An implicit degree condition for hamiltonian cycles

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Abstract

In [11], Zhu, Li and Deng introduced the definition of implicit degree of a vertex v, denoted by $id(v)$. In this paper, we consider implicit degrees and the hamiltonicity of graphs and obtain that: If G is a 2-connected graph of order n such that $id(u) + id(v) \geq n - 1$ for each pair of vertices u and v at distance 2, then G is hamiltonian with some exceptions.

Keywords: Implicit degree; Hamiltonian cycles; Graph

1 Introduction

Throughout this paper, we consider only finite, undirected and simple graphs. We will generally follow the notation and terminology of Bondy and Murty in [3]. For a graph $G = (V(G), E(G))$, $V(G)$ and $E(G)$ denote the vertex-set and edge-set of G respectively. Let H be a subgraph of G, $G[H]$ denotes the subgraph of G induced by $V(H)$. The neighborhood in H of a vertex $u \in V(G)$ is $N_H(u) = \{v \in V(H) : uv \in E(G)\}$ and the degree of u in H is $d_H(u) = |N_H(u)|$. If $H = G$, we can use $N(v)$ and $d(v)$ in place

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of $N_G(v)$ and $d_G(v)$, respectively. Let $N_2(v) = \{u \in V(G) : d(u,v) = 2\}$, where $d(u,v)$ indicates the distance from u to v in G. A and B being the subsets of $V(G)$, $e(A,B)$ is the number of edges ab of G with $a \in A$ and $b \in B$. We write $e(A,b)$ instead of $e(A,\{b\})$.

For a cycle (or a path) C in G with a given orientation and a vertex y in C, y^+ and y^- denote the successor and the predecessor of y in C, respectively. Define $y^{(h+1)+} = (y^{h+})^+$ for every integer $h \geq 0$, with $y^{0+} = y$. y^{h-} is defined analogously. And for any $I \subseteq V(C)$, let

$$I^- = \{y : y^+ \in I\} \ \ \textit{and} \ \ I^+ = \{y : y^- \in I\}.$$

A cycle (or a path) containing all vertices of G is called a hamiltonian cycle (or a hamiltonian path). A graph G is called hamiltonian if it contains a hamiltonian cycle. A cycle C is called an l-cycle if $|V(C)| = l$.

In order to give the results of this paper, we define some special graphs.

(1) The join of two disjoint graphs G and H, denoted by $G \vee H$, is defined as: $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

(2) Let $n \geq 7$ be an odd integer. By $\mathscr{G}_n$ we denote the family of graphs such that $G \in \mathscr{G}_n$ if and only if $|V(G)| = n$ and the vertex-set of G is the disjoint union of the sets A_1, A_2, B_1, B_2 and $\{a_1, a_2, b\}$ so that

(i) $|A_i \cup B_i| = \frac{n-3}{2}, i = 1,2$;

(ii) $|A_i| \geq 2, i = 1,2$;

(iii) $G[A_i \cup B_i]$ and $G[A_i \cup \{a_j\}]$ are both complete subgraphs of G for $i = 1,2$ and $j = 1,2$;

(iv) $e(a_1,a_2) \leq 1$;

(v) $|A_1 \cup A_2| \geq \frac{n-3}{2} - e(a_1,a_2)$; and

(vi) $d(b) = 2$ and the neighbors of b are a_1 and a_2. (See Fig.1.)

(3) H is the graph of order 9 depicted in Fig.2.

(4) $\mathscr{H}_n = (kK_1 \cup 2K_{\frac{n-1}{2}-k}) \vee K_{k+1}$.

(5) Let $n \geq 7$ be an odd integer. $\mathscr{B}_n$ denotes the family of graphs such that $G \in \mathscr{B}_n$ if and only if $|V(G)| = n$ and $V(G)$ is the disjoint union of the sets A_1, A_2, B_1, B_2 and $\{a_1, a_2, b\}$ so that they satisfy the above (i),(iv),(v),(vi) and

(vii) $G[A_i \cup \{a_j\}]$ is complete subgraph of G and $uv \in E(G)$ for any vertex $u \in A_i$ and any vertex $v \in B_i$ for $i = 1,2$ and $j = 1,2$;

(viii) $|A_i| \geq \max\{2, |\{b : d(b) < \frac{n-5}{2} \text{ and } b \in B_i\}| + 1\}, i = 1,2$.

The hamiltonian problem is an important problem in graph theory. Various sufficient conditions for a graph to be hamiltonian have been given in terms of degree conditions. We have two classic results due to Dirac and Fan respectively.

Theorem 1. *([6]) If G is a graph of order $n \geq 3$ such that $d(u) \geq \frac{n}{2}$ for each vertex u in G, then G is hamiltonian. The bound is sharp.*

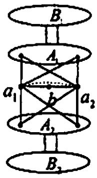

Fig.1

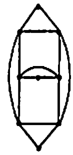

Fig.2

Theorem 2. *([8]) Let G be a 2-connected graph of order $n \geq 3$ such that $\max\{d(u), d(v)\} \geq \frac{n}{2}$ for each pair of vertices u and v at distance 2, then G is hamiltonian. The bound is sharp.*

In order to generalize Theorems 1 and 2, Zhu, Li and Deng proposed the concept of implicit degrees of vertices in [11] as follows.

Definition 1. *([11]) Let v be a vertex of a graph G. If $N_2(v) \neq \emptyset$ and $d(v) \geq 2$, then set $k = d(v) - 1$, $m_2^v = \min\{d(u) : u \in N_2(v)\}$ and $M_2^v = \max\{d(u) : u \in N_2(v)\}$. Suppose $d_1^v \leq d_2^v \leq \cdots \leq d_{k+1}^v \leq \cdots$ is the degree sequence of vertices of $N(v) \cup N_2(v)$. Let*

$$d^*(v) = \begin{cases} m_2^v, & \text{if } d_k^v < m_2^v; \\ d_{k+1}^v, & \text{if } d_{k+1}^v > M_2^v; \\ d_k^v, & \text{otherwise.} \end{cases}$$

Then the implicit degree of v, is defined as $id(v) = \max\{d(v), d^*(v)\}$. If $N_2(v) = \emptyset$ or $d(v) \leq 1$, then we define $id(v) = d(v)$.

Clearly, from the definition of implicit degree, we have $id(v) \geq d(v)$ for every vertex v. The authors in [11] gave a sufficient condition for a graph to be hamiltonian with implicit degrees.

Theorem 3. *([11]) Let G be a 2-connected graph such that $id(u)+id(v) \geq c$ for each pair of nonadjacent vertices u and v in G. Then G contains either a hamiltonian cycle or a cycle of length at least c.*

Chen [4] extended Theorem 3 as follows.

Theorem 4. *([4]) Let G be a 2-connected graph such that $\max\{id(u), id(v)\} \geq c/2$ for each pair of vertices u and v at distance 2. Then G contains either a hamiltonian cycle or a cycle of length at least c.*

In 1987, Benhocine and Wojda [1] extended the result of Fan as follows.

Theorem 5. *([1]) Let G be a 2-connected graph of order $n \geq 3$ with independent number $\alpha(G) \leq \frac{n}{2}$ such that $\max\{d(u), d(v)\} \geq \frac{n-1}{2}$ for each* *pair of vertices u and v at distance 2, then either G is hamiltonian or* $G \in \mathscr{G}_n \cup H$.

Recently, Cai and Dai extended Theorem 3 as follows.

Theorem 6. *([5]) Let G be a 2-connected graph of order* $n \geq 3$ *with independent number* $\alpha(G) \leq \frac{n}{2}$ *such that* $id(v) \geq \frac{n-1}{2}$ *for each vertex v, then either G is hamiltonian or* $G \in \mathscr{B}_n \cup H$ *or G is a subgraph of* $\mathscr{H}_n$.

Motivated by the results of Theorems 4 and 5, we study implicit degrees and the hamiltonicity of graphs and extend Theorem 6 as follows.

Theorem 7. *Let G be a 2-connected graph of order* $n \geq 3$ *such that* $id(u) + id(v) \geq n - 1$ *for each pair of vertices u and v at distance 2, then either G is hamiltonian or* $G \in \mathscr{B}_n \cup H$ *or G is a subgraph of* $\mathscr{H}_n \cup (\frac{n+1}{2}K_1 \vee K_{\frac{n-1}{2}})$.

Remark. The graph G depicted in [11] (see Fig.3) does not satisfy the condition of Theorem 5. It can, however, be easily verified by using Theorem 7 that G is hamiltonian.

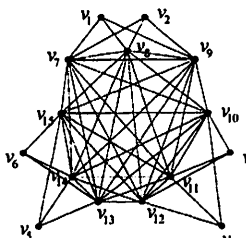

Fig.3

2 Lemmas

Lemma 1. *([7]) Let G be a 2-connected graph of order n, and let* $P(a,b)$ *be a longest path of G with* $d(a) + d(b) \geq n$*, then G is a hamiltonian.*

Lemma 2. *([1]) If a graph G of order n has a cycle C of length* $n-1$*, such that the vertex not in C has degree at least* $\frac{n}{2}$*, then G is hamiltonian.*

Lemma 3. *([11]) Let G be a 2-connected graph and* $P = x_1x_2 \cdots x_p$ *be a longest path of G. If* $d(x_1) < id(x_1)$ *and* $x_1x_p \notin E(G)$*, then either*
(1) there is some $x_j \in N^-(x_1)$ *such that* $d(x_j) \geq id(x_1)$*; or*
(2) $N(x_1) = \{x_2, x_3, \cdots, x_{d(x_1)+1}\}$ *and* $id(x_1) = m_2^{x_1}$.

The proof of the following Lemma is trivial, we omit it here.

Lemma 4. *Let $P = x_1x_2\cdots x_p$ be a path and y_1, y_2 be two vertices not in $V(P)$. If $N_P^-(y_1) \cap N_P(y_2) = \emptyset$ and $x_1y_1 \notin E(G)$, then*

$$d_P(y_1) + d_P(y_2) \le |V(P)|.$$

Lemma 5. *Let G be a 2-connected, non-hamiltonian graph of order n with $id(u) + id(v) \ge n-1$ for each pair of vertices u and v at distance 2 and for every $(n-1)$-cycle C in G, the vertex not in C has degree at most $\frac{n-2}{2}$. Then either $G \in \mathscr{B}_n$ or G is isomorphic to H or G is a subgraph of $\mathscr{H}_n$.*

Proof. By Theorem 4, G contains an $(n-1)$-cycle. Choose an $(n-1)$-cycle C such that the degree of the vertex not in C is as large as possible. Let x be the vertex of G not in C. We must have $2 \le d(x) \le \frac{n-2}{2}$, thus $n \ge 6$. Choosing an arbitrary orientation on C, define $y_1, y_2, \cdots, y_{k+1} (k \ge 1)$ to be the neighbors of x. Since $\{x, y_1^+, y_2^+, \cdots, y_{k+1}^+\}$ is an independent set, $d(x, y_i^+) = 2$ for every $i = 1, 2, \cdots, k+1$.

Claim 1. $id(x) \ge \frac{n-1}{2}$.

Proof. Suppose to the contrary that $id(x) < \frac{n-1}{2}$. Since $d(x, y_i^+) = 2$ for every $i = 1, 2, \cdots, k+1$, we have $id(y_i^+) > \frac{n-1}{2}$ for each i. We consider the following hamiltonian path

$$P = y_1^+ y_1^{2+} \cdots y_1^{h+} y_2 x y_1 y_2^{l+} y_2^{(l-1)+} \cdots y_2^+,$$

where h and l are the minimum integers such that $y_1^{h+} = y_2^-$ and $y_2^{l+} = y_1^-$ respectively. For convenience, let $P = x_1x_2\cdots x_n$, where $x_1 = y_1^+, x_2 = y_1^{2+}$, and so on.

Clearly, we have $x_1x_n \notin E(G)$. By Lemma 1, we can assume, without loss of generality, that $id(x_1) > d(x_1)$. Since $y_1^+y_1 \in E(G)$ and $y_1^+x \notin E(G)$, we have $N(x_1) \ne \{x_2, x_3, \cdots, x_{d(x_1)+1}\}$. By Lemma 3, there exist some $x_i \in N^-(x_1)$ such that $d(x_i) \ge id(x_1)$. Let $P' = x_nx_{n-1}\cdots x_{i+1}x_1x_2\cdots x_i$, and note that P' is another hamiltonian path of G. If $id(x_n) = d(x_n)$, we have $d(x_n) + d(x_i) \ge id(x_n) + id(x_1) > n-1$, and hence by Lemma 1, G contains a hamiltonian cycle, a contradiction. Similarly, if $d(x_n) < id(x_n)$, there is some $x_j \in N_{P'}^-(x_n)$ such that $d(x_j) \ge id(x_n)$. If $j < i$, let

$$P_1 = x_jx_{j-1}\cdots x_1x_{i+1}x_{i+2}\cdots x_nx_{j+1}x_{j+2}\cdots x_i;$$

and if $j > i+1$, let

$$P_2 = x_jx_{j+1}\cdots x_nx_{j-1}x_{j-2}\cdots x_{i+1}x_1x_2\cdots x_i.$$

Since $d(x_i)+d(x_j) \ge id(x_1)+id(x_n) > n-1$, by Lemma 1 again, G contains a hamiltonian cycle, a contradiction. □

By the assumption of Lemma 5 and Claim 1, we know $d(x) < id(x)$.

Since $d(x, y_i^+) = 2$, $|N_2(x)| \geq k+1$. By the definition of implicit degree, we can easily get that $id(x) \neq d_{k+1}^x$. We consider the following two cases.

Case 1. $id(x) = m_2^x$.

Since $d(x, y_i^+) = 2$ for each $i = 1, 2, \cdots, k+1$, we have $d(y_i^+) \geq \frac{n-1}{2}$.
Since G is not hamiltonian, it is easy to check that
(1) $e(y_1^+, z^+) + e(y_2^+, z) \leq 1$ for every $z \in A = \{y_1^+, y_1^{2+}, \cdots, y_1^{h+}\}$, and
(2) $e(y_1^+, z) + e(y_2^+, z^+) \leq 1$ for every $z \in B = \{y_2^+, y_2^{2+}, \cdots, y_2^{l+}\}$, where h and l are defined as in Claim 1.
As $y_1^+ x \notin E(G)$ and $y_2^+ x \notin E(G)$, (1) and (2) imply

$$\begin{aligned} n-1 &\leq d(y_1^+) + d(y_2^+) \\ &= \sum_{z\in A}[e(y_1^+, z^+) + e(y_2^+, z)] + \sum_{z \in B}[e(y_1^+, z) + e(y_2^+, z^+)] \\ &\quad + e(y_1^+, y_1) + e(y_2^+, y_2) \\ &\leq h + l + 2 = n-1, \end{aligned}$$

which implies that all the inequalities above are equalities. In particular, $d(y_1^+) = d(y_2^+) = \frac{n-1}{2}$ and n is odd.

If $d(x) \geq 3$, we have $e(y_1^+, y_3^+) + e(y_2^+, y_3^{2+}) = 1$. As $y_1^+ y_3^+ \notin E(G)$, we deduce $y_2^+ y_3^{2+} \in E(G)$. Then G has a cycle of length $n-1$ avoiding y_3^+ whose degree is at least $\frac{n-1}{2}$, contrary to the hypothesis of Lemma 5. (An analogous argument shows that $y_1^+ y_1^{3+} \in E(G)$ and $y_2^+ y_1^{2+} \notin E(G)$.) So we can assume that $d(x) = 2$ and $h \geq 2, l \geq 2$.

By the choice of C, we can assume that for any an $(n-1)$-cycle, the vertex not in the cycle has degree 2.

Observe that y_1^+ and y_2^+ have degree precisely $\frac{n-1}{2}$ and are joined by a hamiltonian path P in G, where $P = y_1^+ y_1^{2+} \cdots y_1^{h+} y_2 x y_1 y_2^{l+} y_2^{(l-1)+} \cdots y_2^+$.

For convenience, let $P = x_1 x_2 \cdots x_n$, where $x_1 = y_1^+, x_2 = y_1^{2+}$, and so on. We may easily deduce the following useful properties:

(i) $e(x_1, x_{i+1}) + e(x_n, x_i) = 1$ for every $i = 1, 2, \cdots, n-1$;

(ii) If $e(x_1, x_{i+1}) + e(x_n, x_{i-1}) = 2$ for some $i = 2, 3, \cdots, n-1$, then $d(x_i) = 2$. Moreover, similarly as Claim 1, we can get that $id(x_i) \geq \frac{n-1}{2}$. Therefore, by the definition of implicit degrees, we have $d(x_{i-2}) \geq id(x_i) \geq \frac{n-1}{2}$ and $d(x_{i+2}) \geq id(x_i) \geq \frac{n-1}{2}$;

(iii) $x_1 x_{n-1} \notin E(G)$ and $x_n x_2 \notin E(G)$.

Since $x_1 x_3 = y_1^+ y_1^{3+} \in E(G), y_1^+ y_1 \in E(G)$ and $y_1^+ x \notin E(G)$, only two cases can arise.

Case 1.1. There are i and $j, j \geq i+1$, such that $x_1 x_{i-1} \in E(G), x_1 x_{j+1} \in E(G)$ and $x_1 x_s \notin E(G)$ for each $s = i, i+1, \cdots, j$.

Choose such i such that i is as small as possible. We have $i \geq 4$ and $j \leq n-3$ by (i) and (iii); $x_n x_s \in E(G)$ for all $s = i-1, i, \cdots, j-1$ by (i); $d(x_j) = 2$, $d(x_{j-2}) \geq \frac{n-1}{2}$ and $d(x_{j+2}) \geq \frac{n-1}{2}$ by (ii).

Using similar arguments as in [1], we can get the following Statement.

Statement. *If $z_1 z_2 \cdots z_n$ is a hamiltonian path of G such that there are i and $j, i+1 \leq j$, $z_1 z_{i-1} \in E(G), z_1 z_{j+1} \in E(G), z_1 z_s \notin E(G)$ for $s = i, i+1, \cdots, j$ and $d(z_i) \geq \frac{n-1}{2}$, then $j = i+1$, $d(z_{i+3}) \geq \frac{n-1}{2}$ and $d(z_{i-1}) \geq \frac{n-1}{2}$.*

Case 1.1.1. $d(x_i) \geq \frac{n-1}{2}$.

By the Statement, we have $j = i+1$, $d(x_{i+3}) \geq \frac{n-1}{2}$ and $d(x_{i-1}) \geq \frac{n-1}{2}$. Let $P' = x_1 x_2 \cdots x_{i-1} x_n x_{n-1} \cdots x_i$. Since $x_1 x_{i-1} \in E(G), x_1 x_n \notin E(G), x_1 x_{n-1} \notin E(G), x_1 x_{i+2} \in E(G)$ and $d(x_n) \geq \frac{n-1}{2}$, we have $x_1 x_{n-2} \in E(G)$ and $d(x_{n-3}) \geq \frac{n-1}{2}$ by the Statement. Moreover, $d(x_{n-1}) = 2$ since $x_i x_n \in E(G)$. Then using P we can obtain $x_n x_{n-3} \notin E(G)$.

If $i+3 < n-2$, then considering the hamiltonian path

$$x_{i-1} x_{i-2} \cdots x_1 x_{i+2} x_{i+1} x_i x_n x_{n-1} \cdots x_{i+3},$$

we can get $x_{i-1} x_{n-2} \in E(G)$ by (i). So taking the hamiltonian path

$$x_1 x_2 \cdots x_{i-1} x_{n-2} x_{n-1} x_n x_i x_{i+1} \cdots x_{n-3},$$

and observing that $x_{n-3} x_n \notin E(G)$ implies $x_1 x_i \in E(G)$ by (i), but this contradicts the hypothesis of Case 1.1.

Assume $i+3 = n-2$, then i is even. Referring to the hamiltonian path

$$x_i x_{i+1} x_{i+2} x_1 x_2 \cdots x_{i-1} x_{i+5} x_{i+4} x_{i+3},$$

we have $x_i x_{i+2} \in E(G)$ by (i).

Since $x_i x_{i+2} \in E(G), x_1 x_i \notin E(G), x_2 x_i \notin E(G)$ (for $d(x_1) \geq \frac{n-1}{2}$), $x_i x_{i-1} \in E(G)$ and $d(x_1) \geq \frac{n-1}{2}$, we have by the Statement $x_i x_3 \in E(G)$ implying $d(x_2) = 2$ ($x_1 x_{i+3} \in E(G)$). If $i = 4$, we obtain $n = 9$ and G is isomorphic to H.

Then suppose $i \geq 6$. We have $d(x_4) \geq \frac{n-1}{2}$ and $d(x_{i+2}) \geq \frac{n-1}{2}$. Taking the hamiltonian path

$$x_{i+2} x_{i+1} x_i x_3 x_2 x_1 x_{i+3} x_{i+4} x_{i+5} x_{i-1} x_{i-2} \cdots x_4,$$

by (i) and the fact $d(x_{i+4}) = 2$, we obtain $x_{i+2} x_{i+5} \in E(G)$. A hamiltonian cycle is then $x_{i+2} x_{i+1} \cdots x_1 x_{i+3} x_{i+4} x_{i+5} x_{i+2}$, a contradiction.

Case 1.1.2. $d(x_i) < \frac{n-1}{2}$.

We have $x_{i-2} x_i \notin E(G)$, for $x_n x_{i-1} \in E(G), x_1 x_{i-1} \in E(G)$ and G is not hamiltonian.

Claim 2. $d(x_{i-2}) < \frac{n-1}{2}$.

Proof. Suppose to the contrary that $d(x_{i-2}) \geq \frac{n-1}{2}$. By considering the hamiltonian path

$$x_{i-2} x_{i-3} \cdots x_1 x_{j+1} x_j \cdots x_{i-1} x_n x_{n-1} \cdots x_{j+2},$$

and using the fact that $x_{i-2}x_n \notin E(G)$, we deduce $x_{j+2}x_{i-1} \in E(G)$. Then

$$x_1x_2\cdots x_{i-1}x_{j+2}x_{j+3}\cdots x_nx_ix_{i+1}\cdots x_{j+1}x_1,$$

is a hamiltonian cycle of G, a contradiction. □

Claim 3. $j = i+1$.

Proof. Suppose $j \geq i+2$. By considering the hamiltonian path

$$x_{j-2}x_{j-3}\cdots x_1x_{j+1}x_jx_{j-1}x_nx_{n-1}\cdots x_{j+2},$$

and using the fact $x_1x_{j-2} \notin E(G)$, we deduce $x_{j+2}x_2 \in E(G)$. Then

$$x_{j+1}x_j\cdots x_ix_nx_{n-1}\cdots x_{j+2}x_2x_3\cdots x_{i-1}x_1x_{j+1},$$

is a hamiltonian cycle of G, a contradiction. □

Claim 4. $x_1x_s \in E(G)$ for any $4 \leq s \leq i-2$.

Proof. Suppose to the contrary that there exists some $4 \leq s \leq i-2$ such that $x_1x_s \notin E(G)$. We can get that $x_1x_{s-1} \in E(G)$ and $x_1x_{s+1} \in E(G)$. By (i), $x_nx_{s-1} \in E(G)$ and $x_nx_{s-2} \notin E(G)$ by the choice of i; by (ii), $d(x_s) = 2$, thus $d(x_{s+2}) \geq id(x_s) \geq \frac{n-1}{2}$ and $d(x_{s-2}) \geq id(x_s) \geq \frac{n-1}{2}$. So $x_1x_{s-2} \in E(G)$. We consider the following two case.

(a) $x_1x_{s+2} \notin E(G)$.

Then $x_nx_{s+1} \in E(G)$ by (i). Since $d(x_{s+2}) \geq \frac{n-1}{2}$, we have $x_1x_{s+3} \notin E(G)$ by (ii). So $x_nx_{s+2} \in E(G)$ and $x_1x_{s+4} \in E(G)$. By the choice of i, we have $i = s+2$, contrary to $d(x_i) < \frac{n-1}{2}$.

(b) $x_1x_{s+2} \in E(G)$.

Then $x_nx_{s+1} \notin E(G)$ by (i). By considering the hamiltonian path

$$x_{s-2}x_{s-3}\cdots x_1x_{s-1}x_sx_{s+1}\cdots x_n,$$

we deduce $x_{s-2}x_{s+2} \in E(G)$. Then

$$x_{s-2}x_{s-3}\cdots x_1x_{s+1}x_sx_{s-1}x_nx_{n-1}\cdots x_{s+2}x_{s-2}$$

is a hamiltonian cycle of G, a contradiction.

This completes the proof of Claim 4. □

Claim 5. $x_1x_{i+3} \in E(G)$.

Proof. Otherwise, $x_nx_{i+2} \in E(G)$ by (i) and $d(x_{i+3}) \geq \frac{n-1}{2}$ by (ii). Considering the hamiltonian path

$$x_1x_2\cdots x_ix_{i+1}x_{i+2}x_nx_{n-1}\cdots x_{i+3},$$

and using the fact $x_1x_{i+1} \notin E(G)$, we deduce $x_ix_{i+3} \in E(G)$. Then

$$x_1x_2\cdots x_{i-1}x_nx_{n-1}\cdots x_{i+3}x_ix_{i+1}x_{i+2}x_1,$$

is a hamiltonian cycle of G, a contradiction. □

Claim 6. $x_1x_s \notin E(G)$ for any $s = i+4, i+5, \cdots, n$.

Proof. Suppose that there is some s with $i+4 \le s \le n$ such that $x_1x_s \in E(G)$. Clearly, $s \ne n-1, n$. We choose such s such that s is as small as possible. If $s = i+4$, then considering the hamiltonian path

$$x_{i+3}x_{i+2}x_{i+1}x_ix_{i-1}\cdots x_1x_{i+4}x_{i+5}\cdots x_n,$$

and using the fact that $x_{i+1}x_n \notin E(G)$, we deduce $x_ix_{i+3} \in E(G)$. Then

$$x_1x_2\cdots x_{i-1}x_nx_{n-1}\cdots x_{i+3}x_ix_{i+1}x_{i+2}x_1,$$

is a hamiltonian cycle of G, a contradiction.

So we assume $i+5 \le s \le n-2$. By (i) and (ii), we get $d(x_{s+1}) \ge \frac{n-1}{2}$. By considering the hamiltonian path

$$x_{i-1}x_{i-2}\cdots x_1x_sx_{s-1}\cdots x_ix_nx_{n-1}\cdots x_{s+1},$$

and using the fact $x_{i-1}x_{i+1} \notin E(G)$, we deduce $x_{i+2}x_{s+1} \in E(G)$. Then

$$x_{i+2}x_{i+1}\cdots x_1x_sx_{s-1}\cdots x_{i+3}x_nx_{n-1}\cdots x_{s+1}x_{i+2},$$

is a hamiltonian cycle of G, a contradiction. □

We can get that $e(x_n, \{x_{i-1}, x_i, x_{i+3}, x_{i+4}, \cdots, x_{n-1}\}) = n-i-1$ by Claim 6 and (i). The degrees of x_1 and x_n impose $i = \frac{n-1}{2}$. For every $s \le i-2$ and $t \ge i+4$, we have $x_sx_t \notin E(G), x_sx_i \notin E(G)$ and $x_tx_{i+2} \notin E(G)$, for $x_sx_{s-1}\cdots x_1x_{s+1}x_{s+2}\cdots x_{t-1}x_nx_{n-1}\cdots x_t$, $x_sx_{s-1}\cdots x_1x_{s+1}x_{s+2}\cdots x_{i-1}x_nx_{n-1}\cdots x_i$, $x_tx_{t+1}\cdots x_nx_{t-1}x_{t-2}\cdots x_{i+3}x_1x_2\cdots x_{i+2}$ are hamiltonian paths of G, respectively. We deduce that $\{x_{i-1}, x_i, x_{i+1}x_{i+2}, x_{i+3}\}$ is a cut-set of G. Let $U_1 = \{x_1, x_2, \cdots, x_{i-2}\}$ and $U_2 = \{x_{i+4}, x_{i+5}, \cdots, x_n\}$, we see that $|U_1| = |U_2| = \frac{n-5}{2}$. By the above discussion, we can get that $d(x_s) \le \frac{n-1}{2}$ for any $x_s \in U_1 \cup U_2$, and if $d(x_s) = \frac{n-1}{2}$ for some $x_s \in U_1 \cup U_2$, then $N(x_s) = (U_1 \setminus \{x_s\}) \cup \{x_{i-1}, x_{i+2}, x_{i+3}\}$ when $x_s \in U_1$ and $N(x_s) = (U_2 \setminus \{x_s\}) \cup \{x_{i-1}, x_i, x_{i+3}\}$ when $x_s \in U_2$.

Claim 7. $id(x_i) \ge \frac{n-1}{2}$.

Proof. Suppose $id(x_i) < \frac{n-1}{2}$, then $id(x_{i-2}) > \frac{n-1}{2}$. Considering the hamiltonian path

$$P' = x_{i-2}x_{i-3}\cdots x_1x_{i-1}x_i\cdots x_n = z_1z_2\cdots z_n,$$

since $x_i \in N_2(x_{i-2})$ and $d(x_i) < \frac{n-1}{2}$, by Lemma 3, there must exist a vertex $z_s \in N^-(z_1)$ such that $d(z_s) \ge id(z_1) > \frac{n-1}{2}$. Let

$$P'' = z_sz_{s-1}\cdots z_1z_{s+1}z_{s+2}\cdots z_n,$$

which is a hamiltonian path with $d(z_s) + d(z_n) > \frac{n-1}{2} + \frac{n-1}{2} = n-1$. By Lemma 1, there is a hamiltonian cycle in G, a contradiction. □

Claim 8. If $x_ix_t \in E(G)$ for some $x_t \in U_2$, then $x_ix_{t+1}, x_ix_{t+2} \notin E(G)$.

Proof. Otherwise, we can get that $d(x_t) \ge \frac{n-1}{2}$ and $d(x_{t+1}) \ge \frac{n-1}{2}$ or $d(x_{t+2}) \ge \frac{n-1}{2}$. Therefore, $x_{i+3}x_{i+4}\cdots x_tx_ix_{i+1}x_{i+2}x_1x_2\cdots x_{i-1}x_nx_{n-1}\cdots x_{t+1}x_{i+3}$ is a hamiltonian cycle, or $x_{i+3}x_{i+4}\cdots x_tx_ix_{i+1}x_{i+2}x_1x_2\cdots x_{i-1}x_{t+2}x_{t+3}\cdots x_nx_{i+3}$ is an $(n-1)$-cycle of G avoiding x_{t+1} with $d(x_{t+1}) \ge 3$, a contradiction. □

Let $d(x_i) = s+1$. By the above, we can get that $(N^-(x_i) \cup N^+(x_i)) \cap U_2 \subseteq N_2(x_i)$ and $N^-(x_i) \cap N^+(x_i) = \emptyset$. Thus, $|N^-(x_i) \cup N^+(x_i)| \geq 2s-3 \geq s$ and $d(x_t) < \frac{n-1}{2}$ for any $x_t \in N^-(x_i) \cup N^+(x_i)$. It is contrary to the definition of implicit degree.

Case 1.2. $x_1x_{i-1} \in E(G), x_1x_{i+1} \in E(G)$ and $x_1x_i \notin E(G)$ for some $i \in [4, n-3]$.

Choose such i such that i is as small as possible, then $e(x_1, \{x_2, x_3, \cdots, x_{i-1}\}) = i-2$ and $e(x_n, \{x_1, x_2, \cdots, x_{i-2}\}) = 0$.

By (i), $x_nx_{i-1} \in E(G)$ and $x_nx_{i-2} \notin E(G)$; by (ii), $d(x_i) = 2$, thus $d(x_{i+2}) \geq \frac{n-1}{2}$ and $d(x_{i-2}) \geq \frac{n-1}{2}$.

Considering the hamiltonian path

$$x_{i-2}x_{i-3} \cdots x_1x_{i-1}x_i \cdots x_n,$$

and noting that $x_nx_i \notin E(G)$, we get that $x_{i-2}x_{i+1} \in E(G)$ by (i); but since

$$x_1x_2 \cdots x_{i-2}x_{i+1}x_ix_{i-1}x_nx_{n-1} \cdots x_{i+2},$$

is a hamiltonian path of G, we must have $x_1x_{i+2} \notin E(G)$. Therefore, $x_nx_{i+1} \in E(G)$ by (i) and $x_1x_{i+3} \notin E(G)$ by (ii) . Now, we can suppose that $e(x_1, \{x_{i+2}, x_{i+3}, \cdots, x_n\}) = 0$, otherwise Case 1.1 holds. Thus $e(x_n, \{x_{i+1}, x_{i+2}, \cdots, x_{n-1}\}) = n-i-1$. The degrees of x_1 and x_n impose $i = \frac{n+1}{2}$.

For every $s \leq i-2$ and $t \geq i+2$, we have $x_sx_t \notin E(G)$ for $x_sx_{s-1} \cdots x_1 x_{s+1}x_{s+2} \cdots x_{t-1}x_nx_{n-1} \cdots x_t$ is a hamiltonian path of G. We deduce that $\{x_{i-1}, x_i, x_{i+1}\}$ is a cut-set of G, and $d(u) \leq \frac{n-1}{2}$ for any $u \in V_1 \cup V_2$, where $V_1 = \{x_1, x_2, \cdots, x_{i-2}\}$ and $V_2 = \{x_{i+2}, x_{i+3}, \cdots, x_n\}$. We see that $|V_1| = |V_2| = \frac{n-3}{2}$.

Claim 9. *$id(x_{i-1}) \geq \frac{n-1}{2}$ and $id(x_{i+1}) \geq \frac{n-1}{2}$.*

Proof. Suppose, without loss of generality, that $id(x_{i-1}) < \frac{n-1}{2}$. Then, there exists some vertex, say x_j, in $\{x_2, x_3, \cdots, x_{i-1}\}$ such that $d(x_j, x_{i-1}) = 2$. Then $id(x_j) > \frac{n-1}{2}$. Considering the hamiltonian path

$$P' = x_jx_{j-1} \cdots x_1x_{j+1}x_{j+2} \cdots x_n = z_1z_2 \cdots z_n,$$

using the fact that $x_{i-1} \in N_2(x_j)$ and $d(x_{i-1}) < \frac{n-1}{2}$, we can get that there exists some vertex $z_s \in N^-(z_1)$ such that $d(z_s) \geq id(z_1) > \frac{n-1}{2}$ by Lemma 3. Let

$$P'' = z_sz_{s-1} \cdots z_1z_{s+1}z_{s+2} \cdots z_n,$$

is a hamiltonian path with $d(z_s) + d(z_n) > n-1$. Then we can get a Hamilton cycle by Lemma 1, a contradiction. □

Claim 10. *$d(x_{i-1}) \geq \frac{n-1}{2}$ and $d(x_{i+1}) \geq \frac{n-1}{2}$.*

Proof. Suppose, without loss of generality, that $d(x_{i-1}) < \frac{n-1}{2}$. Then

$d(x_{i-1}) < id(x_{i-1})$ by Claim 9.

Let $d(x_{i-1}) = t + 1$. We know $|N(x_{i-1}) \cup N_2(x_{i-1})| = n - 1$. Since all the vertices with degree at least $\frac{n-1}{2}$ must be adjacent to x_{i-1} and x_{i+1}, we get that $d(u) < \frac{n-1}{2}$ for each $u \in N_2(x_{i-1})$. Therefore, $|N_2(x_{i-1})| > t + 1$. It is contrary to the definition of implicit degree. □

For $j = 1, 2, V_j$ can be partitioned into $A_j \cup B_j$ such that $d(a) \geq \frac{n-1}{2}$ for each $a \in A_1 \cup A_2$ and $d(b) < \frac{n-1}{2}$ for each $b \in B_1 \cup B_2$. Since $x_1, x_{i-2}, x_{i+2}, x_n$ have degree at least $\frac{n-1}{2}$, we have $|A_j| \geq 2, j = 1, 2$. Moreover, taking $a \in A_1 \cup A_2$, we must have that a is adjacent to both x_{i-1} and x_{i+1}. Which implies that $e(A_2, \{x_{i-1}, x_{i+1}\}) = 2|A_2|$.

If $B_1 \cup B_2 = \emptyset$, then $G \in \mathscr{B}_n$.

So suppose $B_1 \cup B_2 \neq \emptyset$. Clearly, $e\{B_1 \cup B_2, \{x_{i-1}, x_{i+1}\}\} = 0$. So $d(x_{i-1}) = d(x_{i+1}) = |A_1| + |A_2| + 1 + e$, where $e = e(x_{i-1}, x_{i+1})$. Since $d(x_{i-1}) = d(x_{i+1}) \geq \frac{n-1}{2}$, we get $|A_1| + |A_2| + 1 + e \geq \frac{n-1}{2}$. So $|A_1| + |A_2| \geq \frac{n-3}{2} - e$.

Claim 11. For any two vertices $a, b \in B_1$, if $ab \notin E(G)$, then $id(a) \geq \frac{n-1}{2}$ and $id(b) \geq \frac{n-1}{2}$. Similar for B_2.

Proof. Suppose a, b are two vertices in B_1 with $ab \notin E(G)$, but $id(a) < \frac{n-1}{2}$ or $id(b) < \frac{n-1}{2}$. We assume, without loss of generality, that $id(a) < \frac{n-1}{2}$. Since $d(a, b) = 2$, $id(b) > \frac{n-1}{2}$. Then $id(b) \neq \min\{d(u) : u \in N_2(b)\}$.

Considering the hamiltonian path $bb^- b^{2-} \cdots x_1 b^+ b^{2+} \cdots x_n$, we can get a hamiltonian cycle of G by Lemma 3, a contradiction. □

Choose a vertex b in $B_1 \cup B_2$. By the symmetry, we may assume $b \in B_1$. If $N_2(b) \cap B_1 = \emptyset$. Let $d(b) = s + 1$. And let $|A_1| = m, |N(b) \cap B_1| = n_1$ and $|N_2(b) \cap B_1| = n_2$. Then $n_1 + n_2 + m = \frac{n-5}{2}$ and $s + 1 = m + n_1$. Since $d(x_{i-1}, b) = 2$, $d(x_{i-1}) \geq \frac{n-1}{2}$ and $d(u) \leq \frac{n-1}{2}$ for any $u \in N(b)$, we can easily check that $id(b) \neq d^b_{s+1}$. If $n_2 = 0$, then $G[A_1 \cup B_1]$ is complete subgraphs. If $n_2 \neq 0$, then $id(b) \neq m^b_2$. So $id(b) = d^b_s$, then $d^b_s > m^b_2$ and $d^b_{s+1} \leq M^b_2$. Therefore, $n_1 + n_2 \leq s - 1 = n_1 + m - 2$. Then $1 \leq n_2 \leq m - 2$. By the arbitrary of b, we have $|A_1| \geq \max\{|N_2(b) \cap B_1| + 2 : b \in B_1\} = |\{b : d(b) < \frac{n-5}{2}$ and $b \in B_1\}| + 1$. Similarly, $|A_2| \geq |\{b : d(b) < \frac{n-5}{2}$ and $b \in B_2\}| + 1$. Consequently, $G \in \mathscr{B}_n$.

Case 2. $id(x) = d^x_k$.

Then $d^x_k > m^x_2$ and $k \geq 2$. Let $W_1 = \{y_i : y_i^+ = y_{i+1}^-\}$ and $W_2 = \{y_i : y_i^+ \neq y_{i+1}^-\}$. Set $|W_i| = w_i, i = 1, 2$. Then $w_1 + w_2 = k + 1$. Moreover, $\{y_i^+, y_{i+1}^- : y_i \in W_2\} \cup \{y_i^+ : y_i \in W_1\} \subseteq N_2(x)$, so $|N_2(x)| \geq w_1 + 2w_2$.

By the hypothesis of Lemma 5, we can get that $d(y_i^+) \leq d(x) < \frac{n-1}{2}$ for any $y_i \in W_1$. Since $id(x) = d^x_k$, there are at least $w_2 + 2$ vertices in

$\{y_i^+, y_{i+1}^- : y_i \in W_2\}$ with degree at least $id(x)$.

Claim 12. $w_2 = 2$.

Proof. It is easy to check that $w_2 \geq 2$, otherwise, G contains an $(n-1)$-cycle avoiding a vertex with degree at least $\frac{n-1}{2}$. By the contrary, suppose $w_2 \geq 3$, then there are at least three vertices in $\{y_i^+ : y_i \in W_2\}$ with degrees at least $id(x)$ or at least three vertices in $\{y_{i+1}^- : y_i \in W_2\}$ with degrees at least $id(x)$. Without loss of generality, suppose there are at least three vertices in $\{y_i^+ : y_i \in W_2\}$ with degrees at least $id(x) \geq \frac{n-1}{2}$. Similarly as Case 1, we can get an $(n-1)$-cycle such that the remaining vertex with degree at least $\frac{n-1}{2}$, a contradiction. □

By Claim 12, we assume $W_2 = \{y_i, y_{k+1}\}$. Then $d(y_i^+), d(y_{i+1}^-), d(y_{k+1}^+) \geq id(x)$ and $d(y_1^-) \geq id(x)$.

Claim 13. $id(y_j^+) \geq \frac{n-1}{2}$ for any $y_j \in W_1$.

Proof. Otherwise, $id(x) > \frac{n-1}{2}$. Then $d(y_i^+) \geq id(x) > \frac{n-1}{2}$ and $d(y_{k+1}^+) \geq id(x) > \frac{n-1}{2}$. But $y_i^+ y_i^{2+} \cdots y_{k+1} x y_i y_i^- \cdots y_{k+1}^+$ is a hamiltonian path, by Lemma 1, there is a hamiltonian cycle in G, a contradiction. □

Claim 14. $N(y_j^+) = N(x)$ for any $y_j \in W_1$.

Proof. We assume $y_1 \in W_1$. We just need to prove $N(y_1^+) = N(x)$. Let $d(y_1^+) = s+1$. Since $x \in N_2(y_1^+)$, $d(x) < \frac{n-1}{2}$ and G is not hamiltonian, we can get that $id(y_1^+) \neq m_2^{y_1^+}, d_{s+1}^{y_1^+}$. Then $id(y_1^+) = d_s^{y_1^+}$. If there exists some vertex y_t such that $y_t y_1^+ \in E(G)$ and $y_{t+1} y_1^+ \notin E(G)$ or $y_{t-1} y_1^+ \notin E(G)$, then by similar argument as in Claim 12, we can get that $d(y_t^+) \geq id(y_1^+) \geq \frac{n-1}{2}$, a contradiction. Since $y_1^+ y_1 \in E(G)$, we have $y_1^+ y_s \in E(G)$ for each $s = 2, 3, \cdots, i$.

If $y_1^+ y_{i+1} \notin E(G)$, we can get that $y_1^+ y_t \notin E(G)$ for each $t = i+2, i+3, \cdots, k+1$. Then we can get that there is a vertex y_t^+ with $1 \leq t \leq i-1$ with $d(y_t^+) \geq id(y_1^+) \geq \frac{n-1}{2}$, a contradiction. So $y_1^+ y_{i+1} \in E(G)$. Similarly, $y_1^+ y_t \in E(G)$ for each $i+2, i+3, \cdots, k+1$. Therefore, $N(y_1^+) = N(x)$. □

Claim 15. $N(x) \subseteq N(u)$ *for any* $u \in \{y_i^+, y_{i+1}^-, y_{k+1}^+, y_1^-\}$.

Proof. By symmetry, we just prove $N(x) \subseteq N(y_i^+)$. Considering the Hamilton path $P = y_i^+ y_i^{2+} \cdots y_{k+1} x y_i y_i^- y_i^{2-} \cdots y_{k+1}^+$ and using the fact $d(y_i^+) \geq \frac{n-1}{2}$ and $d(y_{k+1}^+) \geq \frac{n-1}{2}$, we deduce $d(y_i^+) = d(y_{k+1}^+) = \frac{n-1}{2}$. Since $y_s^+ y_{k+1}^+ \notin E(G)$ for any $y_s \in W_1$ and $x y_{k+1}^+ \notin E(G)$, we have $N(x) \backslash \{y_{i+1}\} \subseteq N(y_i^+)$.

By Claim 14, $y_{k+1}^+ y_{k+1}^{2+} \cdots y_1 x y_{k+1} y_{k+1}^- \cdots y_{i+1} y_1^+ y_1^{2+} \cdots y_{i+1}^-$ is a hamiltonian path, then $y_{k+1}^+ y_{i+1}^- \notin E(G)$. Then by using P, we get that $y_i^+ y_{i+1} \in E(G)$. Therefore $N(x) \subseteq N(y_i^+)$. □

Let $C_1 = C[y_i^+, y_{i+1}^-], C_2 = C[y_{k+1}^+, y_1^-]$ and $C_3 = C[y_{i+1}, y_{k+1}] \cup C[y_1, y_i]$. By Claim 15, we can get that $y_{k+1}^+ y_{i+1}^- \notin E(G)$. Since G is non-hamiltonian, we have $N_{C_1}^+(y_{k+1}^+) \cap N_{C_1}(y_i^+) = \emptyset$, by Lemma 4, we can get that $d_{C_1}(y_i^+) + d_{C_1}(y_{k+1}^+) \leq |V(C_1)| - 1$. Similarly, $d_{C_2}(y_i^+) + d_{C_2}(y_{k+1}^+) \leq |V(C_2)| - 1$, $d_{C_2}(y_{i+1}^-) + d_{C_2}(y_1^-) \leq |V(C_2)| - 1$ and $d_{C_1}(y_{i+1}^-) + d_{C_1}(y_1^-) \leq |V(C_1)| - 1$. By the above discussion and Claim 15, we get

$$\begin{aligned}
2(n-1) &\leq d_C(y_i^+) + d_C(y_{k+1}^+) + d_C(y_{i+1}^-) + d_C(y_1^-) \\
&= \sum_{l=1}^{3} d_{C_l}(y_i^+) + d_{C_l}(y_{k+1}^+) + d_{C_l}(y_{i+1}^-) + d_{C_l}(y_1^-) \\
&\leq 4(k+1) + 2(|V(C_1)| - 1) + 2(|V(C_2)| - 1) \\
&\leq 2(n-1),
\end{aligned}$$

which implies that all the inequalities are equalities. If there exists some vertex $x_s \in V(C_1)$ such that $y_{k+1}^+ x_s \in E(G)$, then $y_1^- x_s^-, y_1^- x_s^+, y_1^- y_i^+, y_1^- y_i^{2+} \notin E(G)$ and $y_{i+1}^- x_s^- \notin E(G)$. By Lemma 4, we can get that $d_{C_1}(y_{i+1}^-) + d_{C_1}(y_1^-) < |C_1| - 1$, a contradiction. Hence, $N_{C_1}(y_{k+1}^+) = \emptyset$. Similarly, we can get that $N_{C_1}(y_1^-) = \emptyset, N_{C_2}(y_i^+) = \emptyset$ and $N_{C_2}(y_{i+1}^-) = \emptyset$. Hence, $d_{C_1}(y_i^+) = |V(C_1)| - 1$ and $d_{V(C_2)}(y_{k+1}^+) = |V(C_2)| - 1$. Since $d(y_i^+) = \frac{n-1}{2}$ and $d(y_{k+1}^+) = \frac{n-1}{2}$, we can get that $|V(C_1)| = |V(C_2)| = \frac{n-1}{2} - k$. Therefore, we can get that G is the subgraph of $\mathscr{H}_n$.

3 The proof of Theorem 7

Let G be a non-hamiltonian graph satisfying the hypothesis of Theorem 7. By Theorem 3, G contains a cycle of length $n-1$. We choose an $(n-1)$-cycle C such that the degree of the vertex not in C is as large as possible. Let $C = x_1 x_2 \cdots x_{n-1}$ and x be the vertex of G not in C. By Lemma 2, $d(x) \leq \frac{n-1}{2}$.

If $d(x) = \frac{n-1}{2}$, we can suppose $N(x) = \{x_1, x_3, x_5, \cdots, x_{n-2}\}$. Then $\{x, x_2, x_4, \cdots, x_{n-1}\}$ is an independent set of G with $\frac{n+1}{2}$ elements. It is easy to check that G is the subgraph of $\frac{n+1}{2} K_1 \vee K_{\frac{n-1}{2}}$.

So we can assume that for every cycle C' of length $n-1$, the vertex not in C' has degree at most $\frac{n-2}{2}$. Theorem 7 follows by Lemma 5. □

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References

[1] A. Benhocine and A. Wojda, The Geng-Hua Fan conditions for pancyclic or Hamilton-connected graphs, J. Combin. Theory, Ser. B, 42 (1987) 167-180.

[2] J. Bondy, Pancyclic graphs, I, J. Combin. Theory, Ser. B, 11 (1971) 80-84.

[3] J. Bondy and U. Murty, Graph Theory with Applications, Macmillan Press, London, 1976.

[4] G. Chen, Longest cycles in 2-connected graphs, J. Central China Normal Univ. Natur. Sci. 3 (1986) 39-42.

[5] J. Cai and H. Dai, An implicit degree Dirac condition for Hamilton cycles, Ars Combin, will appear in volume 107, October 2012.

[6] G. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. 2 (1952) 69-81.

[7] G. Dirac, Hamilton circuits and long circuits, Ann. Discrete Math. 3 (1978) 75-92.

[8] G. Fan, New sufficient conditions for cycles in graphs, J.Combin. Theory, Ser. B, 37 (1984) 221-227.

[9] O. Ore, Note on Hamilton circuits, Amer. Math. Montyly, 67 (1960) 55.

[10] E.F. Schmeichel and S.L. Hakimi, A cycle structure theorem for hamiltonian graphs, J. Combin. Theory, Ser. B, 45 (1988) 99-107.

[11] Y. Zhu, H. Li and X. Deng, Implicit-degrees and circumferences, Graphs Combin. 5 (1989) 283-290.