Distance labellings of graphs

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Abstract

An $L(2,1)$-labelling of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x,y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x,y) = 2$, where $d(x,y)$ denotes the distance between x and y in G. The $L(2,1)-$labelling number $\lambda(G)$ of G is the smallest number k such that G has an $L(2,1)-$labelling with $\max\{f(v) : v \in V(G)\} = k$. Griggs and Yeh conjecture that $\lambda(G) \leq \Delta^2$ for any simple graph with maximum degree $\Delta \geq 2$. This article considers the graphs formed by the cartesian product of $n(n \geq 2)$ graphs. The new graph satisfies the above conjecture (with minor exceptions). Moreover, we generalize our results in [19].

1 Introduction

The frequency assignment problem is assigning frequencies one to each radio transmitter so that interfering transmitters are assigned frequencies whose separation is not in a set of disallowed separations. Hale [12] formulated this into a graph vertex coloring problem.

In a private communication with Griggs, Roberts proposed a variation of the channel assignment problem in which "close" transmitters must receive different channels and "very close" transmitters must receive channels that are at least two channels apart. To translate the problem into the language of graph theory, the transmitters are represented by the vertices of a graph; two vertices are "very close" if they are adjacent and "close" if their distance is two in the graph. Motivated by this problem, Griggs and

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Yeh [11] proposed the following labelling on a simple graph. An $L(2,1)$-*labeling* of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x,y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x,y) = 2$, where $d(x,y)$ denotes the distance between x and y in G. A k-$L(2,1)$-*labeling* is an $L(2,1)$-labelling such that no label is greater than k. The $L(2,1)$-*labelling number* of G, denoted by $\lambda(G)$, is the smallest number k such that G has a k-$L(2,1)$-labelling.

There are considerable articles studying the $L(2,1)$-labellings (See references.) Most of papers are considering the values of λ on particular classes of graphs. Griggs and Yeh [11] found an upper bound $\Delta^2+2\Delta$ for a general graph with the maximum degree Δ. Later, Chang and Kuo [4] improved the bound to $\Delta^2+\Delta$. Recently, Král' and Škrekovski [14] reduce the bound to $\Delta^2+\Delta-1$. If the diameter of G is two, then $\lambda(G) \leq \Delta^2$. The upper bound is attainable by *Moore graphs* (diameter 2 graph with order Δ^2+1). (Such graphs exist only if $\Delta = 2, 3, 7$, and possibly 57.) (cf. [11]) Thus Griggs and Yeh [11] conjectured that the best bound is Δ^2 for any graph G with the maximum degree $\Delta \geq 2$ (cf. [11]). (It is not true for $\Delta = 1$. For example, $\Delta(K_2) = 1$ but $\lambda(K_2) = 2$.) To determine the value of λ is proved to be NP-complete. (cf. [11])

Graph products play an important role in connecting many useful networks. In [19], we considered the graph formed by the cartesian product and the composition of two graphs and prove that the $L(2,1)$-labelling number of graph is bounded by the square of its maximum degree. Hence Griggs and Yeh's conjecture holds in both cases (with minor exceptions). In this article, we study the graph formed by the cartesian product of n graphs. The $L(2,1)$-labelling number of graph is bounded by the square of its maximum degree that satisfying Griggs and Yeh's conjecture (with minor exceptions). Moreover, we generalize our results in [19].

2 A Labelling Algorithm

A subset X of $V(G)$ is called an *i-stable set* (or *i-independent set*), if the distance between any two vertices in X is greater than i. An 1-stable (independent) set is a usual independent set. A *maximal* 2-stable subset X of a set Y is a 2-stable subset of Y but X is not a proper subset of any 2-stable subset of X.

Chang and Kuo [4] proposed the following algorithm to obtain an upper bound of the λ-numbers of a given graph.

Algorithm 2.1.

Input: A graph $G = (V, E)$.

Output: The value k is the maximum label.

Idea: In each step, find a maximal 2-stable set from these unlabelled vertices that are distance two away from those vertices labelled in the previous step. Then label all vertices in that 2-stable with the index i in current stage. The index i starts from 0 and then is increasing 1 in each step. The maximum label k is the final value of i.

Initialization: Set $X_{-1} = \emptyset$; $V = V(G)$; $i = 0$.

Iteration:

1. Determine Y_i and X_i.
    - $Y_i = \{x \in V : x$ is unlabelled and $d(x, y) \geq 2$ for all $y \in X_{i-1}\}$.
    - X_i a maximal 2-stable subset of Y_i.
    - If $Y_i = \emptyset$ then set $X_i = \emptyset$.
2. Label these vertices in X_i (if there is any) by i.
3. $V \leftarrow V \backslash X_i$.
4. $V \neq \emptyset$ then $i \leftarrow i + 1$, go to Step 1.
5. Record the current i as k (which is the maximum label). Stop.

Thus k is an upper bound on $\lambda(G)$. Analogously to the chromatic number $\chi(G)$, we would like to find a bound in terms of the maximum degree $\Delta(G)$ of G.

Let x be a vertex with the largest label k obtained by Algorithm 2.1. Denote

$I_1 = \{i : 0 \leq i \leq k-1 \text{ and } d(x,y) = 1 \text{ for some } y \in X_i\}$

$I_2 = \{i : 0 \leq i \leq k-1 \text{ and } d(x,y) \leq 2 \text{ for some } y \in X_i\}$

$I_3 = \{i : 0 \leq i \leq k-1 \text{ and } d(x,y) \geq 3 \text{ for all } y \in X_i\}$

It is clear that $|I_2| + |I_3| = k$.

For any $i \in I_3$, $x \notin Y_i$; otherwise $X_i \cup \{x\}$ is a 2-stable subset of Y_i, which contradicts the choice of X_i. That is, $d(x,y) = 1$ for some vertex y in X_{i-1}; i.e., $i-1 \in I_1$. So, $|I_3| \leq |I_1|$. Hence $k = |I_2| + |I_3| \leq |I_2| + |I_1|$.

In order to find k, it suffices to estimate $B = |I_1| + |I_2|$ in terms of $\Delta(G)$. We will investigate the value B with respect to different cases. For convenience's sake, the notations used in this section are the same as in the following sections.

3 The Cartesian Product of Graphs

The *cartesian product* of two graphs G and H is the graph $G \,\square\, H$ with vertex set $V(G) \times V(H)$, in which the vertex (v, w) is adjacent to the vertex (v', w') if and only if either $v = v'$ and w is adjacent to w' or $w = w'$ and v is adjacent to v'. (See Figure 1 for an example.)

This product (that is commutative and associative in a natural way) is among the most important graph products, with potential applications.

Let $P_{G_1} = G_1, P_{G_1,G_2,...,G_k} = P_{G_1,G_2,...,G_{k-1}} \square G_k, k = 2, ..., n$, the cartesian product of $n (n \geq 2)$ graphs G_1, G_2,..., G_n, $P_{G_1,G_2,...,G_n}$ is the graph $G_1 \,\square\, G_2 \,\square\, ... \square G_n$ with vertex set $V(G_1) \times V(G_2) \times ... \times V(G_n)$ which is a natural generalization of the cartesian product of two graphs.

Whittlesey et al. [20] first consider the labeling on product of paths. Later, Jha et al. [13] gave some values of λ for product of a path and a cycle as well as product of cycles. The product of complete graphs has also been considered by Georges et al. [9]. Chang et al. [6] considered the no-hole 2-distant coloring of Hamming graphs which are the product of complete graphs.

In this section, we obtain an upper bound in terms of the maximum degree on the cartesian product of any n graphs.

Theorem 3.1 *Let Δ, Δ_1, Δ_2, ..., Δ_n be the maximum degree of $P_{G_1,G_2,...,G_n}$, G_1, G_2,..., G_n respectively. For $\Delta_k, k = 1, 2, ..., n$, let t of them be 1 and others be equal or greater than 2. Then $\lambda(P_{G_1,G_2,...,G_n}) \leq \Delta^2 - (n-2)\Delta + \frac{t(2n-t-1)}{2}$.*

Proof. We first apply Algorithm 2.1 to label the graph $P_{G_1,G_2,...,G_n}$ and let k be the maximum label obtained by the algorithm. Let $x = (u_1, u_2, ..., u_n)$ in $V(G_1) \times V(G_2) \times ... \times V(G_n)$ be a vertex with the label k. Then $deg_{P_{G_1,G_2,...,G_n}}(x) = deg_{G_1}(u_1) + deg_{G_2}(u_2) + ... + deg_{G_n}(u_n)$. Denote $d = deg_{P_{G_1,G_2,...,G_n}}(x)$, $d_1 = deg_{G_1}(u_1)$, $d_2 = deg_{G_2}(u_2)$, ..., $d_n = deg_{G_n}(u_n)$. $\Delta_1 = \Delta(G_1)$, $\Delta_2 = \Delta(G_2)$, ..., $\Delta_n = \Delta(G_n)$.

Hence $d = d_1+d_2+...+d_n$ and $\Delta = \Delta(P_{G_1,G_2,...,G_n}) = \Delta_1+\Delta_2+...+\Delta_n$.

A neighbor $(u_1', u_2', ..., u_n')$ of $(u_1, u_2, ..., u_n)$ is called a *G_p-neighbor* if $u_q = u_q'$ in $G_q, q = 1, 2, .., n, q \neq p$ and u_p is adjacent to u_p' in G_p. For each G_p-neighbor of x, there is an G_q-neighbor of x such that they have a common neighbor other than x in $P_{G_1,G_2,...,G_n}$ where $q = 1, 2, ..., n, q \neq p$. By the definition of $P_{G_1,G_2,...,G_n}$, we have $\sum_{k,l=1,2,...,n}^{k<l} d_k d_l$ such "common neighbors".

Let the number of vertices that are distance 2 from x be $d(\Delta-1)-r$ for some $r \geq 0$. (The number $d(\Delta - 1)$ is the best possible.) If two neighbors of x has one common neighbor other than x then it will contribute 1 to r. Hence the number of vertices that are distance 2 from x is at most

$$d(\Delta - 1) - \sum_{k,l=1,2,...,n}^{k<l} d_k d_l = (\sum_{k=1}^{n} d_k)(\sum_{k=1}^{n} \Delta_k - 1) - \sum_{k,l=1,2,...,n}^{k<l} d_k d_l.$$

Hence $|I_1| \leq d, |I_2| \leq d+d(\Delta-1)- \sum_{k,l=1,2,...,n}^{k<l} d_k d_l = d\Delta - \sum_{k,l=1,2,...,n}^{k<l} d_k d_l$.

Then $B = |I_1|+|I_2| \leq d+d\Delta - \sum_{k,l=1,2,...,n}^{k<l} d_k d_l = d(\Delta+1) - \sum_{k,l=1,2,...,n}^{k<l} d_k d_l = (\sum_{k=1}^{n} d_k)(\sum_{k=1}^{n} \Delta_k + 1) - \sum_{k,l=1,2,...,n}^{k<l} d_k d_l$. Define

$$f(x_1, x_2, ..., x_n) = (\sum_{k=1}^{n} x_k)(\sum_{k=1}^{n} \Delta_k + 1) - \sum_{k,l=1,2,...,n}^{k<l} x_k x_l$$

And let

$f_p'(x_1, x_2, ..., x_n) = (\sum_{k=1}^{n} \Delta_k + 1) - \sum_{k=1,2,...,n}^{k\neq p} x_k$

$= (\sum_{k=1}^{n} \Delta_k + 1) - (\sum_{k=1}^{n} x_k - x_p) = 0$, where $p = 1, 2, ..., n$.

Then

$\sum_{k=1}^{n} x_k - (\sum_{k=1}^{n} \Delta_k + 1) = x_p (= x_1 = x_2 = ... = x_n) \Rightarrow n x_p - (\sum_{k=1}^{n} \Delta_k + 1) = x_p \Rightarrow x_p = (\sum_{k=1}^{n} \Delta_k + 1)/(n-1) \Rightarrow n x_p = (\frac{n}{n-1})(\sum_{k=1}^{n} \Delta_k + 1) > (\sum_{k=1}^{n} \Delta_k + 1)$

But $0 \leq x_p \leq \Delta_p \Rightarrow 0 \leq n x_p \leq \sum_{k=1}^{n} \Delta_k$. By contradiction, there doesn't exist $(x_1, x_2, ..., x_n)$ in the domain of $x_1, x_2, ..., x_n$ such that $f_p'(x_1, x_2, ..., x_n) = 0$, where $p = 1, 2, ..., n$. Hence $f(x_1, x_2, ..., x_n)$ has the absolute maximum at the boundary $(\Delta_1, \Delta_2, ..., \Delta_n)$ on $[0, \Delta_1] \times [0, \Delta_2] \times ... \times [0, \Delta_n]$.

$f(\Delta_1, \Delta_2, ..., \Delta_n) = (\sum_{k=1}^{n} \Delta_k)(\sum_{k=1}^{n} \Delta_k + 1) - \sum_{k,l=1,2,...,n}^{k<l} \Delta_k \Delta_l = \Delta(\Delta + 1) - \sum_{k,l=1,2,...,n}^{k<l} \Delta_k \Delta_l$.

Then $\lambda(P_{G_1,G_2,...,G_n}) \leq k \leq B \leq \Delta^2 + \Delta - \sum_{k,l=1,2,...,n}^{k<l} \Delta_k \Delta_l$.

Case 1. If one of Δ_k or Δ_l is 1 then

$(\Delta_k - 1)(\Delta_l - 1) = \Delta_k \cdot \Delta_l - \Delta_k - \Delta_l + 1 = 0$.

This implies $\Delta_k + \Delta_l - \Delta_k \cdot \Delta_l = 1$.

Case 2. Suppose $\Delta_k \geq 2$ and $\Delta_l \geq 2$. Then

$(\Delta_k - 1)(\Delta_l - 1) = \Delta_k \cdot \Delta_l - \Delta_k - \Delta_l + 1 \geq 1$.

This implies $\Delta_k + \Delta_l - \Delta_k \cdot \Delta_l \leq 0$.

For $\Delta_k, k = 1, 2, ..., n$, let t of them be 1 and others be equal or greater than 2, by the analysis of of Case 1 and Case 2, then

$$\sum_{k,l=1,2,\dots,n}^{k\neq l} (\Delta_k + \Delta_l - \Delta_k \cdot \Delta_l) \leq t(n-1) + t(n-t) = t(2n-t-1)$$

But $\sum_{k,l=1,2,\dots,n}^{k\neq l} (\Delta_k + \Delta_l - \Delta_k \cdot \Delta_l) = 2(n-1)\sum_{k=1}^{n} \Delta_k - \sum_{k,l=1,2,\dots,n}^{k\neq l} \Delta_k\Delta_l =$
$2(n-1)\Delta - \sum_{k,l=1,2,\dots,n}^{k\neq l} \Delta_k\Delta_l$

Hence $\sum_{k,l=1,2,\dots,n}^{k\neq l} \Delta_k\Delta_l \geq 2(n-1)\Delta - t(2n-t-1)$

Hence $\sum_{k,l=1,2,\dots,n}^{k<l} \Delta_k\Delta_l \geq (n-1)\Delta - \frac{t(2n-t-1)}{2}$

Hence
$\lambda(P_{G_1,G_2,\dots,G_n}) \leq \Delta^2 + \Delta - \sum_{k,l=1,2,\dots,n}^{k<l} \Delta_k\Delta_l \leq \Delta^2 - (n-2)\Delta + \frac{t(2n-t-1)}{2}$.

Therefore the results follow. ■

For $\Delta_k \geq 1, k = 1, 2, \dots, n$, then $\Delta \geq n$.

Define $g(t) = -(n-2)\Delta + \frac{t(2n-t-1)}{2}$, where $0 \leq t \leq n$.

Let $g'(t) = \frac{2n-2t-1}{2} = 0$, then $t = n - \frac{1}{2}$.

Notice that t is an integer and $g(n) = g(n-1) = -(n-2)\Delta + \frac{n(n-1)}{2}$, hence

$$-(n-2)\Delta + \frac{t(2n-t-1)}{2} \leq -n(n-2) + \frac{n(n-1)}{2} = -\frac{n(n-3)}{2} = \begin{cases} 1 & (n=2) \\ 0 & (n=3) \\ <0 & (n \geq 4) \end{cases}.$$

Hence $\lambda(P_{G_1,G_2,\dots,G_n}) \begin{cases} \leq \Delta^2 + 1 & (n=2) \\ \leq \Delta^2 & (n=3) \\ \leq \Delta^2 - \frac{(n-3)}{2}\Delta & (n \geq 4) \end{cases}.$

Moreover, if we require that $\Delta_k \geq 2, k = 1, 2, \dots, n$, then $\lambda(P_{G_1,G_2,\dots,G_n}) \leq \Delta^2 - (n-2)\Delta$.

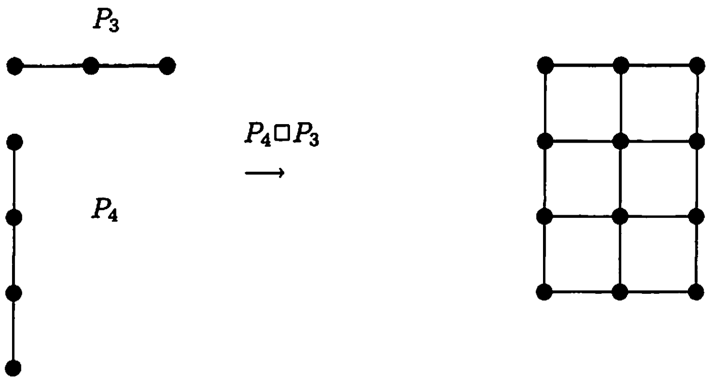

Figure 1. (*Cartesian*) product of 2 Graphs

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