Multidecompositions of complete bipartite graphs into cycles and stars

Hung-Chih Lee*
Department of Information Technology
Ling Tung University
Taichung 40852, Taiwan
E-mail: birdy@teamail.ltu.edu.tw

Abstract

Let C_k denote a cycle of length k and let S_k denote a star with k edges. For graphs F, G and H, a (G, H)-multidecomposition of F is a partition of the edge set of F into copies of G and copies of H with at least one copy of G and at least one copy of H. In this paper, necessary and sufficient conditions for the existence of the (C_k, S_k)-multidecomposition of a complete bipartite graph are given.

1 Introduction and preliminaries

For positive integers m and n, $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. A k-cycle, denoted by C_k, is a cycle of length k. A k-star, denoted by S_k, is the complete bipartite graph $K_{1,k}$.

Let F, G and H be graphs. A *G-decomposition* of F is a partition of the edge set of F into copies of G. If F has a G-decomposition, we say that F is *G-decomposable* and write $G|F$. A *(G, H)-multidecomposition* of F is a partition of the edge set of F into copies of G and copies of H with at least one copy of G and at least one copy of H. If F has a (G, H)-multidecomposition, we say that F is *(G, H)-multidecomposable* and write $(G, H)|F$.

A great deal of work has been done on G-decompositions of graphs (see survey articles [7, 9, 11, 14, 25] and a book [8]). In particular, C_k-decompositions of graphs have attracted considerable attention. The

*This research was supported by NSC of R.O.C. under grant NSC 99-2115-M-275-001

reader can refer to [10, 13, 16] for surveys of this topic. Decompositions of graphs into k-stars have also attracted a fair share of interest. Articles of interest include [12, 15, 23, 24, 26, 27]. It is natural to consider the problem for decomposing a graph into copies of two different graphs. The study of the (G, H)-multidecomposition was introduced by Abueida and Daven in [2]. Abueida and Daven [3] investigated the problem of the (K_k, S_k)-multidecomposition of the complete graph K_n. Abueida and O'Neil [6] settled the existence problem of the (C_k, S_{k-1})-multidecomposition of the complete multigraph λK_n for $k = 3, 4$ and 5. Priyadharsini and Muthusamy [17] established necessary and sufficient conditions for the existence of the (G_n, H_n)-multidecomposition of λK_n where $G_n, H_n \in \{C_n, P_{n-1}, S_{n-1}\}$. A *graph-pair* (G, H) *of order* m is a pair of non-isomorphic graphs G and H on m non-isolated vertices such that $G \cup H$ is isomorphic to K_m. Abueida and Daven [2] and Abueida, Daven and Roblee [4] completely determined the values of n for which λK_n admits a (G, H)-multidecomposition where (G, H) is a graph-pair of order 4 or 5. Abueida, Clark and Leach [1] and Abueida and Hampson [5] considered the existence of multidecompositions of $K_n - F$ for the graph-pair of order 4 and 5, respectively, where F is a Hamiltonian cycle, a 1-factor or almost 1-factor. Recently, Shyu [19] investigated the problem of decomposing K_n into k-paths and k-stars, and gave a necessary and sufficient condition for $k = 3$. In [20], Shyu considered the existence of a decomposition of K_n into k-paths and k-cycles, and established a necessary and sufficient condition for $k = 4$. Shyu [21] investigated the problem of decomposing K_n into k-cycles and k-stars, and settled the case $k = 4$.

In this paper, we investigate the problem of the multidecomposition of a complete bipartite graph into k-cycles and k-stars, and give necessary and sufficient conditions for such a multidecomposition to exist.

2 Main results

First we give necessary conditions of the (C_k, S_k)-multidecomposition of $K_{m,n}$. Before going on, some terms and notations are introduced. Let $deg_G(x)$ denote the degree of a vertex x in a graph G. The vertex of degree k in S_k is called the *center* of S_k. Suppose that $G_1, G_2, \ldots, G_t$ are graphs. Then $G_1 + G_2 + \cdots + G_t$, or $\sum_{i=1}^{t} G_i$, denotes the graph G with vertex set $V(G) = \bigcup_{i=1}^{t} V(G_i)$, and edge set $E(G) = \bigcup_{i=1}^{t} E(G_i)$. Thus, if a graph G can be decomposed into subgraphs $G_1, G_2, \ldots, G_t$, we write $G = G_1 + G_2 + \cdots + G_t$, or $G = \sum_{i=1}^{t} G_i$. Since each cycle uses two edges incident with a vertex, the following is trivial.

Lemma 2.1. *If a graph G can be decomposed into cycles, then the degree of each vertex of G must be even.*

Throughout the paper, we use (A, B) to denote the bipartition of $K_{m,n}$ where $A = \{a_0, a_1, \cdots, a_{m-1}\}$ and $B = \{b_0, b_1, \cdots, b_{n-1}\}$. Now we show the necessary conditions.

Lemma 2.2. *Let m and n be positive integers with $m \geq n$. If $K_{m,n}$ is (C_k, S_k)-multidecomposable, then $k \equiv 0 \pmod 2$, $4 \leq k \leq \min\{m, 2n\}$ and $mn \equiv 0 \pmod k$. Furthermore, $K_{m,n}$ is not (C_k, S_k)-multidecomposable in the following cases:* (1) $m \equiv 1 \pmod 2$ *and* $n < k$, (2) $(m,n) = (k, k/2+1)$ *for* $k \equiv 2 \pmod 4$ *or* $(m,n) = (k, k/2)$.

Proof. First, bipartite graphs contain no odd cycle, hence $k \equiv 0 \pmod 2$. Secondly, the minimum length of a cycle and the maximum size of a star in $K_{m,n}$ are 4 and m, respectively, we have $4 \leq k \leq m$. Moreover, each k-cycle in $K_{m,n}$ uses $k/2$ vertices of each partite set, which implies that $k \leq 2n$. Thirdly, the size of each member in the multidecomposition is k and $|E(K_{m,n})| = mn$, the condition $mn \equiv 0 \pmod k$ follows. Finally, we disprove the existence of the multidecomposition for the cases (1) and (2). Suppose, on the contrary, that there exists a (C_k, S_k)-multidecomposition $\mathscr{D}$ of $K_{m,n}$ if m and n belong to one of the cases (1) and (2). Since $n < k$ in those cases, each S_k in $\mathscr{D}$ must have its center in B. Let $H_1, H_2, \ldots, H_t$ be all of the k-stars in $\mathscr{D}$. We distinguish two cases.

Case 1. $m \equiv 1 \pmod 2$ and $n < k$.

Let $G = K_{m,n} - E(\sum_{i=1}^{t} H_i)$. Suppose that are c_j S_k's with centers at b_j for $j = 0, 1, \ldots, n-1$. Then $\deg_G b_j = m - kc_j$, which is odd for each $b_j \in B$ since m is odd and k is even. By Lemma 2.1, G is not C_k-decomposable, which leads to a contradiction.

Case 2. $(m,n) = (k, k/2+1)$ for $k \equiv 2 \pmod 4$ or $(m,n) = (k, k/2)$.

Note that $K_{m,n} - E(\sum_{i=1}^{t} H_i) = K_{m,n-t} + K_t^c$ where K_t^c is the complement of the complete graph K_t. Since $n \in \{k/2, k/2+1\}$ and $t \geq 1$, we have $n - t \leq k/2$. If $n - t < k/2$, then $K_{m,n-t}$ contains none of k-cycles. This is a contradiction. If $n - t = k/2$, then $k \equiv 2 \pmod 4$. This implies that $n - t$ is odd. Hence $K_{m,n-t}$ can not be decomposed into k-cycles by Lemma 2.1. We obtain a contradiction. □

From now on, we will show that the necessary conditions are also sufficient. The proof is divided into four cases: (i) $m \equiv 0 \pmod k$ or $n = k$, (ii) $m \geq 2k$ and $n > k$, (iii) $2k > m \geq n > k$, and (iv) $m > k > n \geq k/2$.

The following results due to Yamamoto et al. and Sotteau are essential for our discussions.

Proposition 2.3. (Yamamoto et al. [27]) *Let $m \geq n \geq 1$ be integers. Then $K_{m,n}$ is S_k-decomposable if and only if $m \geq k$ and*

$$\begin{cases} m \equiv 0 \pmod k & \text{if } n < k \\ mn \equiv 0 \pmod k & \text{if } n \geq k. \end{cases}$$

Proposition 2.4. (Sotteau [22]) *Let m, n and k be positive integers. Then there exists a C_k-decomposition of $K_{m,n}$ if and only if m, n and k are even, $k \geq 4$, $\min\{m,n\} \geq k/2$ and $mn \equiv 0 \pmod{k}$.*

For our discussions, more notations are needed. Suppose that G is a graph. Let V and E be subsets of the vertex set and the edge set of G, respectively. We use $G[V]$ to denote the subgraph of G induced by V and $G - E$ to denote the subgraph obtained from G by deleting E. Moreover, $\lceil x \rceil$ denotes the smallest integer not less than x and $\lfloor x \rfloor$ denotes the largest integer not greater than x. Let $(v_1, v_2, \ldots, v_k)$ denote the k-cycle with edges $v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_1$. Before plunging into the proof of the sufficiency, we need a result due to Ma, Pu and Shen.

Proposition 2.5. ([18]) *Let k and n be positive integers and let I be a 1-factor. Then there exists a k-cycle decomposition of $K_{n,n} - I$ if and only if $n \equiv 1 \pmod{2}$, $k \equiv 0 \pmod{2}$, $4 \leq k \leq 2n$ and $n(n-1) \equiv 0 \pmod{k}$.*

Lemma 2.6. *Let k be a positive even integer and let p be a positive integer. Then there exist $pk/2 - p$ edge-disjoint k-cycles in $K_{pk,k/2}$ (also in $K_{k/2,pk}$).*

Proof. It suffices to show the result holds for $K_{pk,k/2}$. If $k \equiv 0 \pmod{4}$, then $k/2$ is even. By Proposition 2.4, there exists a C_k-decomposition $\mathscr{D}$ of $K_{pk,k/2}$ with $|\mathscr{D}| = pk/2$, in which k-cycles are edge-disjoint. If $k \equiv 2 \pmod{4}$, then $k/2$ is odd. By Proposition 2.5, there exists a C_k-decomposition $\mathscr{D}'$ of $K_{k/2,k/2} - I$ with $|\mathscr{D}'| = (k-2)/4$. Since $K_{pk,k/2}$ can be decomposed into $2p$ copies of $K_{k/2,k/2}$, there exist $2p|\mathscr{D}'| = pk/2 - p$ edge-disjoint k-cycles in $K_{pk,k/2}$. This completes the proof. □

Lemma 2.7. *Let $k \geq 4$ be a positive even integer. Then $K_{m,n}$ has a (C_k, S_k)-multidecomposition if one of the following conditions holds:*

(1) $m \equiv 0 \pmod{k}$, $k/2 \leq n \leq k$, *and* $(m,n) \neq (k, k/2+1)$ *for* $k \equiv 2 \pmod{4}$ *and* $(m,n) \neq (k, k/2)$,
(2) $n = k < m$.

Proof. We distinguish two cases.
Case 1. $m \equiv 0 \pmod{k}$, $k/2 \leq n \leq k$ and $(m,n) \neq (k, k/2+1)$ for $k \equiv 2 \pmod{4}$ and $(m,n) \neq (k, k/2)$.

Let $m = pk$ where p is a positive integer. Note that for $n > 2$

$$\begin{aligned} K_{m,n} = K_{pk,n} &= K_{pk,n-2} + K_{pk,2} \\ &= K_{pk,n-1} + K_{pk,1}. \end{aligned}$$

By Proposition 2.4, $C_k \mid K_{pk,n-2}$ when n is even and $n \geq k/2+2$, and $C_k \mid K_{pk,n-1}$ when n is odd and $n \geq k/2+1$. By Proposition 2.3, $S_k \mid K_{pk,2}$

and $S_k \mid K_{pk,1}$. Thus, $K_{m,n}$ is (C_k, S_k)-multidecomposable when n is even with $n \geq k/2+2$ or n is odd with $n \geq k/2+1$. Since $(m,n) \neq (k, k/2+1)$ for $k \equiv 2 \pmod 4$ and $(m,n) \neq (k, k/2)$, we have that $n \geq k/2+2$ for even n and $n \geq k/2+1$ for odd n when $m = k$. So it remains to consider the cases that $m = pk$ with $p \geq 2$ and $n = k/2+1$ for $k \equiv 2 \pmod 4$ and $n = k/2$. We distinguish two subcases according to the parity of n.

Subcase 1.1. $n \in \{k/2, k/2+1\}$ and n is even.

Note that $K_{m,n} = K_{pk,n} = K_{(p-1)k,n} + K_{k,n}$. Since n is even and $n \geq k/2$, we have $C_k \mid K_{(p-1)k,n}$ by Proposition 2.4. On the other hand, $S_k \mid K_{k,n}$ by Proposition 2.3, we have the result.

Subcase 1.2. $n = k/2$ for $k \equiv 2 \pmod 4$.

Note that $pk/2 - p = p(k-2)/2 \geq k-2 \geq k/2$ for $p \geq 2$ and $k \geq 4$. By Lemma 2.6, there exist $k/2$ edge-disjoint k-cycles $Q_1, Q_2, \ldots, Q_{k/2}$ in $K_{pk,k/2}$ for $p \geq 2$. Let $G = K_{pk,k/2} - E(\sum_{i=1}^{k/2} Q_i)$. For each $b_j \in B$, since $\deg_{K_{pk,k/2}} b_j = pk$ and each Q_i uses two edges incident with b_j, we have $\deg_G b_j = pk - k = (p-1)k$. Thus, G can be decomposed into k-stars with centers in B. This settles Case 1.

Case 2. $n = k < m$.

Note that $K_{m,n} = K_{m,k} = K_{k,k} + K_{m-k,k}$. By Proposition 2.4, $C_k \mid K_{k,k}$, and by Proposition 2.3, $S_k \mid K_{m-k,k}$. Thus, $(C_k, S_k) \mid K_{m,n}$ and the proof is complete. □

Lemma 2.8. *Let k be a positive even integer and let m and n be positive integers with $m \geq n > k \geq 4$. If $m \geq 2k$ and $mn \equiv 0 \pmod k$ then $K_{m,n}$ has a (C_k, S_k)-multidecomposition.*

Proof. Let $m = pk + r$ where p and r are integers with $0 \leq r < k$. Note that $p \geq 2$ for $m \geq 2k$, and

$$\begin{aligned} K_{m,n} = K_{pk+r,n} &= K_{(p-1)k,n} + K_{k+r,n} \\ &= K_{(p-1)k,n-1} + K_{(p-1)k,1} + K_{k+r,n}. \end{aligned}$$

Since $n > k \geq 4$, we have $n-1 > k/2$. Thus, $C_k \mid K_{(p-1)k,n}$ for even n and $C_k \mid K_{(p-1)k,n-1}$ for odd n by Proposition 2.4. On the other hand, $|E(K_{k+r,n})| = n(k+r) \equiv 0 \pmod k$ from the assumption $mn \equiv 0 \pmod k$. This implies that $S_k \mid K_{k+r,n}$ by Proposition 2.3. Trivially, $S_k \mid K_{(p-1)k,1}$. Hence, $(C_k, S_k) \mid K_{m,n}$ and the proof is complete. □

Lemma 2.9. *Let k be a positive even integer and let m and n be positive integers with $2k > m \geq n > k \geq 4$. If $mn \equiv 0 \pmod k$, then $K_{m,n}$ has a (C_k, S_k)-multidecomposition.*

Proof. Suppose that $m = k + r$ and $n = k + s$. Then $k > r \geq s > 0$ from the assumption $2k > m \geq n > k$. Let $A_0 = \{a_0, a_1, \ldots, a_{k/2-1}\}$,

$A_1 = \{a_{k/2}, a_{k/2+1}, \dots, a_{k-1}\}$, $A' = A - (A_0 \cup A_1)$, $B_0 = \{b_0, b_1, \dots, b_{s-1}\}$ and $B' = B - B_0$. Let $G_i = K_{m,n}[A_i \cup B']$ for $i = 0, 1$, $F = K_{m,n}[A' \cup B']$ and $H = A \cup B_0$. Then $K_{m,n} = G_0 + G_1 + F + H$. Note that G_0 and G_1 are isomorphic to $K_{k/2,k}$, F is isomorphic to $K_{r,k}$ and H is isomorphic to $K_{m,s}$. Since $k \mid mn$, we have $k \mid rs$, which implies $t = rs/k$ is a positive integer. Let $p_0 = \lceil t/2 \rceil$ and $p_1 = \lfloor t/2 \rfloor$. Then $p_0 = 1$ and $p_1 = 0$ for $t = 1$ and $p_0 \geq p_1 \geq 1$ for $t \geq 2$. Trivially, F is S_k-decomposable. In the following, we will show that, for $0 \leq i \leq \delta$ where $\delta = 0$ if $p_1 = 0$ and $\delta = 1$ if $p_1 \geq 1$, G_i can be decomposed into p_i copies of C_k and $k/2$ copies of S_{k-2p_i}, and H can be decomposed into $k/2$ copies of S_{2p_i} and s copies of S_k where the $(k-2p_i)$-stars and $2p_i$-stars have their centers in A_i. In particular, G_1 is decomposed into k-stars if $p_1 = 0$.

We first show the required multidecomposition of G_i. Since $r < k$, we have $t < s$. Thus, $t + 1 \leq s$; in turn, $p_0 = \lceil t/2 \rceil \leq (t+1)/2 \leq s/2 < k/2$, which implies $p_i \leq k/2 - 1$ for $i = 0, 1$. This assures us that there exist p_i edge-disjoint k-cycles in G_i by Lemma 2.6. Suppose that $Q_{i,0}, Q_{i,1}, \dots, Q_{i,p_i-1}$ are edge-disjoint k-cycles in G_i for $0 \leq i \leq \delta$ where $\delta = 0$ if $p_1 = 0$ and $\delta = 1$ if $p_1 \geq 1$. Let $F_i = G_i - E(\sum_{h=0}^{p_i-1} Q_{i,h})$ and $X_{i,j} = F_i[\{a_{ik/2+j}\} \cup B']$ where $j = 0, 1, \dots, k/2 - 1$. Since $\deg_{G_i} a_{ik/2+j} = k$ and each $Q_{i,h}$ uses two edges incident with $a_{ik/2+j}$ for each i and j, we have $\deg_{F_i} a_{ik/2+j} = k - 2p_i$. Hence, $X_{i,j}$ is a $(k-2p_i)$-star with the center at $a_{ik/2+j}$.

Now we show the required star-decomposition of H by orienting the edges of H. For any vertex x of H, we use $\deg^+ x$ ($\deg^- x$, respectively) to denote the outdegree (indegree, respectively) of x in an orientation of H. It is sufficient to show that there exists an orientation of H such that, for $0 \leq i \leq \delta$ where $\delta = 0$ if $p_1 = 0$ and $\delta = 1$ if $p_1 \geq 1$, $j = 0, 1, \dots, k/2 - 1$ and $w = 0, 1, \dots, s - 1$,

$$\deg^+ a_{ik/2+j} = 2p_i \quad (1)$$

$$\deg^+ b_w = k. \quad (2)$$

First, the edges $a_j b_{2jp_0}, a_j b_{2jp_0+1}, \dots, a_j b_{2(j+1)p_0-1}$, and in case $p_1 \geq 1$ $a_{k/2+j} b_{2jp_1+kp_0}$, $a_{k/2+j} b_{2jp_1+kp_0+1}, \dots, a_{k/2+j} b_{2(j+1)p_1+kp_0-1}$ are all oriented outward from $a_{ik/2+j}$ where the subscripts of b's are taken modulo s. Note that from each $a_{ik/2+j}$, we orient $2p_i$ edges. Since $2p_1 \leq 2p_0 \leq t + 1 \leq s$, this assures us that there are enough edges for the above orientation. Finally, the edges which are not oriented yet are all oriented from B_0 to A.

From the construction of the orientation, it is easy to see that (1) is satisfied, and for all $b_w, b_{w'} \in B_0$, we have

$$|\deg^- b_w - \deg^- b_{w'}| \leq 1. \quad (3)$$

So, we only need to check (2).

Since $\deg^+ b_w + \deg^- b_w = k + r$ for $b_w \in B_0$, it follows from (3) that $|\deg^+ b_w - \deg^+ b_{w'}| \leq 1$ for $b_w, b_{w'} \in B_0$. Furthermore,

$$\begin{aligned}\sum_{w=0}^{s-1} \deg^+ b_w &= |E(K_{k+r,s})| - \sum_{i=0}^{\delta} \sum_{j=0}^{k/2-1} \deg^+ a_{ik/2+j} \\ &= (k+r)s - (2p_0 + 2p_1)k/2 \\ &= ks + rs - tk \\ &= ks\end{aligned}$$

where $\delta = 0$ if $p_1 = 0$ and $\delta = 1$ if $p_1 \geq 1$. Thus $\deg^+ b_w = k$ for $b_w \in B_0$. This proves (2). Hence, there exists a decomposition $\mathscr{D}$ of H into $k/2$ copies of S_{2p_i} with center at A_i and s copies of S_k with center at B_0. Let $X'_{i,j}$ be the $2p_i$-star with center at $a_{ik/2+j}$ in $\mathscr{D}$. Then $X_{i,j} + X'_{i,j}$ is a k-star. This completes the proof. □

Lemma 2.10. *Let k and m be positive even integers and n be a positive integer with $m > k > n \geq k/2 \geq 2$ and $k \nmid m$. If $mn \equiv 0 \pmod{k}$, then $K_{m,n}$ has a (C_k, S_k)-multidecomposition.*

Proof. Let $m = uk + r$ where u and r are integers with $0 < r < k$. Since k and m are even, r is even. Hence $2 \leq r \leq k - 2$. Note that $k \mid rn$ from the assumption $k \mid mn$. In the following we will prove that $K_{m,n}$ can be decomposed into rn/k copies of C_k and nu copies of S_k.

Let $k = 2x$, $r = 2y$ and $d = \gcd(n, x)$. Then $d > 1$ from the assumption $k \mid mn$ and $k \nmid m$. Take $n = dp$ and $x = ds$. Then p and s are coprime. This implies $s \mid y$ since $k \mid rn$. Let $y = sq$, we have $rn/k = pq$. Moreover, since $q = r/(2s)$ and $2 \leq r \leq k - 2 = 2ds - 2$, we have $1 \leq q \leq d - 1$; in turn, $kpq < kpd = kn$, this assures us that there are enough edges for constructing pq edge-disjoint k-cycles in $K_{k,n}$-subgraph of $K_{m,n}$. Let $t = \min\{q, \lfloor d/2 \rfloor\}$. Define pq C_k's as follows. For $i = 0, 1, \ldots, p-1$, $j = 0, 1, \ldots, t-1$ and $h = 0, 1, \ldots, q - \lfloor d/2 \rfloor - 1$, let

$$\begin{aligned}C_{i,j} &= (b_{xi+2j}, a_0, b_{xi+2j+1}, a_1, \ldots, b_{x(i+1)+2j-1}, a_{x-1}), \text{ and} \\ C'_{i,h} &= (b_{xi+2h}, a_x, b_{xi+2h+1}, a_{x+1}, \ldots, b_{x(i+1)+2h-1}, a_{2x-1}) \text{ if } q > \lfloor d/2 \rfloor,\end{aligned}$$

where the subscripts of b's are taken modulo n. Let $\mathscr{C}$ be the set of the pq C_k's defined above, and let H be the spanning subgraph of $K_{m,n}$ with

$$E(H) = \begin{cases} \bigcup E(C_{i,j}) & \text{if } q \leq \lfloor d/2 \rfloor, \\ \bigcup E(C_{i,j} + C'_{i,h}) & \text{if } q > \lfloor d/2 \rfloor. \end{cases}$$

where $0 \leq i \leq p-1$, $0 \leq j \leq t-1$ and $0 \leq h \leq q - \lfloor d/2 \rfloor - 1$.

We first check that cycles in $\mathscr{C}$ are edge-disjoint. Observe that in $C_{i,j}$, a_v is adjacent to $b_{xi+2j+v}$ and $b_{xi+2j+v+1}$ for $v = 0, 1, \ldots, x-2$, and a_{x-1} is adjacent to b_{xi+2j} and $b_{x(i+1)+2j-1}$. For $0 \le i, i' \le p-1$ and $0 \le j, j' \le t-1$, $1-p \le i'-i \le p-1$ and $2-d \le 2-2t \le 2(j'-j)+\delta \le 2t-1 \le d-1$ where $\delta \in \{0,1\}$. If $i' \ne i$ or $j' \ne j$, then $n \nmid x(i'-i)+2(j'-j)+\delta$ and $n \nmid x(i'-i+1)+2(j'-j)-1$, and hence $xi'+2j'+v+\delta$ is not congruent to $xi+2j+v$ modulo n when $v = 0, 1, \ldots, x-2$, and $x(i'+1)+2j'-1$ is not congruent to both of $x(i+1)+2j-1$ and $xi+2j$ modulo n. This implies $C_{i,j}$'s are edge-disjoint. Similarly, $C'_{i,h}$'s are also edge-disjoint. Clearly, $E(C_{i,j}) \cap E(C'_{z,h}) = \emptyset$. Thus, cycles in $\mathscr{C}$ are edge-disjoint.

Let $G = K_{m,n} - E(H)$. Now we show that $S_k|G$. It is not difficult to verify that each vertex in B appears in $pqx/n = sq$ cycles in $\mathscr{C}$. Thus, $\deg_H b_w = 2sq = 2y = r$ for each $b_w \in B$. It implies $\deg_G b_w = m - r = uk$ and hence G can be decomposed into k-stars with centers in B. This completes the proof. □

Now, we are ready for the main result. It is obtained by combining Lemmas 2.2, 2.7 to 2.10.

Theorem 2.11. *Let k, m and n be positive integers with $m \ge n$. Then $K_{m,n}$ has a (C_k, S_k)-multidecomposition if and only if $k \equiv 0 \pmod 2$, $4 \le k \le \min\{m, 2n\}$ and $mn \equiv 0 \pmod k$ except for the following cases:* (1) *$m \equiv 1 \pmod 2$ and $n < k$,* (2) *$(m,n) = (k, k/2+1)$ for $k \equiv 2 \pmod 4$ or $(m,n) = (k, k/2)$.*

Acknowledgment

The author is grateful to the referee for the valuable comments.

References

[1] A. Abueida, S. Clark, D. Leach, Multidecomposition of the complete graph into graph pairs of order 4 with various leaves, Ars Combin. **93** (2009), 403–407.

[2] A. Abueida and M. Daven, Multidesigns for graph-pairs of order 4 and 5, Graphs Combin. **19** (2003), 433–447.

[3] A. Abueida and M. Daven, Mutidecompositons of the complete graph, Ars Combin. **72** (2004), 17–22 .

[4] A. Abueida, M. Daven and K. J. Roblee, Multidesigns of the λ-fold complete graph for graph-pairs of order 4 and 5, Australas. J. Combin. **32** (2005), 125–136.

[5] A. Abueida and C. Hampson, Multidecomposition of $K_n - F$ into graph-pairs of order 5 where F is a Hamilton cycle or an (almost) 1-factor, Ars Combin. **97** (2010), 399–416

[6] A. Abueida and T. O'Neil, Multidecomposition of λK_m into small cycles and claws, Bull. Inst. Combin. Appl. **49** (2007), 32–40.

[7] P. Adams, D. Bryant, M. Buchanan, A survey on the existence of G-designs, J. Combin. Des. **16** (2008), 373–410.

[8] J. Bosák, Decompositions of Graphs, Kluwer, Dordrecht, Netherlands, 1990.

[9] J. C. Bermond and D. Sotteau, Graph decompositions and G-designs, Congr. Numer. **15** (1976), 53–72.

[10] D. Bryant, Cycle decompositions of complete graphs, Surveys in Combinatorics 2007, A. Hilton and J. Talbot (Editors), London Mathematical Society Lecture Note Series 346, Proceedings of the 21st British Combinatorial Conference, Cambridge University Press, 2007, pp. 67–97.

[11] D. Bryant and S. El-Zanati, Graph decompositions, In: The CRC Handbook of Combinatorial Designs, Second Edition, C. J. Colbourn and J. H. Dinitz (Editors), CRC Press, Boca Raton, 2007, pp. 477–486.

[12] D. E. Bryant, S. El-Zanati, C.V. Eyden and D.G. Hoffman, Star decompositions of cubes, Graphs Combin. **17** (2001), 55–59.

[13] D. Bryant and C. A. Rodger, Cycle decompositions, In: The CRC Handbook of Combinatorial Designs, Second Edition, C. J. Colbourn and J. H. Dinitz (Editors), CRC Press, Boca Raton, 2007, pp. 373–382.

[14] K. Heinrich, Graph decompositions, In: The CRC Handbook of Combinatorial Designs, C. J. Colbourn and J. H. Dinitz (Editors), CRC Press, Boca Raton, 1996, pp. 361–366.

[15] C. Lin, J.-J. Lin, T.-W. Shyu, Isomorphic star decomposition of multicrowns and the power of cycles, Ars Combin. **53** (1999) 249–256.

[16] C. C. Lindner and C. A. Rodger, Decomposition in cycles II: Cycle systems, In: Contemporary design theory: A collection of surveys, J. H. Dinitz and D. R. Stinson (Editors), Wiley, New York, 1992, pp. 325–369.

[17] H. M. Priyadharsini and A. Muthusamy, (G_m, H_m)-multifactorization of λK_m, J. Combin. Math. Combin. Comput. **69** (2009), 145–150.

[18] J. Ma, L. Pu and H. Shen, Cycle decompositions of $K_{n,n} - I$, SIAM J. Discrete Math. **20** (2006), 603–609.

[19] T.-W. Shyu, Decomposition of complete graphs into paths and stars, Discrete Math. **310** (2010), 2164–2196.

[20] T.-W. Shyu, Decompositions of complete graphs into paths and cycles, Ars Combin. **97** (2010), 257–270.

[21] T.-W. Shyu, Decomposition of complete graphs into cycles and stars, Graphs Combin. (in press) DOI: 10.1007/s00373-011-1105-3.

[22] D. Sotteau, Decomposition of $K_{m,n}$ ($K^*_{m,n}$) into cycles (circuits) of length $2k$, J. Combin. Theory, Ser. B **30** (1981), 75–81.

[23] M. Tarsi, Decomposition of complete multigraphs into stars, Discrete Math. **26** (1979), 273-278.

[24] S. Tazawa, Decomposition of a complete multipartite graph into isomorphic claws, SIAM J. Algebraic Discrete Methods **6** (1985), 413-417.

[25] K. Ushio, G-designs and related designs, Discrete Math **116** (1993), 299–311.

[26] K. Ushio, S. Tazawa, S. Yamamoto, On claw-decomposition of complete multipartite graphs, Hiroshima Math. J. **8** (1978) 207–210.

[27] S. Yamamoto, H. Ikeda, S. Shige-ede, K. Ushio and N. Hamada, On claw decomposition of complete graphs and complete bipartie graphs. Hiroshima Math. J. **5** (1975), 33–42.