A Note on Leaf-constrained Spanning Trees in a Graph

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Abstract

An independent set S of a connected graph G is called a *frame* if $G - S$ is connected. If $|S| = k$, then S is called a *k-frame*. We prove the following theorem. Let $k \geq 2$ be an integer, G be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, and $\deg_G(u)$ denote the degree of a vertex u. Suppose that for every 3-frame $S = \{v_a, v_b, v_c\}$ such that $1 \leq a < b < c \leq n$, $\deg_G(v_a) \leq a$, $\deg_G(v_b) \leq b - 1$ and $\deg_G(v_c) \leq c - 2$, it holds that $\deg_G(v_a) + \deg_G(v_b) + \deg_G(v_c)$ $-|N_G(v_a) \cap N_G(v_b) \cap N_G(v_c)| \geq |G| - k + 1$. Then G has a spanning tree with at most k-leaves. Moreover, the condition is sharp. This theorem is a generalization of the results of E. Flandrin, H.A. Jung and H. Li (*Discrete Math.* 90 (1991), 41–52) and of A. Kyaw (*Australasian Journal of Combinatorics.* 37 (2007), 3–10) for traceability.

1 Introduction

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. In this paper, we consider only simple graphs, which has neither loops nor

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multiple edges. We write $|G|$ for the order of G, that is, $|G| = |V(G)|$. For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G, and by $N_G(v)$ the neighborhood of v in G. A vertex of degree one is called an *end-vertex*, and an end-vertex of a tree is usually called a *leaf*. Let X be a nonempty subset of $V(G)$. We write

$$N_G(X) = \bigcup_{x \in X} N_G(x) \qquad \text{and} \qquad \deg_G(X) = \sum_{x \in X} \deg_G(x).$$

The subgraph of G induced by X is denoted by $\langle X \rangle_G$. We write $G - X$ for $\langle V(G) - X \rangle_G$, and for a vertex v of G, write $G - v$ for $G - \{v\}$. For an integer $i \geq 1$, define

$$N_G(X; i) = \{x \in V(G);\ |N_G(x) \cap X| = i\}.$$

In particular,

$$N_G(\{u, v, w\}; 3) = N_G(u) \cap N_G(v) \cap N_G(w).$$

Let H be a sugraph of a graph G. If xy is an edge of G not contained in H, then $H + xy$ denotes the subgraph of G obtained from H by adding xy. For an edge uv of H, $H - uv$ is defined analogously. A subset $S \subseteq V(G)$ is called *independent* if no two vertices of S are adjacent in G. An independent set S of G is called a *frame* if $G - S$ is connected. A frame S with $|S| = k$ is called a *k-frame*. For sets X and Y, the cardinality of X is denote by $|X|$, and $X \setminus Y$ is denoted by $X - Y$ if $Y \subseteq X$. For further explanation of terminology and notation, we refer to [2].

In [4], E. Flandrin, H.A. Jung and H. Li obtained the following theorem for a graph to have a hamiltonian path.

Theorem 1 ([4]) *Let G be a connected graph. If $\deg_G(\{u, v, w\}) - |N_G(\{u, v, w\}; 3)| \geq |G| - 1$ for every independent set $\{u, v, w\}$ of G, then G has a hamiltonian path.*

A. Kyaw [5] improved the previous result in the following way.

Theorem 2 ([5]) *Let G be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Suppose that for every 3-frame $S = \{v_a, v_b, v_c\}$ of G such that $1 \leq a < b < c \leq n$, $\deg_G(v_a) \leq a$, $\deg_G(v_b) \leq b - 1$ and $\deg_G(v_c) \leq c - 2$, it holds that $\deg_G(\{v_a, v_b, v_c\}) - |N_G(\{v_a, v_b, v_c\}; 3)| \geq |G| - 1$. Then G has a hamiltonian path.*

Generalizing Theorem 1 and Theorem 2, we prove the following result.

Theorem 3 *Let $k \geq 2$ be an integer, and G be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Suppose that for every 3-frame $S = \{v_a, v_b, v_c\}$ of G such that $1 \leq a < b < c \leq n$, $\deg_G(v_a) \leq a$, $\deg_G(v_b) \leq b - 1$ and $\deg_G(v_c) \leq c - 2$, it holds that*

$$\deg_G(\{v_a, v_b, v_c\}) - |N_G(\{v_a, v_b, v_c\}; 3)| \geq |G| - k + 1. \tag{1}$$

Then G has a spanning tree with at most k leaves.

We first show that the condition $|G| - k + 1$ in (1) is sharp. Consider a complete bipartite graph $H = K_{m,m+k}$. It has no spanning tree with at most k leaves, and for any numbering of vertices of H, H satisfies $\deg_H(\{v_a, v_b, v_c\}) - |N_H(\{v_a, v_b, v_c\}; 3)| \geq |H| - k$. Hence the condition is sharp.

Since

$$\begin{aligned} & \deg_G(\{x, y, z\}) - |N_G(\{x, y, z\}; 3)| \\ = \ & \deg_G(\{x, y\}) + |N_G(z) - N_G(\{x, y, z\}; 3)|, \end{aligned}$$

Theorem 3 includes the following theorem of H. Broersma and H. Tuinstra [1].

Theorem 4 ([1]) *Let G be a connected graph. If $\deg_G(\{u, v\}) \geq |G| - k + 1$ for every independent set $\{u, v\}$ of G, then G has a spanning tree with at most k leaves.*

Some other results related to our theorem can be found in [3], [6], [7] and others.

2 Proof of Theorem 3

Let G be a connected graph. We call a tree T of G a *maximum tree with 3 leaves* if there exists no tree T' with 3 leaves in G such that $|T| < |T'|$. To prove Theorem 3, we need the following lemmas.

Lemma 5 *Suppose that a connected graph G has no hamiltonian path. Let T be a maximum tree with 3 leaves of G, which might be spanning, r be the unique vertex of T with $\deg_T(r) = 3$, and V_1, V_2, V_3 be the vertex sets of components of $T - r$. For every $1 \leq i \leq 3$, let u_i be the leaf of T contained in V_i, w_i be the vertex of V_i adjacent to r in T, and $U = \{u_1, u_2, u_3\}$. For each vertex $x \in V_i$, the vertex that precedes x on the path from r to x is denoted by x^-. Then the following holds:*

(i) U *is an independent set of* G.
(ii) For all two distinct $i, j \in \{1,2,3\}$, *if* $x \in V_i \cap N_G(u_j)$, *then* $x \neq w_i$ *and* $x^- \notin N_G(U - \{u_j\})$.
(iii) For every $1 \le i \le 3$, $|V_i| \ge 1 + \sum_{j=1}^{3} |N_G(u_j) \cap V_i| - |N_G(U;3) \cap V_i|$.
(iv) $|T| \ge 2 + \deg_G(U) - |N_G(U;3)|$.

Proof. (i) Suppose $u_iu_j \in E(G)$ for some $1 \le i < j \le 3$. Then $T' = T + u_iu_j - rw_i$ is a path of G. Since G does not have a hamiltonian path, there exist two adjacent vertices $x \in V(T')$ and $y \in V(G) - V(T')$. Then $T' + xy$ is a tree with 3 leaves, which contradicts the maximality of T. Hence (i) is proved.

(ii) Suppose that a vertex $x \in V_i$ is adjacent to u_j for some $j \neq i$. If $x = w_i$, then $T + w_iu_j - rw_i$ is a path of G, and as in the proof of (i) we derive a contradiction. Hence we may assume $x \neq w_i$. Then $T + xu_j - xx^-$ is a maximum tree with 3 leaves, whose leaf set is $U - u_j + x^-$. Thus by (i) x^- and $u_\ell \in U - \{u_j\}$ are not adjacent in G. Hence (ii) holds.

(iii) Let $\{i, j, \ell\} = \{1,2,3\}$ and $(N_G(\{u_j, u_\ell\}))^- = \{x^- : x \in N_G(\{u_j, u_\ell\})\}$. By (i) and (ii), it follows that $\{u_i\}$, $N_G(u_i) \cap V_i$, $(N_G(\{u_j, u_\ell\}))^- \cap V_i$ and $(N_G(u_j) \cap N_G(u_\ell) - N_G(U;3)) \cap V_i$ are pair-wise disjoint. So we have

$$\begin{aligned}
|V_i| &\ge |\{u_i\}| + |N_G(u_i) \cap V_i| + |(N_G(\{u_j, u_\ell\}))^- \cap V_i| \\
&\quad + |(N_G(u_j) \cap N_G(u_\ell) - N_G(U;3)) \cap V_i| \\
&= 1 + |N_G(u_i) \cap V_i| + |(N_G(\{u_j, u_\ell\})) \cap V_i| \\
&\quad + |(N_G(u_j) \cap N_G(u_\ell) - N_G(U;3)) \cap V_i| \\
&= 1 + \sum_{j=1}^{3} |N_G(u_j) \cap V_i| - |(N_G(U;3) \cap V_i)|.
\end{aligned}$$

(iv) Since $2 \ge \sum_{j=1}^{3} |N_G(u_j) \cap \{r\}| - |N_G(U;3) \cap \{r\}|$, and by (iii) we obtain

$$\begin{aligned}
\sum_{i=1}^{3} |V_i| + 1 &\ge 2 + \sum_{i=1}^{3}\sum_{j=1}^{3} |N_G(u_j) \cap V_i| - \sum_{i=1}^{3} |N_G(U;3) \cap V_i| \\
&\quad + \sum_{j=1}^{3} |N_G(u_j) \cap \{r\}| - |N_G(U;3) \cap \{r\}| \\
&= 2 + \sum_{j=1}^{3} |N_G(u_j) \cap V(T)| - |N_G(U;3) \cap V(T)|.
\end{aligned}$$

Since $N_G(u_j) \subset V(T)$ for every $1 \leq j \leq 3$ by the maximality of T, we have

$$|T| = \sum_{i=1}^{3} |V_i| + 1 \geq 2 + \sum_{j=1}^{3} |N_G(u_j)| - |N_G(U;3)|.$$

□

By using Lemma 5, we can measure the order of a tree with at most 3 leaves in G.

Lemma 6 *Let $m \geq 1$ be an integer and G be a connected graph with the vertex set $\{v_1, v_2, \ldots, v_n\}$. Assume that for every 3-frame $S = \{v_a, v_b, v_c\}$ such that $a < b < c$, $\deg_G(v_a) \leq a$, $\deg_G(v_b) \leq b-1$ and $\deg_G(v_c) \leq c-2$, it holds that $\deg_G(S) - |N_G(S;3)| \geq m$. Then G has either a hamiltonian path or a tree with 3 leaves and of order at least $m+2$.*

Proof. Assume that G does not have a hamiltonian path. Let T be a maximum tree with 3 leaves of G, and denote the leaves of T by v_a, v_b, v_c, where $1 \leq a < b < c \leq n$. Choose such a tree T so that

$$a + b + c \quad \text{is maximum} \tag{2}$$

among all the maximum trees with 3 leaves of G. Let r be the unique vertex of T with $\deg_T(r) = 3$, $U = \{v_a, v_b, v_c\}$ be the set of leaves of T, and $N_T(r) = \{w_a, w_b, w_c\}$, which lie on the paths from r to v_a, v_b, v_c, respectively. By the maximality of T, we have $N_G(U) \subset V(T)$, and so by Lemma 5 (i), U is a 3-frame of G.

We now show that $\deg_G(v_a) \leq a$, $\deg_G(v_b) \leq b-1$ and $\deg_G(v_c) \leq c-2$. We first consider v_c. For every $v_t \in N_G(v_c) - N_T(v_c)$, it follows that $v_t \notin U \cup \{w_a, w_b\}$ from Lemma 5, and choose an edge $v_t v_x$ of T in the cycle of $T + v_c v_t$. Then $T + v_c v_t - v_t v_x$ is a tree with the leaf set $\{v_a, v_b, v_x\}$. By (2) we have $x < c$. Since $a, b < c$, there exist at least $|N_G(v_c) - N_T(v_c)| + 2 = \deg_G(v_c) + 1$ vertices v_y whose indexes y are less than c. Hence $\deg_G(v_c) + 1 < c$, which implies $\deg_G(v_c) \leq c-2$. By the same argument as above, we can show that $\deg_G(v_b) \leq b-1$ and $\deg_G(v_a) \leq a$.

By Lemma 5 (iv), we obtain

$$|T| \geq 2 + \deg_G(U) - |N_G(U;3)| \geq 2 + m.$$

Consequently the lemma is proved. □

Proof of Theorem 3. If G has a hamiltonian path, then this path is the desired tree. So we may assume that G does not have a hamiltonian path. Choose a maximal tree T with 3 leaves as in Lemma 6. Then

$$|T| \geq |G| - k + 1 + 2 = |G| - k + 3.$$

This implies $k \geq 3$, and also the theorem is proved when $k = 2$ or 3. Assume $k \geq 4$. By connecting all the vertices in $V(G) - V(T)$ to T by edges or paths, we can obtain a spanning tree of G with at most $3 + |G| - |T|$ leaves, which is the desired spanning tree of G as $3 + |G| - |T| \leq k$. □

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