Knight's Tours on Cylindrical and Toroidal Boards with One Square Removed

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Abstract

The following two theorems are proved: A closed knight's tour exists on all $m \times n$ boards wrapped onto a cyclinder so that the m rows go around the cylinder, with one square removed, with the exception of the following boards:

(a) n is even,
(b) $m \in \{1,2\}$
(c) $m = 4$ and the removed square is in row 2 or 3;
(d) $m \geq 5$, $n = 1$, and the removed square is in row 2, 3, ..., or m - 1.

A closed knight's tour exists on all $m \times n$ boards wrapped onto a torus with one square removed except boards with m and n both even and 1×1, 1×2, and 2×1 boards.

1. Introduction

We consider $m \times n$ boards with m rows and n columns of squares. Square (i, j) is in the ith row and the jth column using matrix notation. A knight's move is two squares in either the vertical or horizontal direction, followed by one square in the perpendicular direction. An otherwise unoccupied board is available to the knight. The steps in a knight's path are designated 1, 2, ..., mn - 1 on the boards, as in Figure 1. A tour of a board visits every square exactly once, except that the first and the last squares may be the same. A closed tour is a tour in which the last square visited is the first square. For simplicity, a board with one square has no tour. Taking the squares as the vertices and the possible knight's moves as the edges creates a graph. A closed knight's tour is a Hamiltonian cycle of the graph.

Investigating knight's tours on chessboards is an old problem. Euler used a technique that is used today, including in the present proofs. The idea is to create paths on two or more boards and to attach or stitch together the boards into the required board by knight's moves from one board's knight's path to another board's path and back again. The history and general discussions of knight's tours appear in [1], [3], and [7] and their many references.

Schwenk [5] discovered which sizes of at rectangular boards have closed knights tours. Watkins [6] and Watkins and Hoenigman [8] found which sizes have closed tours when the boards are wrapped onto a cylinder or a torus. Interest in boards on cylinders may be very old [3, p. 173]. Watkins points out that allowing a knight to reenter a board from the opposite side after leaving from the other, as if it were wrapped onto a cylinder or a torus, is not a surprise to video gamers [6, p. 118], [7, p. 9]. Besides their intrinsic interest and mathematics, closed knight's tours of a torus have been used to create magic squares [2], [7, p. 56].

For the last two decades, there has been interest in boards with one or more squares removed. The knight is allowed to pass over, but not land on, a removed square. DeMaio and Hippchen [4] devised a function giving the minimum number of squares that must be removed from an $m \times n$ flat board so that there is a closed knight's tour. The function has value 0 for sizes of boards that have a closed knight's tour in Schwenk's theorem.

Our concern is with $m \times n$ boards that are wrapped onto a cylinder or a torus and have each one of its squares removed in turn. Cylindrical boards are investigated in Section 2, and toroidal boards are investigated in Section 3. In Section 4, we consider bipartite graphs and knight's tours.

2. Boards with one square removed and wrapped onto a cylinder

In this section, we show that all $m \times n$ boards that are wrapped onto a cylinder and have one square removed possess a closed knight's tour, except those in Lemma 1.

Lemma 1. *An $m \times n$ chessboard wrapped onto a cylinder so that the m rows go around the cylinder and one square is removed does not have a closed knight's tour if one or more of the following conditions hold:*

(a) n is even;

(b) $m \in \{1,2\}$;

(c) $m = 4$ and the removed square is in row 2 or 3;

(d) $m \geq 5$, $n = 1$, and the removed square is in row 2, 3, ..., or m - 1.

Proof. For n even, the wrapping does not disturb the standard parity argument, which is often used for flat boards. For $m \geq 2$, before wrapping, alternately color the squares black and white with the (1,1) square colored black. Each row has $n/2$ squares of each color, and $n/2$ is an integer. A knight's move is from a white to a black or from a black to a white square. When a white or a black square is removed, there is one more square of the other color, so that the tour cannot be closed. For $m = 1$, there is just one direction for the knight to move for any n, but the knight requires two directions. For $m = 2$ and $n = 1$, removing a square leaves just one square. For $m = 2$ and $n \geq 3$, beginning in square (1,1), the only move is to squares (2,3) or (2,n - 2). Without loss of generality, move to (2,3), which forces all subsequent moves. If no square is removed, a closed knight's tour is obtained. Removing a square irreparably breaks the tour.

For $m = 4$ and the removed square in rows 2 or 3, a parity argument can be made. In order to have a closed knight's tour, each of the $2n$ squares in rows 1 and 4 must have a move in and out for a total of $4n$ moves. All of those moves must be to or from a square in rows 2 or 3, requiring at least $2n$ squares in rows 2 and 3. But, after the removal of a square, rows 2 and 3 together contain only $2n$ - 1 squares.

For $m \geq 5$ and $n = 1$, if the removed square is in rows 2 through m - 1, going over the removed square is irreversible, since it must be accomplished from a square next to the removed square and go to the other square next to the removed square. Both of those squares have been visited. The proposed tour cannot go back to the initial square in order to be completed. ■

The proof of Lemma 2 is by construction. Wrapping flat boards with knights' tours onto a cylinder creates boards with knights' tours on cylinders. For many sizes, flat boards are available. For boards wrapped onto a cylinder, for each row, the column from which the square is removed is immaterial, since the cylinder can be revolved to move the removed square to any column.

Lemma 2. *All 3 × n boards with n ≥ 3 and n odd, wrapped onto a cylinder with the rows going around the cylinder and one square removed, have a closed knight's tour.*

Proof. Since boards with a square removed from row 3 can be obtained by inverting the boards that have a square removed from row 1, only removal of squares from rows 1 and 2 is considered. Figure 1 displays closed knight's tours on $3 \times n$ boards for $n \leq 13$ and n odd. The removed square is designated "Hole." Some sizes require that the boards be wrapped in order to have the tour, while others can be at boards. In Figure 1, boards (d), (h), (i), (k), (l), (m), and (n) can be flat.

Boards for $n \geq 15$ are created by attaching copies of the 3×4 board in Figure 2 from the right to boards (k), (l), (m), and (n) in Figure 1. The 3×4 board, called an extender board, has an open knight's tour that can be joined to each of the flat boards. Copies of the 3×4 extender board can be attached to each other, adding four additional columns to the board with each attachment. The extender board in Figure 2 was used similarly in [7, p. 46].

Two extender boards can be joined to create a 3×8 extender board with the path: $1 \to 7$ in the left board, $12 \to 1$ in reverse numerical order in the right board, then $8 \to 12$ in the left board. "$1 \to 7$" indicates 1, 2, 3, 4, 5, 6, then 7. This process can be continued to create an $3 \times 4k$ extender board for $k = 1, 2, 3, \ldots$.

Hole
1
2

(a)

1
Hole
2

(b)

1	Hole	3
7	2	5
4	6	8

(c)

1	6	3
4	Hole	8
7	2	5

(d)

1	5	Hole	13	8
11	9	7	2	4
6	3	12	10	14

(e)

1	10	8	13	6
12	Hole	5	3	9
4	2	11	7	14

(f)

1	4	10	Hole	18	7	16
11	15	2	8	13	20	5
3	9	12	19	6	17	14

(g)

1	18	3	6	9	12	15
4	Hole	20	17	14	7	10
19	2	5	8	11	16	13

(h)

1	4	25	10	Hole	8	19	14	17
24	11	2	5	22	13	16	7	20
3	26	23	12	9	6	21	18	15

(i)

1	4	7	10	16	19	22	25	14
24	Hole	12	3	6	9	15	18	21
13	2	5	8	11	17	20	23	26

(j)

Hole	1	4	23	20	29	8	27	18	15	12
5	22	31	2	7	24	19	10	13	26	17
32	3	6	21	30	9	28	25	16	11	14

(k)

1	30	3	6	9	12	27	24	21	18	15
4	Hole	32	29	26	7	10	13	16	23	20
31	2	5	8	11	28	25	22	19	14	17

(l)

Hole	1	4	31	36	33	8	11	26	23	20	17	14
5	30	37	2	7	28	25	34	9	12	15	22	19
38	3	6	29	32	35	10	27	24	21	18	13	16

(m)

1	4	7	10	35	12	19	32	17	14	25	30	27
6	Hole	2	37	20	9	34	13	22	31	28	15	24
3	38	5	8	11	36	21	18	33	16	23	26	29

(n)

Figure 1: Wrapped $3 \times n$ boards with $n \leq 13$ and n odd have a closed knight's tour

1	4	7	10
12	9	2	5
3	6	11	8

Figure 2: The 3 × 4 extender board, which is used to create boards with a larger numbers of columns

An extender board can be attached from the right to the 3 × 11 and 3 × 13 boards (k), (l), (m), and (n) in Figure 1. For example, as shown in Figure 3, for (k), a path is: break the tour in (k) at 14, go to 1 → 12 in the extender board, then 15 → 32 in (k). Similarly, break the tours in boards (l), (m), and (n) at 17, 16, and 29, respectively. By attaching additional extender boards from the right, closed knight's tours can be created for all 3 × n boards with one square removed for $n \geq 15$ and n odd. ■

Hole	1	4	23	20	29	8	27	18	15	12
5	22	31	2	7	24	19	10	13	26	17
32	3	6	21	30	9	28	25	16	11	14

1	4	7	10
12	9	2	5
3	6	11	8

Figure 3: Joining the 3 × 4 extender board in Figure 2 and the 3 × 11 board in Figure 1(k) to create a 3 × 15 board

Lemma 3. *A closed knight's tour exists on all 4 × n boards, n odd, $n \geq 3$, wrapped onto a cylinder so that the rows go around the cylinder with one square removed from rows 1 or 4.*

Proof. Remove a square in row 1 without loss of generality. A closed knight's tour is created on the 4 × n wrapped board without any square removed, using the pattern in Figure 4. Since n is odd, the path visits every square and is closed. Figure 5 contains these boards for $n \leq 11$ and n odd.

1		5		9				
				4		8	and so forth →	
	2		6		10			
			3		7			

Figure 4: The pattern for the closed knight's tour on a $4 \times n$ wrapped board, n odd, with no square removed

Square (1,3), which is step 5 in the tour, is removed from each board and a new closed knight's tour is constructed. For $n \geq 11$, the closed knight's tour is

$1 \rightarrow 4$,
7, 8, then 11, 12, ..., to $2n$ - 11, $2n$ - 10 in pairs,
$4n$ - 2 $\rightarrow$ $2n$ - 9 in reverse numerical order,
$2n$ - 12, $2n$ - 13, then $2n$ - 16, $2n$ - 17, ..., to 10, 9, in pairs,
6, $4n$ - 1, $4n$ to 1, completing the tour.

For example, the 4×11 and 4×13 boards have tours $1 \rightarrow 4$, 7, 8, 11, 12, 42 $\rightarrow$ 13, 10, 9, 6, 43, 44, 1 and 1 $\rightarrow$ 4, 7, 8, 11, 12, 15, 16, 50 $\rightarrow$ 17, 14, 13, 10, 9, 6, 51, 52, 1, respectively.

The closed knight's tours for the 4×3, 4×5, 4×7, and 4×9 boards are similar, but abbreviated in their formats, because of the boards' smaller sizes. For them, tours are: 1 $\rightarrow$ 4, 10 $\rightarrow$ 6, 11, 12, 1; 1 $\rightarrow$ 4, 7, 8, 18 $\rightarrow$ 9, 6, 19, 20, 1; 1 $\rightarrow$ 4, 26 $\rightarrow$ 6, 27, 28, 1; and 1 $\rightarrow$ 4, 7, 8, 34 $\rightarrow$ 9, 6, 35, 36, 1, respectively. ■

Lemma 4. *A closed knight's tour exists on all $m \times 1$ boards, $m \geq 4$, wrapped onto a cylinder so that the m rows go around the cylinder, with one square removed from rows 1 or m.*

Proof. Removing a square from row 1 or row m leaves an (m - 1) $\times$ 1 board. Starting in row 1 of the reduced board and noticing that the knight can move either one or two rows, move two rows at each step to the odd numbered rows 3, 5, and so forth. If m is even, when reaching row m - 1, make a move to row m - 2, then two rows at a time move to row 2, from which a one-row move reaches row 1. If m is odd, when reaching row m - 2, make a move to row m - 1, then move two rows at a time to row 2. ■

1	9	5
8	4	12
6	2	10
3	11	7

1	13	5	17	9
16	8	29	12	4
10	2	14	6	18
7	19	11	3	15

1	17	5	21	9	25	13
24	12	28	16	4	20	8
14	2	18	6	22	10	26
11	27	15	3	19	7	23

1	21	5	25	9	29	13	33	17
32	16	36	20	4	24	8	28	12
18	2	22	6	26	10	30	14	34
15	35	19	3	23	7	27	11	31

1	25	5	29	9	33	13	37	17	41	21
40	20	44	24	4	28	8	32	12	36	16
22	2	26	6	30	10	34	14	38	18	42
19	43	23	3	27	7	31	11	35	15	39

Figure 5: Wrapped $4 \times n$ boards, $n \leq 11$, n odd, with closed knight's tours, constructed using the pattern in Figure 4

Lemma 5. *A closed knight's tour exists on all $m \times n$ boards, m and n odd, $m \geq 5$, $n \geq 3$, wrapped onto a cylinder so that the rows go around the cylinder, with one square removed.*

Proof. The $3 \times n$ boards in Lemma 2 are the starter boards. Figure 1 displays closed knight's tours for $n \leq 13$. Extender boards are attached above and below the starter boards. The extender boards are $2 \times n$ and wrapped around the cylinder. Their tours are just a knight moving to the right. There is no choice for each move. Figure 6 contains a 2×9 extender board.

The attachment can be accomplished since the boards have sequential squares, which allow the joining in a manner analogous to Figure 3. All starter boards in Figure 1 have sequential steps in certain squares, which make the boards very flexible. Separately, in the top two rows and the bottom two rows of each board, there is a pair with squares that are two columns apart with a rising slope and a pair with the falling slope. For example, in the 3×9 starter board in Figure 1(i), in the top two rows, pairs are 11 rising to 10 and 4 falling to 5, and in the bottom two rows, pairs are 3 rising to 2 and 11 falling to 12. There are numerous other pairs. An additional adaptation of the boards is to rotate them around the cylinder to align pairs for joining.

1	15	11	7	3	17	13	9	5
10	6	2	16	12	8	4	18	14

Figure 6: Wrapped, extender board for $n = 9$

To illustrate the joining process, join two 2×9 extender boards from Figure 6. Using falling pairs in both boards, a path is: 1 in the upper board, $15 \to 16$ in reverse numerical order in the lower board, then $2 \to 18$ in the upper board. Alternatively, using rising pairs in both boards, a path is: 18 in the upper board, $4 \to 5$ in reverse numerical order in the lower board, then $1 \to 17$ in the upper board.

Join u extender boards above and v extender boards below the starter board with $m = 2u + 2v + 3$, u and v non-negative integers. To arrange for a hole in row $r \neq m$, select $u = (r - 1)/2$ for r odd and $u = (r - 2)/2$ for r even and $v = (m - 2u - 3)/2$. The starter board has the hole in row 1 if r is odd and in row 2 if r is even. For $r = m$, turn the board top-to-bottom and call the row with the hole row 1.

For example, for $r = 6$, $u = (6 - 2)/2 = 2$. Two extender boards are attached above, the starter board has a hole in its row 2, and $v = (m - 7)/2$ starter boards are attached below. ■

Lemma 6. *A closed knight's tour exists on all $m \times n$ boards, m even and n odd, $m \geq 6$, $n \geq 3$, wrapped onto a cylinder so that the rows go around the cylinder, with one square removed.*

Proof. The proof is similar to the proof of Lemma 5. Three cases are considered separately. One is r even and $r \geq 4$. Another is r odd and $r \neq m - 1$. The third contains the cases $r = 2$ and $r = m - 1$.

For r even and $r \geq 4$, the $4 \times n$ boards in Lemma 3 with the hole in row 4 are the starter boards. The extender boards are 2 × n extender boards and are the same ones used in Lemma 5. Attach u extender boards above and v extender boards below the starter board with $m = 2u + 2v + 4$, and $u, v \in \{0, 1, 2, ...\}$. Select $u = (r - 4)/2$ and $v = (m - r)/2$. For example, for $m = 14$ and $r = 10$, we have $u = 3$ extender boards attached above and $v = 2$ extender boards attached below.

For r odd and $r \neq m - 1$, the 4 × n boards in Lemma 3 with the hole in row 1 are starter boards. The extender boards are the same 2 × n extender boards. Attach u extender boards above and v extender boards below the starter board with $m = 2u + 2v + 4$. Select $u = (r - 2)/2$ and $v = (m - r - 3)/2$. For example, for $m = 12$ and $r = 7$, we have $u = 3$ extender boards attached above and $v = 1$ extender board attached below.

The third case does not fit these patterns since the 4 × n starter board cannot have a hole in row 2 or row 3 by Lemma 1. For $r = 2$, use a 6 × n starter board. These new 6 × n starter boards can be created by attaching a 3 × n extender board below a 3 × n starter board with a hole in its row 2 from Lemma 2. An example of a 6 × 9 starter board is created by attaching the 3 × 9 extender board in Figure 7 from below to the board in Figure 1(j). The attachment is done with the path $1 \rightarrow 2$ in the upper starter board, $8 \rightarrow 7$ in reverse numerical order in the lower extender board, then $3 \rightarrow 26$ to 1 in the upper board. All the 3 × n extender boards are created with the pattern in Figure 7. Finally, using this 6 × n starter board, attach v 2 × n extender boards from below with $v = (m - 6)/2$. If $r = m - 1$, turn the board top-to-bottom to place the hole in row 2. ■

1	4	7	10	13	16	19	22	25
8	11	14	17	20	23	26	2	5
3	6	9	12	15	18	21	24	27

Figure 7: Wrapped, 3 × 9 extender board with a closed knight's tour for the case $r = 2$ of Lemma 6

Combining Lemmas 1 through 6 gives Theorem 1.

Theorem 1 **(Cylinders)** *Except for those boards listed in Lemma 1, a closed knight's tour exists on all $m \times n$ boards wrapped onto a cylinder so that the m rows go around the cylinder, with one square removed.*

3. Boards with one square removed and wrapped onto a torus

A board on a torus is created by wrapping a flat board onto a cylinder and joining the ends of the cylinder without twisting. Any board on a cylinder that has a closed knight's tour continues to have a tour when it is joined so that it is on a torus. We must examine only boards that do not have tours on cylinders, which are listed in Lemma 1.

Any square can be moved to any other square's position by rotating the cylinder and spinning the torus. Hence, the factor that determines whether a board on a torus has a tour is its size, not the location of the removed square before the board was formed onto a torus.

Theorem 2 **(Tori)** *A closed knight's tour exists on all $m \times n$ boards wrapped onto a torus with one square removed, except:*

(a) boards with m and n both even;
(b) 1×1, 1×2, and 2×1 boards.

Proof. The standard parity argument says that both m and n cannot be even. By Lemmas 5 and 6, for n odd, $n \geq 3$, $m \geq 5$, there is a tour.

On a cylinder, boards with $m = 1$ do not have tours. The distinction between rows and columns is lost on the torus, so $1 \times n$ boards are the same as $n \times 1$ boards. The removed square can be moved to row 1. Lemmas 3 and 4 say that there is a tour, unless $n = 1$ or 2.

On a cylinder, $m \times n$ boards with $m = 2$ do not have tours. However, for $n \geq 3$ and n odd, they have tours on a torus. Tours are easy to construct. Figure 8 has a closed knight's tour on a 2×13 board, which illustrates the pattern for tours on boards for $n \geq 9$.

Steps 1 through 6 are in row 1, then steps 7 through 12 are in row 2, in pairs the steps 13 and 14 are in row 1, steps 15 and 16 in row 2, and so forth. Boards with $n = 3$, 5, and 7 containing tours are in Figure 9.

1	2	3	4	5	6	13	14	17	18	21	22	Hole
23	24	25	7	8	9	10	11	12	15	16	19	20

Figure 8: 2 × 13 board on a torus with a closed knight's tour

1	2	Hole
3	4	5

1	2	3	4	Hole
6	5	9	8	7

1	2	3	4	5	6	Hole
11	12	13	7	8	9	10

Figure 9: 2 × 3, 2 × 5, and 2 × 7 boards on a torus with closed knight's tours

On a cylinder, if $m = 4$ and the removed square is in rows 2 or 3, there is no tour. But, Lemma 3 says that a closed knight's tour exists if $m = 4$ and the removed square is in rows 1 or 4. On the torus, the removed square could have come from rows 2 or 3, since squares in rows 1 and 4 can be moved to positions in row 2 or 3.

There is no tour on a cylinder if $m \geq 4$ and $n = 1$ and the removed square is in rows 2 through m - 1, inclusively. But, on a torus the removed square can be moved to a position in row 1 or row m, for which there is a tour by Lemma 4.

All other boards on a torus have a closed knight's tour except boards that do not have a sufficient number of squares for two moves, that is 1 × 2 and 2 × 1 boards, which have only one square after removal of a square, and 1 × 1 boards, which have no square after a square is removed. ■

4. Bipartite graphs and knight's tours

A graph is bipartite if its vertices are the union of two disjoint sets such that there are no edges connecting vertices that are in the same set. If there is a knight's tour on a board, it creates a graph with the vertices being the visited squares and the edges being the knight's moves. If a board possessing a knight's tour has an even number of squares, the tour produces a bipartite graph. The knight visits squares in the two sets alternatively. If there are an odd number of squares, the graph is not bipartite.

The standard parity argument shows how all knights' tours on $m \times n$ flat chessboards produce a bipartite graph and also that some boards do not have a knight's tour. Color the squares, alternating black and white. Since a knight's move is always from one colored square to a differently colored square, in order to have a closed knight's tour, there must be the same number of black and white squares. The black squares comprise one of the sets, and the white squares comprise the other set in the bipartite graph. Using this reasoning, no rectangular flat board with both m and n odd can have a closed knight's tour [7, pp. 8, 9, and 142].

This parity argument was used in Lemma 1 to show that some wrapped boards do not have a tour. However, wrapping can make this parity argument ineffective. In Lemma 3, consider a 4×3 board, wrapped around a cylinder with a square removed in row 1, so that it has 11 squares. This board has a knight's tour. If it were colored in the usual way and subsequently wrapped, squares of the same color would be joined, destroying the coloring scheme.

For wrapped $m \times n$ boards with one square removed and possessing a knight's tour, the graph is bipartite if and only if $mn - 1$ is even, since the knight enters and leaves each square once and there is an even number of squares.

For example, in Lemma 3 the $4 \times n$ boards, n odd, $n \geq 3$, with one square removed from rows 1 or 4, have tours but do not yield a bipartite graph since $mn - 1$ is odd. On the other hand, all the boards described in Lemma 5 have $mn - 1$ even, so the corresponding graph is bipartite. The alternating black and white coloring scheme that might be imposed before the wrapping does not give the two sets. The sets are defined by the knight's moves used in the tour.

In Theorem 2, for boards on a torus with a knight's tour, there are different combinations of having a bipartite graph and the possibility of successfully coloring the board with alternating colors, depending on the dimensions m and n being odd or even. If both m and n are odd, then the board cannot be colored with alternating colors because of the joining and a bipartite graph is created by the tour, since $mn - 1$ is even. If one of m and n is odd and the other is even, then the board cannot be colored with alternating colors because of the joining of the odd sides and a bipartite graph is not created by the tour, since mn - 1 is odd. The third possibility for these boards is that both m and n are even, then the board can be colored with alternating colors and a bipartite graph is not created by the tour, since mn - 1 is odd.

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