.Skoviera[4] has studied those problem. But, there are few results on the estimation of maximum genus orientable embedding. S.Stahl[5] has proved that the number of maximum genus orientable embeddings of almost all graphs G is not less than $(d_1-5)!(d_2-5)!(d_3-5)!(d_4-5)!\prod_{i=5}^{v}(d_i-2)!$ where v is the number of vertices of G; $d_1, d_2, ..., d_p$ is a suitable recordering of the degree sequence of G; If m is a non-positive integer, $m! = 1$. In this paper, we find an exponential lower bound of the number of maximum genus orientable embeddings. For cubic graphs, our results are better than the results of S.Stahl[5]. For cubic graphs, Theorem 4 of this paper give out an exponential lower bound of the number of maximum genus orientable embeddings; but the lower bound of S.Stahl[5] is 1.

In history, finding a genus embedding for a complete graph was a long and difficult way, as surveyed in Ringel's monograph[6], completed the proof of the well known Heawood Conjecture and gave the birth of modern topological graph. Since then, few attention has been payed to the estimation the number of such embedding, until very recently some sparse, but crucial, progresses were made. In S.Lawrencenko[7], K_7 has an unique genus embedding on orientable surfaces; S.lawrencenko, S.Negami and A.T.White[8], K_{19} has at least three genus embeddings on orientable surfaces; C.P.Bonningtong, M.J.Grannel, T.S.Griggs and J.Siran[9], for $s \geq 2$, K_{12s+7} has at least two genus embeddings on orientable surfaces; C.P.Bonningtong, M.J.Grannel, T.S.Griggs and J.Siran[10], for $n \equiv 7, 19(mod36)$ and $n \equiv 19, 55(mod108)$, K_n has, respectively, $2^{\frac{n^2}{54}-O(n)}$ and $2^{\frac{2n^2}{81}-O(n)}$ genus embeddings on orientable surfaces; V.Korzhik and H.J.Voss[11], both K_{12s+4} and K_{12s+7} has at least 4^s genus embeddings on orientable surfaces; V.Korzhik and H.J.Voss[12], for any $i \in \{1, 2, 3, ..., 11\} - \{3, 4, 7\}$, K_{12+i} (for $s \geq d(i) \in 1, 2, 3, 4$), has at least $h(i)4^s$ distinct genus embeddings on orientable surfaces; Ren and Bai[13], the complete graph K_n with order $n \equiv 4, 7, 10(mod2)$ has at least $C2^{\frac{n}{4}}$ distinct genus embeddings on orientable surfaces, where $C = 1, 2^{\frac{-3}{4}}$ or $2^{-\frac{3}{2}}$ with respect to $n \equiv 4, 7, 10(mod12)$, respectively.

II. Main Result

Theorem 1[14,15] The maximum genus of a graph G is

$$\gamma_M(G) = \frac{\beta(G) - \omega(T)}{2}$$

where $\beta(G)$ is the Betti-number of G and $\omega(T)$ is the number of odd components in an optimal tree T in G.

Theorem 2[14,15] Let T be an optimal tree in a graph G with $\omega(T)$ odd components in $G - T$. Then edges of $E(G) - E(T)$ may be partitioned as follows

$$E(G) - E(T) = \bigcup_{i=1}^{s}\{e_{2i-1}, e_{2i}\} \bigcup \{f_1, f_2, ..., f_m\}$$

where for each $i : 1 \leq i \leq s$, $e_{2i-1} \bigcap e_{2i} \neq \phi$ and $\{f_1, f_2, ..., f_m\}$ is a matching of G and $s = \gamma_M(G)$, $m = \omega(T)$.

Theorem 3 For any graph $G = (V, E)$,$v \in V$,$d(v) \geq 3$, let T be an optimal of G with an edge-partition

$$E(G) - E(T) = \bigcup_{i=1}^{s}\{e_{2i-1}, e_{2i}\} \bigcup \{f_1, f_2, ..., f_m\}$$

such that $e_i = (x_i, y_i)$, $1 \leq i \leq 2s$ and $x_{2i-1} = x_{2i} (1 \leq i \leq s)$. Then there are at least

$$4^m \prod_{i=1}^{s} d_{G_{i-1}}(x_{2i-1}) d_{G_{i-1}}(y_{2i-1}) d_{G_{i-1}}(y_{2i})$$

distinct MOGE for G,where $G_i = G_{i-1} + \{e_{2i-1}, e_{2i}\}$ $(1 \leq i \leq s)$ and $G_0 = T$.

Proof Consider an optimal tree T of G, $v_1, v_2, ..., v_\alpha$ being its inner vertices such that each v_i $(1 \leq i \leq \alpha)$ has a rotation scheme π_i of edges of T incident to it. Then $\pi_1, \pi_2, ..., \pi_\alpha$ determine a planar embedding of T with a single region (face) whose boundary is W_0. Let $e_i = (x_i, y_i)(i \leq i \leq 2s)$ and $x_j \in e_{2j-1} \bigcap e_{2j} (1 \leq j \leq s)$ be as assumed in Theorem 3.

Consider the vertex $x_1 \in e_1 \bigcap e_2$ and fix it at a copy of x_1 on W_0. Then choose two copies of y_1 and y_2, respectively, on W_0 (note that there are,respectively, $d_{G_0}(x_1)$, $d_{G_0}(y_1)$ and $d_{G_0}(y_2)$ ways to do so), where $G_0 = T$. Then we find a new facial walk W_1 (containing W_0 and the edges e_1, e_2) by changing the local rotation at x_1 as shown in Fig.1.

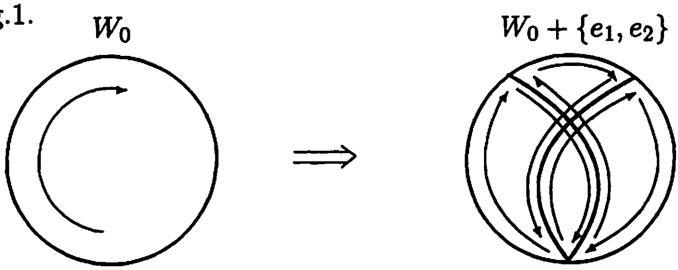

Fig.1 Adding a pair of "V" type of edges into a single face will result in another one-face graph in higher surface

Now we obtain $G_1 = G_0 + \{e_1, e_2\}$ to have its maximum genus embedding on surface S_1 with a single region (face) bounded by edges of W_1. It is clear that there are at least $d_{G_0}(x_1)d_{G_0}(y_1)d_{G_0}(y_2)$ ways to construct the facial walk W_1. Repeat this procedure until we arrive at G_s and W_s, the since for any pair of edges $f_i, f_j \in \{f_1, f_2, ..., f_m\}, f_i \bigcap f_j = \phi$, we see that there are $d_{G_s}(x_{2s+i})d_{G_s}(y_{2s+i}) (\geq 2 \times 2 = 4)$ ways to add f_i into W_s, $1 \leq i \leq m$. This completes the proof of Theorem 3. □

Applying Theorem 3, we have the following

Theorem 4[13] Let G be a cubic graph of order n with an optimal tree T having α inner vertices and $\omega(T)$ odd components in $E(G) - E(T)$. Then G has at least $(\sqrt{2})^{n+\frac{\alpha}{2}+\omega(T)}$ MOGE for G. In particular, if $E(G) - E(T)$ induces an m-cycle or an m-path, then G has at least $(\sqrt{2})^{n+\alpha}$ distinct MOGE, where $n = |V(G)|$.

Rule Δ^* If in line i has: $...jk...$, then in line k must have: $...ij....$

Rule R^* If in row i has :$...jkl...$, then row k appears as : $...lij....$

Each of the current graphs has the property that the given rotation induces one single circuit. The circuit provides row 0, denote row 0 by $\langle 0 \rangle$.

Additive Rule: $\langle i \rangle = \langle 0 \rangle + i; * + i = *, * = x, y, z, ..., 1 \leq i \leq n$, the number row $\langle i \rangle$ is obtained; the letter row $(\langle x \rangle, \langle y \rangle, \langle z \rangle, ...)$ can be obtained from number rows by Rule Δ^*.

Theorem 5[6] S_p is an orientable surface, G can triangular embedded into S_p, if and only if there exists a rotation of G such that the scheme of this rotation satisfies Rule R^*.

Theorem 6[16] Triangular embeddings is minimum genus embeddings.

Theorem 7 For $s \geq 2$, s is a natural number. Then $n = 12s+1$, K_n has at least $2^{\frac{n-5}{4}}$ distinct minimum genus embeddings on orientable surfaces.

Proof Let $n = 12s-2$ and associated group is $Z_2 \times Z_{6s-1}$ which, together with a system of rules on current graphs. At each vertex of valency three, Kirchhoff's Current Law holds. The first component being represented by a heavily drawn arc if it is 1, and a lightly drawn one if it is 0. For the first component, at each vertex of valence three there are either two heavily drawn arcs or none. The distribution of the second component, which consists of the nonzero elements of the group Z_{6s-1}.

If s is an odd number, the currents on the rungs are

$$3s-3, \overline{3s-6}, ..., 12, \overline{9}, 6, \overline{3}, \overline{1}, 3s-3, 6, \overline{9}, 12, ..., \overline{3s-6}.$$

The bar means that the orientation on the rung points down.

If s is a even number, the sequence of the currents for the rungs is

$$3s-3, ..., \overline{12}, 9, \overline{6}, 3, \overline{1}, 3s-6, 6, \overline{9}, 12, ..., \overline{3s-9}, \overline{3s-3}.$$

The currents graphs are, respectively

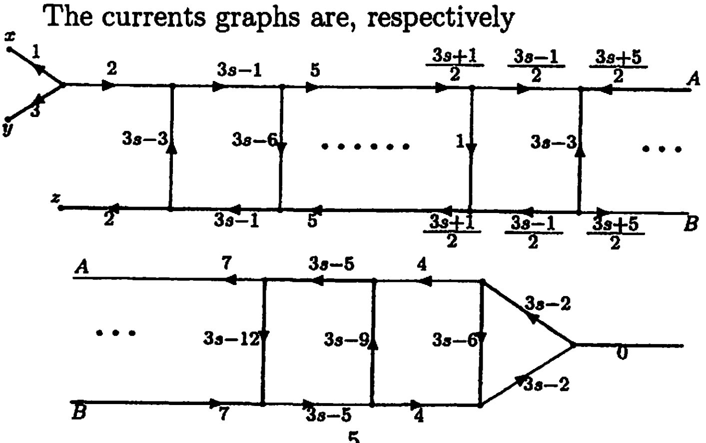

Fig.2 A current graph G with the group $Z_2 \times Z_{6s-1} (s \equiv 1 (mod 2))$

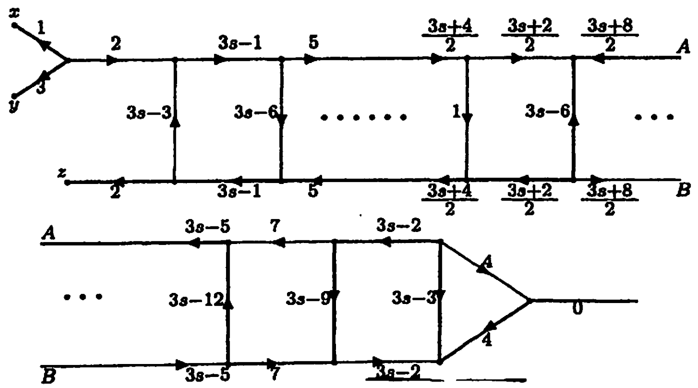

Fig.3 A current graph G with the group $Z_2 \times Z_{6s-1}(s \equiv 0(mod2))$

The current graph G has the property that the given rotation induces one single circuit. The log of the circuit provides row 0 and all the other rows are determined by the additive Rule(or for the extra row x, y, z by Rule R^*). We say row 0 generates the whole scheme. The scheme satisfies Rule R^*. According to Theorem 5, the scheme presents a triangular embeddings of the graph $K_n - K_3$ into an orientable surface S_p. We can only use one handle to connected the vertices x, y, z with each other by an arc. We can handle it in the following way. Consider the dual map of the embedding $K_n - K_3$ into orientable surface S_p. This map has only vertices of valence 3 and the countries x, y, z are not adjacent to each other. Consider another map on a torus, this map has only vertices of valence 3 and five countries, the five countries are denoted by $x, y, z, t, 0$, and the five countries are adjacent to each other. Go to the surface S_p and excise the country 0 of the dual map of the embedding $K_n - K_3$ into S_p. Do the same with country t of the torus. Identify the boundaries of the two resulting surfaces in the obvious way. After this there is a new country named 0 that is adjacent to the same countries as the old county 0 as before. Notice that we have gained the adjacencies between x, y and z. So we have constructed a map with n mutually adjacent countries on the orientable surface S_{p+1}. Consider the dual map of this map. We have a triangular embedding of the graph K_n into an orientable surface S_{p+1}. According to Theorem 6, the triangular embedding is minimum genus embedding.

If consider an optimal tee T of the current graph G, the component $E(G) - E(T)$ is a path with exactly $2s - 1$ vertices.It is clear that there are $2s + 1$ inner vertices in T , except two inner vertices valency two, all other inner vertices of T having valency three.

By Theorem 3 and 4, there are at least $2^{\frac{n-5}{4}}$ distinct minimum genus embedding on orientable surfaces. This completes the proof of Theorem 7. □

Theorem 8 If s is a natural number, $n = 12s + 9$. Then K_n has at least $2^{\frac{n-5}{4}}$ distinct minimum genus embeddings on orientable surfaces.

Proof Consider the current graph G of Fig 4 using the group Z_{12s+8}, the elements $1, \hat{2}, 3, ..., 6s + 4$ are used as currents. As usual the notation $\hat{2}$ means that the element 2 is excluded. Therefore row 0 of the produced scheme doses not contain the elements 2 and -2.

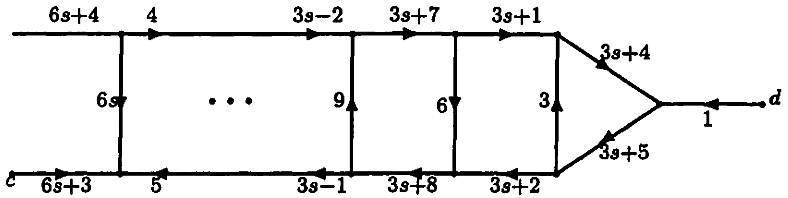

Fig.4 A current graph G with the group Z_{12s+8}

From this scheme we produce a very useful scheme by the following operation in row j:

If j is even insert $2 + j$,x,$j - 2$ in place of c and omit d.

If j is odd insert $j - 2$,x,$j + 2$ in place of d and omit c.

After doing this we obtain the scheme, the Rule R^* holds in the final scheme. Similar to the proof of Theorem 7, the final scheme determines a minimum genus embedding on orientable surface.

If consider an optimal tree T of the current graph G, the component $E(G) - E(T)$ is a path with exactly $2s + 1$ vertices. It is clear that there are $2s + 2$ inner vertices in T, except two inner vertices valency two, all other inner vertices of T having valency three. By Theorem 3 and Theorem 4, there are at least $2^{\frac{n-5}{4}}$ distinct minimum genus embedding on orientable surface. This complete the proof of Theorem 8. □

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