(

Enumeration of boundary cubic inner-forest maps *

Shude Long[a] **Junliang Cai**[b]

[a] *Department of Mathematics, Chongqing University of Arts and Sciences, Chongqing 402160, P.R.China*

[b] *School of Mathematical Sciences, Beijing Normal University, Beijing 100875, P.R.China*

(longshude@163.com; caijunliang@bnu.edu.cn)

Abstract

This paper investigates the number of boundary cubic inner-forest maps and presents some formulae for such maps with the size (number of edges) and the valency of the root-face as two parameters. Further, by duality, some corresponding results for rooted outer-planar maps are obtained. It is also an answer to the open problem in [15] and corrects the result on boundary cubic inner-tree maps in [15].

MSC: O5C45; 05C30

Keywords:Boundary cubic; Enumerating function; Functional equation; Lagrangian inversion

1. Introduction

The concept of rooted map was first introduced by Tutte. His series of census papers [21–24] laid the foundation for the theory. Since then, the theory has been developed by many scholars such as Arquès [1], Brown [7,8], Mullin et al. [20], Tutte [25], Bender et al. [2–6], Liskovets et al. [13,14], Gao [9,10] and Liu [16–19].

In 2007, Wenzhong Liu, Yanpei Liu and Yan Xu [15] investigated the enumeration of boundary cubic rooted planar maps and obtained some formulae for the number of boundary cubic rooted planar maps with the valency of the root-face and the size (number of edges) as two parameters. But the formula for the number of boundary cubic inner-forest maps with the size and the root-face as parameters could not have been obtained at that time and the result on boundary cubic inner-tree maps is error.

*Supported by the NNSFC (No. 10271017), Chongqing Municipal Education Commission (No. KJ101204) and Chongqing University of Arts and Sciences.

Now, on the basis of what was obtained in [15] we obtain the parametric expressions of the functional equations presented as by (2.8) and (2.12) in [15]. By employing Lagrangian inversion [11,26] the solutions may be found. Further, formulae for the numbers of boundary cubic inner-forest maps and boundary cubic inner-tree maps with the size and the valency of the root-face as two parameters can be obtained. One of these formulae corrects the result on boundary cubic inner-tree maps in [15]. In addition, by duality, some corresponding results for rooted outer-planar maps are also obtained.

Now, we define some basic concepts and terms. A *map* on an orientable surface is a connected graph cellularly embedded on the surface. A map is *rooted* if an edge and a direction along the edge are distinguished. If the root is the oriented edge from u to v and then u is the *root-vertex* while the face on the oriented side of the edge is defined as the *root-face*. In this paper, maps are always rooted and planar (that is, imbedded on a sphere).

An *outer-planar map* is a planar map such that the boundary of the root-face contains all the vertices. A *boundary cubic map* is a map such that all the vertices on the root-face boundary are of valency 3. A *boundary cubic inner-forest map* is a boundary cubic map such that the map obtained by deleting all the edges on the root-face boundary is a forest. A *boundary cubic inner-tree map* is defined similarly.

For a boundary cubic map M, we operate on it as follows: the edges on the boundary of the root-face are contracted to a point. This operation continues until the boundary of the root-face becomes a vertex. The map M' obtained by this operation is called the contracted map of M, where the vertex obtained is the new root-vertex and the edge incident with the root-vertex of M and not on the boundary of the root-face of M is the new root-edge.

Conversely, a map M' with the valency $m(M')$ of the root-vertex can be extended to a boundary cubic map M by splitting the root-vertex into $m(M')$ vertices and joining the vertices by new edges in turn, where the new vertex incident with the root-edge of M' is the root-vertex of M and the added edge incident with the root-vertex and along the orientable direction is the root-edge of M. The map M is called the extended map of M'.

For convenience, we introduce the following generating function for the set $\mathcal{M}$, the set of rooted planar maps:

$$f_{\mathcal{M}}(x,y,z) = \sum_{M\in\mathcal{M}} x^{m(M)}y^{l(M)}z^{n(M)},$$

where $m(M), l(M)$ and $n(M)$ denote the root-vertex valency, the root-face valency and the size of M, respectively. In addition, we write that

$$F_{\mathcal{M}}(x,z) = f_{\mathcal{M}}(x,1,z), \quad H_{\mathcal{M}}(y,z) = f_{\mathcal{M}}(1,y,z), \quad h_{\mathcal{M}}(z) = f_{\mathcal{M}}(1,1,z).$$

For the power series $f(x)$, $f(x,y)$ and $f(x,y,z)$, we employ the following notations:

$$\partial_x^m f(x), \quad \partial_{(x,y)}^{(m,l)} f(x,y) \quad and \quad \partial_{(x,y,z)}^{(m,l,n)} f(x,y,z)$$

to represent the coefficients of x^m in $f(x)$, x^my^l in $f(x,y)$ and $x^my^lz^n$ in $f(x,y,z)$, respectively. Terminologies and notations not explained here can be found in [16].

2. Boundary cubic inner-tree maps

Let $\mathscr{M}'$ be the set of all boundary cubic inner-tree maps and $\mathscr{M}^{c'}$ be the contraction of $\mathscr{M}'$, that is, the set of all the rooted maps obtained by contracting all the members of $\mathscr{M}'$. In this section we will solve the the following functional equation with three variables as shown by (2.12) in [15]:

$$\begin{aligned}(1-y+xyz)f_{\mathscr{M}^{c'}} =&(1-y)x^2yz+\frac{x(1-y)}{2}\left(1-\sqrt{1-4y^2z}\right)\\&+xy^2zF_{\mathscr{M}^{c'}},\end{aligned} \tag{1}$$

where $F_{\mathscr{M}^{c'}} = f_{\mathscr{M}^{c'}}(x,1,z)$.

Before stating our results, we introduce the following lemma.

Lemma 1 (Liu [15]). Let $\mathscr{M}(m,n)$ and $\mathscr{R}(m,n)$ be the sets of all planar maps of size n with root-vertex valency m and all boundary cubic maps of size n with root-face valency m for $m,n \geq 1$, respectively. Then

$$\mid \mathscr{M}(m,n) \mid=\mid \mathscr{R}(m,m+n) \mid, \tag{2}$$

and there exists a 1-to-1 correspondence between $\mathscr{M}(m,n)$ and $\mathscr{R}(m,m+n)$.

Let ξ be the root of the characteristic equation of (1) solved for y. Then we have

$$\begin{cases} 1-\xi+\xi xz=0; \\ (1-\xi)\xi x^2z+\dfrac{(1-\xi)x}{2}(1-\sqrt{1-4\xi^2z})+\xi^2xzF_{\mathscr{M}^{c'}}=0. \end{cases} \tag{3}$$

By (3) we get

$$xz=\frac{\xi-1}{\xi}, \quad F_{\mathscr{M}^{c'}}-x^2z=\frac{(\xi-1)}{2\xi^2z}(1-\sqrt{1-4\xi^2z}). \tag{4}$$

If we introduce a new parameter θ such that

$$z=\frac{\theta(1-\theta)}{\xi^2}, \tag{5}$$

then the second part of (4) becomes

$$F_{\mathscr{M}^{c'}}-x^2z=\frac{\xi-1}{1-\theta}. \tag{6}$$

Further, let $\xi = 1+\eta$. By (4–6), one may find the parametric expression of $F_{\mathscr{M}^{c'}} = F_{\mathscr{M}^{c'}}(x, z)$ as follows:

$$xz = \frac{\eta}{1+\eta}, \quad z = \frac{\theta(1-\theta)}{(1+\eta)^2}, \quad F_{\mathscr{M}^{c'}} - x^2 z = \frac{\eta}{1-\theta}. \tag{7}$$

By (7) we have

$$\Delta_{(\eta,\theta)} = \begin{vmatrix} \frac{1}{1+\eta} & 0 \\ * & \frac{1-2\theta}{1-\theta} \end{vmatrix} = \frac{1-2\theta}{(1+\eta)(1-\theta)}. \tag{8}$$

Applying Lagrangian inversion with two parameters [11], from (7) and (8) one may find that

$$\begin{aligned} \partial^{(m,k)}_{(xz,z)}(F_{\mathscr{M}^{c'}} - x^2 z) =& \partial^{(m-1,k)}_{(\eta,\theta)} \frac{(1+\eta)^{m+2k-1}(1-2\theta)}{(1-\theta)^{k+2}} \\ =& \binom{m+2k-1}{m-1} \partial^k_\theta \frac{1-2\theta}{(1-\theta)^{k+2}} \\ =& \frac{(2k)!}{k!(k+1)!} \binom{m+2k-1}{m-1}. \end{aligned}$$

Let $n = m + k$. Then we have

$$\partial^{(m,n)}_{(x,z)}(F_{\mathscr{M}^{c'}} - x^2 z) = \frac{(2n-m-1)!}{(n-m)!(n-m+1)!(m-1)!}, \tag{9}$$

which proves

Theorem 1. The enumerating function $F_{\mathscr{M}^{c'}}$ determined by (1) has the following explicit expression:

$$F_{\mathscr{M}^{c'}} = 1 + x^2 z + \sum_{n\geq 1} \sum_{m=1}^{n} \frac{(2n-m-1)!}{(n-m)!(n-m+1)!(m-1)!} x^m y^n. \tag{10}$$

By substituting m and n for l and $n-l$, respectively, from Lemma 1 and (10) we can obtain

Theorem 2. The number of boundary cubic inner-tree maps with size n and the root-face valency l is

$$\frac{(2n-3l-1)!}{(n-2l)!(n-2l+1)!(l-1)!} \tag{11}$$

for $1 \leq l \leq \lfloor \frac{n}{2} \rfloor$; 1 for $l = 2, n = 3$.

By (1) we have

$$f_{\mathscr{M}^{c'}} = \frac{(1-y)x^2 yz + xy^2 z F_{\mathscr{M}^{c'}}}{1-y+xyz} + \frac{x}{2(1+\frac{xyz}{1-y})}\left(1 - \sqrt{1-4y^2 z}\right). \tag{12}$$

Now, let

$$P(x,y,z) = \frac{(1-y)x^2yz + xy^2zF_{\mathscr{M}^{c'}}}{1-y+xyz}. \qquad (13)$$

By introducing another parameter λ such that

$$y = \lambda(1+\eta), \qquad (14)$$

from (7), (13) and (14) one may find the following parametric expression of the function $P = P(x,y,z)$:

$$\begin{aligned} xz &= \frac{\eta}{1+\eta}, \quad y = \lambda(1+\eta), \\ z &= \frac{\theta(1-\theta)}{(1+\eta)^2}, \quad P - x^2yz = \frac{\lambda^2\eta^2(1+\eta)}{(1-\theta)(1-\lambda)}, \end{aligned} \qquad (15)$$

from which we get

$$\Delta_{(\eta,\lambda,\theta)} = \begin{vmatrix} \frac{1}{1+\eta} & 0 & 0 \\ * & 1 & 0 \\ * & * & \frac{1-2\theta}{1-\theta} \end{vmatrix} = \frac{1-2\theta}{(1+\eta)(1-\theta)}. \qquad (16)$$

By employing Lagrangian inversion with three variables [11], from (15) and (16) one may find that

$$\begin{aligned} \partial^{(m,l,k)}_{(xz,y,z)}(P - x^2yz) &= \partial^{(m-2,l-2,k)}_{(\eta,\lambda,\theta)} \frac{(1+\eta)^{m+2k-l}(1-2\theta)}{(1-\theta)^{k+2}(1-\lambda)} \\ &= \binom{m+2k-l}{m-2} \partial^k_\theta \frac{1-2\theta}{(1-\theta)^{k+2}} \\ &= \frac{(2k)!}{k!(k+1)!}\binom{m+2k-l}{m-2}. \end{aligned} \qquad (17)$$

Let $n = m+k$. Then we get

$$\partial^{(m,l,n)}_{(x,y,z)}(P - x^2yz) = \frac{(2n-2m)!}{(n-m)!(n-m+1)!}\binom{2n-m-l}{m-2}. \qquad (18)$$

In addition, we have

$$\begin{aligned} \frac{x}{2(1+\frac{xyz}{1-y})}\left(1-\sqrt{1-4y^2z}\right) = &\sum_{\substack{n\geq 1 \\ l\geq 2n-m+1}} \sum_{m=1}^{n} (-1)^{m-1} \frac{(2n-2m)!}{(n-m+1)!(n-m)!} \\ &\times \binom{l+2m-2n-3}{l+m-2n-1} x^m y^l z^n. \end{aligned} \qquad (19)$$

Combining (12),(13),(18) with (19), we can obtain

Theorem 3. The enumerating function $f_{\mathscr{M}^{c'}} = f_{\mathscr{M}^{c'}}(x,y,z)$ has the following explicit expression:

$$\begin{aligned} f_{\mathscr{M}^{c'}}(x,y,z) =& 1 + x^2yz + \sum_{n,l\geq 2}\sum_{m=2}^{n} \frac{(2n-2m)!}{(n-m)!(n-m+1)!} \\ & \times \binom{2n-m-l}{m-2} x^m y^l z^n \\ & + \sum_{\substack{n\geq 1 \\ l\geq 2n-m+1}} \sum_{m=1}^{n} (-1)^{m-1} \frac{(2n-2m)!}{(n-m+1)!(n-m)!} \\ & \times \binom{l+2m-2n-3}{l+m-2n-1} x^m y^l z^n. \end{aligned} \tag{20}$$

3. Boundary cubic inner-forest maps

Let $\mathscr{M}$ be the set of all boundary cubic inner-forest maps and $\mathscr{M}^c$ denote the set each of whose elements is the contracted map of some map in $\mathscr{M}$. In this section we will solve the following equation with three variables as shown by (2.8) in [15]:

$$\begin{aligned} &\left\{(1-y)\left[1 - \frac{x(1-\sqrt{1-4y^2z})}{2} + (1-x)xyzF_{\mathscr{M}^c}\right] + xy^2z\right\} f_{\mathscr{M}^c} \\ &= 1 - y + xyzF_{\mathscr{M}^c}, \end{aligned} \tag{21}$$

where $F_{\mathscr{M}^c}(x,z) = f_{\mathscr{M}^c}(x,1,z)$.

Lemma 2. The enumerating function $H_{\mathscr{M}^c} = H_{\mathscr{M}^c}(y,z)$ satisfies the following equation:

$$\left[(1-y)\left(1 - \frac{1-\sqrt{1-4y^2z}}{2}\right) + y^2z\right] H_{\mathscr{M}^c} = 1 - y + yzh_{\mathscr{M}^c}, \tag{22}$$

where $H_{\mathscr{M}^c} = f_{\mathscr{M}^c}(1,y,z), h_{\mathscr{M}^c} = f_{\mathscr{M}^c}(1,1,z) = F_{\mathscr{M}^c}(1,z)$.

Proof. It follows immediately from (21) by putting $x = 1$. □

Let α be the root of the characteristic equation of (22). Then we get

$$\begin{cases} (1-\alpha)\left(1 - \dfrac{1-\sqrt{1-4\alpha^2z}}{2}\right) + \alpha^2 z = 0; \\ 1 - \alpha + \alpha z h_{\mathscr{M}^c} = 0. \end{cases} \tag{23}$$

By (23) we have

$$\alpha^2 z = (\alpha-1)(2-\alpha), \quad zh_{\mathscr{M}^c} = \frac{\alpha-1}{\alpha}. \tag{24}$$

Now, let $\alpha = 1+\eta$. By (24) we have the following parametric expression of the function $h_{\mathscr{M}^c} = h_{\mathscr{M}^c}(z)$:

$$z = \frac{\eta(1-\eta)}{(1+\eta)^2}, \quad zh_{\mathscr{M}^c} = \frac{\eta}{1+\eta}. \tag{25}$$

Theorem 4. The enumerating function $h_{\mathscr{M}^c} = h_{\mathscr{M}^c}(z)$ has the following explicit expression:

$$h_{\mathscr{M}^c}(z) = \sum_{n\geq 0} \frac{2^n(2n)!}{n!(n+1)!} z^n. \tag{26}$$

Proof. By employing Lagrangian inversion with one parameter [26] for (25), one may find that

$$\begin{aligned} h_{\mathscr{M}^c}(z) &= \sum_{n\geq 1} \frac{z^{n-1}}{n!} \frac{d^{n-1}}{d\eta^{n-1}} \frac{(1+\eta)^{2n-2}}{(1-\eta)^n}\bigg|_{\eta=0} \\ &= \sum_{n\geq 1} \frac{2^{n-1}(2n-2)!}{(n-1)!n!} z^{n-1} \\ &= \sum_{n\geq 0} \frac{2^n(2n)!}{n!(n+1)!} z^n. \end{aligned}$$

This completes the proof of Theorem 4. □

Let $\mathscr{M}_0$ denote the dual set of the contraction set $\mathscr{M}^c$ of all the boundary cubic inner-forest maps. It is seen that $\mathscr{M}_0$ is the set of all outer-planar maps.

Corollary 1. The number of rooted outer-planar maps with size n is

$$\frac{2^n(2n)!}{n!(n+1)!} \tag{27}$$

for $n \geq 0$.

Proof. It follows easily by duality and (26). □

By (22) we have

$$H_{\mathscr{M}^c} = \frac{1-y+yzh_{\mathscr{M}^c}}{(1-y)(2-y)+y^2z}\left[1 + \frac{(1-y)(1-\sqrt{1-4y^2z})}{2y^2z}\right]. \tag{28}$$

Now, let

$$Q(y,z) = \frac{1-y+yzh_{\mathscr{M}^c}}{(1-y)(2-y)+y^2z}. \tag{29}$$

By introducing a new parameter η such that

$$y = \lambda(1+\eta), \tag{30}$$

one may find from (25) and (29) that

$$Q(y,z) = \frac{1}{2-\lambda(1+3\eta)}. \tag{31}$$

By (25), (30) and (31), we have the parametric expression of $Q = Q(y,z)$ as follows:

$$y = \lambda(1+\eta), \quad z = \frac{\eta(1-\eta)}{(1+\eta)^2}, \quad Q = \frac{1}{2-\lambda(1+3\eta)}. \tag{32}$$

By (32) we get

$$\Delta_{(\lambda,\eta)} = \begin{vmatrix} 1 & * \\ 0 & \frac{1-3\eta}{(1+\eta)(1-\eta)} \end{vmatrix} = \frac{1-3\eta}{(1+\eta)(1-\eta)}. \tag{33}$$

By employing Lagrangian inversion with two variables [11], from (32) and (33) one may find that

$$\begin{aligned} B(l,n) =& \partial_{(y,z)}^{(l,n)} Q = \partial_{(\lambda,\eta)}^{(l,n)} \frac{(1+\eta)^{2n-l-1}(1-3\eta)}{(1-\eta)^{n+1}[2-\lambda(1+3\eta)]} \\ =& \frac{1}{2^{l+1}} \partial_\eta^n \frac{(1+\eta)^{2n-l-1}(1+3\eta)^l(1-3\eta)}{(1-\eta)^{n+1}} \\ =& \sum_{i=0}^{\min\{l,n\}} \frac{1}{2^{l-i+1}} \binom{l}{i} \partial_\eta^{n-i} \frac{(1+\eta)^{2n-i-1}(1-3\eta)}{(1-\eta)^{n+1}} \\ =& \sum_{i=0}^{\min\{l,n\}} \sum_{j=0}^{n-i} \frac{1}{2^{l-i+1}} \binom{l}{i} \binom{2n-i-1}{j} \partial_\eta^{n-i-j} \frac{1-3\eta}{(1-\eta)^{n+1}} \\ =& \frac{l!}{2^{l-n+1} n!} \sum_{i=1}^{\min\{l,n\}} \frac{(2n-i-1)!}{(l-i)!(i-1)!(n-i)!}. \end{aligned} \tag{34}$$

In addition, we have

$$\begin{aligned} 1 + \frac{(1-y)(1-\sqrt{1-4y^2z})}{2y^2z} =& 1 + (1-y)\sum_{k\geq 0} \frac{(2k)!}{(k+1)!k!} y^{2k} z^k \\ =& 1 + \sum_{k\geq 0} \frac{(2k)!}{(k+1)!k!} y^{2k} z^k \\ & - \sum_{k\geq 0} \frac{(2k)!}{(k+1)!k!} y^{2k+1} z^k. \end{aligned} \tag{35}$$

Let

$$H_{\mathscr{M}^c}(y,z) = 1 + \sum_{l,n\geq 1} A(l,n)y^l z^n. \tag{36}$$

By (28), (29), (34), (35) and (36), we have

$$\begin{aligned} A(l,n) =& B(l,n) + \sum_{k=0}^{\lfloor \frac{l}{2} \rfloor} \frac{(2k)!}{(k+1)!k!} B(l-2k, n-k) \\ & - \sum_{k=0}^{\lfloor \frac{l-1}{2} \rfloor} \frac{(2k)!}{(k+1)!k!} B(l-2k-1, n-k), \end{aligned} \tag{37}$$

which proves

Theorem 5. The enumerating function $H_{\mathscr{M}^c} = H_{\mathscr{M}^c}(y,z)$ has the following explicit expression:

$$H_{\mathscr{M}^c}(y,z) = 1 + \sum_{l,n\geq 1} A(l,n)y^l z^n, \tag{38}$$

where

$$\begin{aligned} A(l,n) =& B(l,n) + \sum_{k=0}^{\lfloor \frac{l}{2} \rfloor} \frac{(2k)!}{(k+1)!k!} B(l-2k, n-k) \\ & - \sum_{k=0}^{\lfloor \frac{l-1}{2} \rfloor} \frac{(2k)!}{(k+1)!k!} B(l-2k-1, n-k), \end{aligned} \tag{39}$$

in which

$$B(l,n) = \frac{l!}{2^{l-n+1}n!} \sum_{i=1}^{\min\{l,n\}} \frac{(2n-i-1)!}{(l-i)!(i-1)!(n-i)!}. \tag{40}$$

Now, we present a corollary of Theorem 5.

Corollary 2. The number of rooted outer-planar maps with size $n (n \geq 1)$ and the root-vertex valency $m (m \geq 1)$ is

$$\begin{aligned} A(m,n) =& B(m,n) + \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(2k)!}{(k+1)!k!} B(m-2k, n-k) \\ & - \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(2k)!}{(k+1)!k!} B(m-2k-1, n-k), \end{aligned} \tag{41}$$

where

$$B(m,n) = \frac{m!}{2^{m-n+1}n!} \sum_{i=1}^{\min\{m,n\}} \frac{(2n-i-1)!}{(m-i)!(i-1)!(n-i)!}. \tag{42}$$

Proof. By duality and substituting l for m, from (38–40) the corollary can be obtained. □

Let θ be the root of the characteristic equation of (21). Then we have

$$\begin{cases} (1-\theta)\Big[1 - \frac{x(1-\sqrt{1-4\theta^2 z})}{2} + \theta(1-x)xzF_{\mathscr{M}^c}\Big] + \theta^2 xz = 0; \\ 1-\theta+\theta xzF_{\mathscr{M}^c} = 0. \end{cases} \tag{43}$$

By the second part of (43) we get

$$xzF_{\mathscr{M}^c} = \frac{\theta-1}{\theta}. \tag{44}$$

Now, by introducing a new parameter ξ such that

$$z = \frac{\xi(1-\xi)}{\theta^2}, \tag{45}$$

from the first part of (43), (44) and (45) one may find that

$$x = \frac{\theta}{\xi+\theta-1+\frac{\xi(1-\xi)}{\theta-1}}. \tag{46}$$

Further, let $\theta = 1+\xi\eta$. By (44–46) we have the following parameter expression of the function $F_{\mathscr{M}^c} = F_{\mathscr{M}^c}(x,z)$:

$$x = \frac{\eta(1+\xi\eta)}{1-\xi+\xi\eta+\xi\eta^2}, \quad z = \frac{\xi(1-\xi)}{(1+\xi\eta)^2}, \quad xzF_{\mathscr{M}^c} = \frac{\xi\eta}{1+\xi\eta}, \tag{47}$$

from which we get

$$\begin{aligned} \Delta_{(\eta,\xi)} &= \begin{vmatrix} \frac{(1-\xi)(1+2\xi\eta-\xi\eta^2)}{(1+\xi\eta)(1-\xi+\xi\eta+\xi\eta^2)} & \frac{\xi(1-\eta^2)}{(1+\xi\eta)(1-\xi+\xi\eta+\xi\eta^2)} \\ -\frac{2\xi\eta}{1+\xi\eta} & \frac{1-2\xi-\xi\eta}{(1-\xi)(1+\xi\eta)} \end{vmatrix} \\ &= \frac{1-2\xi-\xi\eta^2}{(1+\xi\eta)(1-\xi+\xi\eta+\xi\eta^2)}. \end{aligned} \tag{48}$$

Theorem 6. The enumerating function $F_{\mathscr{M}^c} = F_{\mathscr{M}^c}(x,z)$ has the following explicit expression:

$$\begin{aligned} F_{\mathscr{M}^c}(x,z) =& 1 + \sum_{n\geq 1}\sum_{m=1}^{2n-1}\sum_{k=0}^{\min\{m,n\}}\sum_{i=0}^{\min\{m-k,n-k\}} \frac{m!}{(m-k)!i!(n-k-i)!} \\ & \times \frac{(2n-m-1)!J_{m,n}(k,i)}{(n-m+k)!(m-k-i)!(2k+i-m+2)!} x^m z^n, \end{aligned} \tag{49}$$

where

$$J_{m,n}(k,i) = (2k+i-m)(2k+i-m+1)(2k+i-m+2) - (n-k-i)(m-k-i)(m-k-i-1). \tag{50}$$

Proof. Applying Lagrangian inversion with two parameters [11] to formulae (47) and (48) we obtain

$$\begin{aligned}
F_{\mathscr{M}^c}(x,z) &= \sum_{m,n\geq 1} \partial_{(\eta,\xi)}^{(m-1,n-1)} \frac{(1-\xi+\xi\eta+\xi\eta^2)^{m-1}(1+\xi\eta)^{2n-m-2}}{(1-\xi)^n} \\
&\quad \times (1-2\xi-\xi\eta^2)x^{m-1}z^{n-1} \\
&= \sum_{m,n\geq 0} \partial_{(\eta,\xi)}^{(m,n)} \frac{(1-\xi+\xi\eta+\xi\eta^2)^{m}(1+\xi\eta)^{2n-m-1}}{(1-\xi)^{n+1}} \\
&\quad \times (1-2\xi-\xi\eta^2)x^{m}z^{n} \\
&= 1 + \sum_{m,n\geq 1} \sum_{k=0}^{\min\{m,n\}} \binom{m}{k} \partial_{(\eta,\xi)}^{(m-k,n-k)} \frac{(1+\eta)^k(1+\xi\eta)^{2n-m-1}}{(1-\xi)^{n-m+k+1}} \\
&\quad \times (1-2\xi-\xi\eta^2)x^{m}z^{n} \\
&= 1 + \sum_{m,n\geq 1} \sum_{k=0}^{\min\{m,n\}} \sum_{i=0}^{\min\{m-k,n-k\}} \binom{m}{k}\binom{2n-m-1}{i} \\
&\quad \times \partial_{(\eta,\xi)}^{(m-k-i,n-k-i)} \frac{(1+\eta)^k(1-2\xi-\xi\eta^2)}{(1-\xi)^{n-m+k+1}} x^m z^n \\
&= 1 + \sum_{m,n\geq 1} \sum_{k=0}^{\min\{m,n\}} \sum_{i=0}^{\min\{m-k,n-k\}} \binom{m}{k}\binom{2n-m-1}{i} \\
&\quad \times \left[\partial_{(\eta,\xi)}^{(m-k-i,n-k-i)} \frac{(1+\eta)^k(1-2\xi)}{(1-\xi)^{n-m+k+1}}\right. \\
&\quad \left. - \partial_{(\eta,\xi)}^{(m-k-i-2,n-k-i-1)} \frac{(1+\eta)^k}{(1-\xi)^{n-m+k+1}}\right] x^m z^n \\
&= 1 + \sum_{m,n\geq 1} \sum_{k=0}^{\min\{m,n\}} \sum_{i=0}^{\min\{m-k,n-k\}} \binom{m}{k}\binom{2n-m-1}{i} \\
&\quad \times \left[\binom{k}{m-k-i} \partial_{\xi}^{n-k-i} \frac{1-2\xi}{(1-\xi)^{n-m+k+1}}\right. \\
&\quad \left. - \binom{k}{m-k-i-2} \partial_{\xi}^{n-k-i-1} (1-\xi)^{-(n-m+k+1)}\right] x^m z^n
\end{aligned}$$

$$
\begin{aligned}
=&1+\sum_{m,n\geq 1}\sum_{k=0}^{\min\{m,n\}}\sum_{i=0}^{\min\{m-k,n-k\}}\binom{m}{k}\binom{2n-m-1}{i}\\
&\times\left[\frac{(2n-m-i-1)!(2k+i-m)}{(n-k-i)!(n-m+k)!}\binom{k}{m-k-i}\right.\\
&\left.-\binom{k}{m-k-i-2}\binom{2n-m-i-1}{n-k-i-1}\right]x^m z^n,
\end{aligned}
$$

which is equivalent to the theorem. □

By duality and substituting m for l, from (49) and (50) we can obtain

Corollary 3. The number of rooted outer-planar maps with size $n(n \geq 1)$ and the root-face valency $l(1 \leq l \leq 2n-1)$ is

$$
\begin{aligned}
&\sum_{k=0}^{\min\{l,n\}}\sum_{i=0}^{\min\{l-k,n-k\}}\frac{l!(2n-l-1)!}{(l-k)!i!(n-k-i)!(n-l+k)!(l-k-i)!}\\
&\times\frac{J_{l,n}(k,i)}{(2k+i-l+2)!},
\end{aligned}
\tag{51}
$$

where

$$
\begin{aligned}
J_{l,n}(k,i) =&(2k+i-l)(2k+i-l+1)(2k+i-l+2)\\
&-(n-k-i)(l-k-i)(l-k-i-1).
\end{aligned}
\tag{52}
$$

Theorem 7. The number of boundary cubic maps with size $n(n \geq 2)$ and the root-face valency $l(1 \leq l \leq \lfloor\frac{2n-1}{3}\rfloor)$ is

$$
\begin{aligned}
&\sum_{k=0}^{\min\{l,n-l\}}\sum_{i=0}^{\min\{l-k,n-l-k\}}\frac{l!}{(l-k)!i!(n-l-k-i)!}\\
&\times\frac{(2n-3l-1)!R_{l,n}(k,i)}{(n-2l+k)!(l-k-i)!(2k+i-l+2)!},
\end{aligned}
\tag{53}
$$

where

$$
\begin{aligned}
R_{l,n}(k,i) =&(2k+i-l)(2k+i-l+1)(2k+i-l+2)\\
&-(n-l-k-i)(l-k-i)(l-k-i-1).
\end{aligned}
\tag{54}
$$

Proof. According to Lemma 1, (49) and (50), the theorem can be deduced by substituting m and n for l and $n-l$, respectively. □

References

[1] D. Arquès, Relations fonctionelles et dénombremant des cartes pointées sur le tore, J. Combin. Theory Ser. B 43 (1987) 253–274.

[2] E.A. Bender, Asymptotic methods in enumeration, SIAM Rev. 16 (1974) 485–515.

[3] E.A. Bender, E.R. Canfield, R.W. Robinson, The enumeration of maps on the torus and the projective plane, Canad. Math. Bull. 31 (1988) 257–271.

[4] E.A. Bender, E.R. Canfield, The asymptotic number of rooted maps on a surface, J. Combin. Theory Ser. A 43 (1986) 244–257.

[5] E.A. Bender, L.B. Richmond, A survey of the asymptotic behaviour of maps, J. Combin. Theory Ser. B 40 (1986) 297–329.

[6] E.A. Bender, N.C. Wormald, The asymptotic number of rooted nonseparable maps on a given surface, J. Combin. Theory Ser. A 49 (1988) 370–380.

[7] W.G. Brown, Enumeration of nonseparable planar maps, Canad. J. Math. XV (1963) 526–545.

[8] W.G. Brown, On the number of nonplanar maps, Mem. Amer. Math. Soc. 65 (1966) 1–42.

[9] Z.C. Gao, The number of rooted 2-connected triangular maps on the projective plane, J. Combin. Theory Ser. B 53 (1991) 130–142.

[10] Z.C. Gao, The asymptotic number of rooted 2-connected triangular maps on a surface, J. Combin. Theory Ser. B 54 (1992) 102–112.

[11] I.J. Good, Generalizations to several variables of Lagrange's expansion, with applications to stochastic processes, Proc. Camb. Phil. Soc. 56 (1960) 367–380.

[12] I.P. Goulden, D.M. Jackson, Combinatorial Enumeration, Wiley, New York, 1983.

[13] V.A. Liskovets, Enumeration of nonisomorphic planar maps, Selecta Math. Soviet. 4 (1985) 304–323.

[14] V.A. Liskovets, T.R.S. Walsh, Enumeration of eulerian and unicursal planar maps, Discrete Math. 282 (2004) 209–221.

[15] Wenzhong Liu, Yanpei Liu, Yan Xu, A census of boundary cubic rooted planar maps, Discrete Appl. Math. 155 (2007) 1678-1688.

[16] Yanpei Liu, Enumerative Theory of Maps, Kluwer, Boston, 1999.

[17] Yanpei Liu, On the number of rooted c-nets, J. Combin. Theory Ser. B 36 (1984) 118–123.

[18] Yanpei Liu, On functional equations arising from map enumerations, Discrete Math. 123 (1993) 93–109.

[19] Yanpei Liu, On the number of Eulerian planar maps, Acta Math. Sci. 12 (1992) 418–423.

[20] R.C. Mullin, P.J. Schellenberg, The enumeration of c-nets via quadrangulations, J. Combin. Theory 4 (1964) 256–276.

[21] W.T. Tutte, A census of planar triangulations, Canad. J. Math. 14 (1962) 21–38.

[22] W.T. Tutte, A census of slicings, Canad. J. Math. 14 (1962) 708–722.

[23] W.T. Tutte, A census of hamiltonian polygons, Canad. J. Math. Soc. 68 (1962) 402–417.

[24] W.T. Tutte, A census of planar maps, Canad. J. Math. 15 (1963) 249–271.

[25] W.T. Tutte, On the enumeration of planar maps, Bull. Amer. Math. Soc. 74 (1968) 64–74.

[26] E.T. Whittaker, G.N. Watson, A course of modern analysis, Cambridge, 1940.