The Fibonacci Length of Amalgamated Free Products of Dihedral Groups

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Abstract

In this paper we obtain the Fibonacci length of amalgamated free products having as factors dihedral groups.
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1 Introduction and Preliminaries

By considering finite cyclic groups, the concept of the Fibonacci length of groups has been introduced by Wall [10], and then this concept has been extended to abelian groups by Wilcox [11]. After that, in [3], Campbell et. al. defined the Fibonacci orbit and the Fibonacci length of 2-generator groups. We may refer to [4, 6] for the Fibonacci length of more than two generator groups. We may also refer to [1, 2, 7, 8] for some other known results.

For a finitely generated group G with a generating set $A = \{a_1, ..., a_n\}$, the *Fibonacci orbit* $F_A(G)$ of G with respect to the A is the sequence $x_1 = a_1, \cdots, x_n = a_n$, $x_{i+n} = \prod_{j=1}^{n} x_{i+j-1}$, where $i \geq 1$. Moreover, if $F_A(G)$ is periodic, then the length of the period of the sequence is called the *Fibonacci length* of G with respect to the A, denoted by $LEN_A(G)$. We note that the Fibonacci length of a group depends on the *chosen* generating set A and its order.

For the dihedral group D_{2n} with a presentation $< a_1, b_1; a_1^n = b_1^2 = (a_1 b_1)^2 = 1 >$, and for any generating set $\{a_1, b_1\}$ (with has an independent ordering) of D_{2n}, it has been showed that $LEN_{(a_1,b_1)}(D_{2n}) = 6$ (see [3] for the details). Later, this result was generalized to the powers of dihedral groups in [4], and was generalized to polyhedral and binary polyhedral groups in [5].

Let $A =< a_1, \cdots, a_k \, ; \, R_1, \cdots, R_l >$ and $B =< b_1, \cdots, b_p \, ; \, S_1, \cdots, S_t >$ be two groups with the proper subgroups $H \leq A$, $K \leq B$, respectively. Suppose $f : H \rightarrow K$ is an isomorphism. The *free product of* A and B *amalgamating* H to K is the group G with presentation $A *_H B = G =< a_{i_1}, b_{j_1} \, ; \, R_{i_2}, S_{j_2}, H =$

$f(H)>$, where $1 \leq i_1 \leq k$, $1 \leq i_2 \leq l$, $1 \leq j_1 \leq p$ and $1 \leq j_2 \leq t$. The groups A and B are called *factors* while H and K are called *amalgamated subgroups.* As a special case, if $H = 1$ then the amalgamated free product becomes the free product.

2 Amalgamated Free Products of Dihedral Groups

For $n \geq 3$, let us consider the dihedral group D_{2n} with a generating set $\{a_1, b_1\}$. Since all subgroups of D_{2n} are either cyclic or dihedral, these cyclic subgroups of order n/d are generated by $< a_1^d >$, where $d|n$, and the dihedral subgroups of order $2n/d$ are generated by $< a_1^d, a_1^r b_1 >$, where $d|n$ and $0 \leq r < d$. Firstly, by considering the subgroup $H =< a_1 b_1 >$ of D_{2n} and considering another dihedral group D_{2m} $(m \geq 3)$ with a generating set $\{a_2, b_2\}$, we obtain the Fibonacci length for $G = D_{2n} *_{<a_1 b_1>} D_{2m}$ with all related subgroups of D_{2m}. We should note that since the ordering in the generating set is important while obtaining LEN, we will strictly use the notation $LEN_{(A)}(G)$ to reveal the importance of the ordering that we use.

Then the first main result is the following.

Theorem 1 *Let $G = D_{2n} *_{<a_1 b_1>} D_{2m}$ and K be a subgroup of D_{2m}.*

i) For $0 \leq r' < m$, $r' \neq 1$, if $K =< a_2^{r'} b_2 >$ and $m \equiv l \mod (2r'-2)$, then

$$LEN_{(a_1,b_1,a_2,b_2)}(G) = \frac{10m}{(l, 2r'-2)}.$$

ii) If $K =< a_2 b_2 >$, then $LEN_{(a_1,b_1,a_2,b_2)}(G) = 10$.

iii) For an even positive integer m and a subgroup $K =< a_2^{m/2} >$,

$$LEN_{(a_1,b_1,a_2,b_2)}(G) = 10.$$

Proof. *i*) By a calculation using the sequence shows that the Fibonacci orbit starts with the words

$a_1,\ b_1,\ a_2,\ b_2,$

$a_1 b_1 a_2 b_2 = a_2^{r'-1},$

$b_1 a_2 b_2 a_2^{r'-1} = b_1 b_2 a_2^{r'-2},$

$b_1 a_2 b_2 a_2^{r'-1} = b_1 b_2 a_2^{r'-2},$

$a_2 b_2 a_2^{r'-1} b_1 b_2 a_2^{r'-2} = b_2 a_2^{r'-2} b_1 b_2 a_2^{r'-2},$

$b_2 a_2^{r'-1} b_1 b_2 a_2^{r'-2} b_2 a_2^{r'-2} b_1 b_2 a_2^{r'-2} = b_2 a_2^{r'-1} b_2 a_2^{r'-2} = a_2^{-1},$

$a_2^{r'-1} b_1 b_2 a_2^{r'-2} b_2 a_2^{r'-2} b_1 b_2 a_2^{r'-2} a_2^{-1} = a_2 a_2^{r'-2} b_2 a_2^{r'-2} a_2^{-1} = a_2^2 b_2,$

$b_1 b_2 a_2^{r'-2} b_2 a_2^{r'-2} b_1 b_2 a_2^{r'-2} a_2^{-1} a_2^2 b_2 = b_2 a_2^{r'-1} b_2 = a_2^{-r'+1},$

$b_2 a_2^{r'-2} b_1 b_2 a_2^{r'-2} a_2^{-1} a_2^2 b_2 a_2^{-r'+1} = b_2 a_2^{r'-2} b_1 a_2^{-2r'+2} = a_2^{-2r'+2} a_1 a_2^{-2r'+2},$

$a_2^{-1} a_2^2 b_2 a_2^{-r'+1} a_2^{-2r'+2} a_1 a_2^{-2r'+2} = a_2 b_2 a_2^{-2r'+3} b_2 b_1 a_2^{-2r'+2} = a_2^{2r'-2} b_1 a_2^{-2r'+2},$

$$a_2^2b_2a_2^{-r'+1}a_2^{-2r'+2}a_1a_2^{-2r'+2}a_2^{2r'-2}b_1a_2^{-2r'+2}=a_2^2b_2a_2^{-2r'+3}b_2a_2^{-2r'+2}=a_2,$$
$$a_2^{-r'+1}a_2^{-2r'+2}a_1a_2^{-2r'+2}a_2^{2r'-2}b_1a_2^{-2r'+2}a_2=a_2^{-3r'+3}a_1b_1a_2^{-2r'+3}=b_2,\ldots$$

Now, by replacing $-2r'+2$ with α, we can consider what happens to the orbit when we have a section of the form $\cdots,\ a_2^{\alpha}a_1a_2^{\alpha},\ a_2^{-\alpha}b_1a_2^{\alpha},\ a_2,\ b_2,\ \cdots$.

$a_2^{\alpha}a_1a_2^{\alpha},$
$a_2^{-\alpha}b_1a_2^{\alpha},$
$a_2,$
$b_2,$
$a_2^{\alpha}a_1a_2^{\alpha}a_2^{-\alpha}b_1a_2^{\alpha}a_2b_2=a_2^{\alpha}a_2^{r'}b_2a_2^{\alpha}a_2b_2=a_2^{r'}b_2a_2b_2=a_2^{r'-1},$
$a_2^{-\alpha}b_1a_2^{\alpha}a_2b_2a_2^{r'-1}=a_2^{-\alpha}b_1b_2a_2^{-\alpha+r'-2},$
$a_2b_2a_2^{r'-1}a_2^{-\alpha}b_1b_2a_2^{-\alpha+r'-2}=b_2a_2^{-\alpha+r'-2}b_1b_2a_2^{-\alpha+r'-2},$
$b_2a_2^{r'-1}a_2^{-\alpha}b_1b_2a_2^{-\alpha+r'-2}b_2a_2^{-\alpha+r'-2}b_1b_2a_2^{-\alpha+r'-2}=a_2^{-1},$
$a_2^{r'-1}a_2^{-\alpha}b_1b_2a_2^{-\alpha+r'-2}b_2a_2^{-\alpha+r'-2}b_1b_2a_2^{-\alpha+r'-2}a_2^{-1}=a_2^2b_2,$
$a_2^{-\alpha}b_1b_2a_2^{-\alpha+r'-2}b_2a_2^{-\alpha+r'-2}b_1b_2a_2^{-\alpha+r'-2}a_2^{-1}a_2^2b_2=a_2^{-r'+1},$
$b_2a_2^{-\alpha+r'-2}b_1b_2a_2^{-\alpha+r'-2}a_2^{-1}a_2^2b_2a_2^{-r'+1}=a_2^{\alpha-2r'+2}a_1a_2^{\alpha-2r'+2},$
$a_2^{-1}a_2^2b_2a_2^{-r'+1}a_2^{\alpha-2r'+2}a_1a_2^{\alpha-2r'+2}=a_2^{-\alpha+2r'-2}b_1a_2^{\alpha-2r'+2},$
$a_2^2b_2a_2^{-r'+1}a_2^{\alpha-2r'+2}a_1a_2^{\alpha-2r'+2}a_2^{-\alpha+2r'-2}b_1a_2^{\alpha-2r'+2}=a_2,$
$a_2^{-r'+1}a_2^{\alpha-2r'+2}a_1a_2^{\alpha-2r'+2}a_2^{-\alpha+2r'-2}b_1a_2^{\alpha-2r'+2}a_2=b_2,$
$\cdots$

As in [5], the Fibonacci orbit can be said to form layers of length ten. In fact, by using the above, the orbit becomes:

$$x_1=a_1,\ x_2=b_1,\ x_3=a_2,\ x_4=b_2,\ \cdots,$$
$$x_{11}=a_2^{-2(r'-1)}a_1a_2^{-2(r'-1)},x_{12}=a_2^{2(r'-1)}b_1a_2^{-2(r'-1)},$$
$$x_{13}=a_2,\ x_{14}=b_2,\ \cdots,$$
$$x_{21}=a_2^{-4(r'-1)}a_1a_2^{-4(r'-1)},x_{22}=a_2^{4(r'-1)}b_1a_2^{-4(r'-1)},$$
$$x_{23}=a_2,\ x_{24}=b_2,\ \cdots,$$
$$x_{10i+1}=a_2^{-2i(r'-1)}a_1a_2^{-2i(r'-1)},x_{10i+2}=a_2^{2i(r'-1)}b_1a_2^{-2i(r'-1)},$$
$$x_{10i+3}=a_2,\ x_{10i+4}=b_2,\cdots$$

So we need the smallest positive value of i such that $2i(r'-1)=km$, for $k\in\mathbb{N}$. If $m\equiv l \bmod (2r'-2)$, then $i=\frac{m}{(l,2r'-2)}$ and the Fibonacci length is $\frac{10m}{(l,2r'-2)}$.

ii) We prove this by a direct calculation. We have the sequence

$a_1,\ b_1,\ a_2,\ b_2,\ a_1b_1a_2b_2=1,\ b_1a_2b_2=a_1^{-1},\ a_2b_2a_1^{-1}=a_1^2b_1,\ b_2a_1^{-1}a_1^2b_1=a_2^{-1},$
$a_1^{-1}a_1^2b_1a_2^{-1}=a_2^2b_2,\ a_1^{-1}a_1^2b_1a_2^{-1}a_2^2b_2=1,\ a_1^2b_1a_2^{-1}a_2^2b_2=a_1,$
$a_2^{-1}a_2^2b_2a_1=b_1,\ a_2^2b_2a_1b_1=a_2,\ a_1b_1a_2=b_2,\ \cdots$

and the Fibonacci length is 10.

iii) The members of the Fibonacci orbit are

$$a_1,\ b_1,\ a_2,\ b_2,\ a_2^{\frac{m}{2}+1}b_2,\ a_1^{-1},\ a_1^2b_1,\ a_2^{-1},\ a_2^2b_2,\ a_2^{\frac{m}{2}+1}b_2,\ a_1,\ b_1, a_2,\ b_2\cdots.$$

Hence the Fibonacci length is 10, as required. ∎

We can show that for the subgroup $K=<a_2^{r'}b_2>$, $LEN_{(b_2,a_1,b_1,a_2)}(G)=\frac{10m}{(l,4)}$, where $m\equiv l \mod 4$. Also, if $K=<a_2^{m/2}>$, then $LEN_{(a_2,b_2,a_1,b_1)}(G)=LEN_{(b_2,a_1,b_1,a_2)}(G)=10$.

Now let us consider the subgroup $H=<a_1^d, a_1b_1>$ of D_{2n}, where $d|n$, $d\neq 1$ and $d\neq n$. We should note that if there exists a positive divisor d' of m such that $d'=\frac{dm}{n}$, then one can pick the subgroup

$$K=<a_2^{d'}, a_2^{r'}b_2> \tag{1}$$

of D_{2m}, where $0\leq r'<d'$.

Corollary 2 *Let $G=D_{2n}*_{<a_1^d,a_1b_1>}D_{2m}$ and let K be as in (1).*

i) If $r'\neq 1$ and $m\equiv l \mod (2r'-2)$, then

$$LEN_{(a_1,b_1,a_2,b_2)}(G)=\frac{10m}{(l,2r'-2)}.$$

ii) If $r'=1$, then $LEN_{(a_1,b_1,a_2,b_2)}(G)=10$.

Proof. In the proof of Theorem 1, the relation $a_1^n=1$ is never used in the calculation of the Fibonacci orbit. Thus $D_{2n}*_{<a_1^d,a_1b_1>}D_{2m}$ has the same Fibonacci length of the group given by Theorem 1. ∎

Let us take the dihedral subgroups $H=<a_1^rb_1>$ and cyclic subgroup $H=<a_1^{n/2}>$. Then we have the following result.

Theorem 3 *Let $K=<a_2b_2>$ be a subgroup of D_{2m}. Let $G=D_{2n}*_H D_{2m}$ where $f:H\to K$ is an isomorphism.*

i) If $H=<a_1^rb_1>$, where $0\leq r<n$, $r\neq 1$ and $n\equiv l \mod(4r-4)$, then

$$LEN_{(a_1,b_1,a_2,b_2)}(G)=\frac{10n}{(l,4r-4)}.$$

ii) Let n be even. If $H=<a_1^{n/2}>$, then $LEN_{(a_1,b_1,a_2,b_2)}(G)=10$.

Proof. We need to show that the Fibonacci orbit is of the form

$$x_j = \begin{cases} a_1^{\frac{(2r-2)(j-1)}{5}+1}, & j \equiv 1 \bmod 10, \\ a_1^{-\frac{(2r-2)(j-2)}{5}} b_1, & j \equiv 2 \bmod 10, \\ a_2, & j \equiv 3 \bmod 10, \\ b_2, & j \equiv 4 \bmod 10, \\ a_1^{1-r}, & j \equiv 5 \bmod 10, \\ a_1^{-\frac{(2r-2)(j-6)}{5}-(2r-1)}, & j \equiv 6 \bmod 10, \\ a_1^{\frac{(2r-2)(j-7)}{5}+4r-2} b_1, & j \equiv 7 \bmod 10, \\ a_2^{-1}, & j \equiv 8 \bmod 10, \\ a_2^2 b_2, & j \equiv 9 \bmod 10, \\ a_1^{r-1}, & j \equiv 0 \bmod 10. \end{cases}$$

We use induction on j.

$x_1 = a_1,$
$x_2 = b_1,$
$x_3 = a_2,$
$x_4 = b_2,$
$x_5 = a_1 b_1 a_2 b_2 = a_1^{1-r},$
$x_6 = b_1 a_2 b_2 a_1^{1-r} = a_1^{1-2r},$
$x_7 = a_2 b_2 a_1^{1-r} a_1^{1-2r} = a_1^r b_1 a_1^{2-3r} = a_1^{4r-2} b_1,$
$x_8 = b_2 a_1^{1-r} a_1^{1-2r} a_1^{4r-2} b_1 = b_2 a_1^r b_1 = a_2^{-1},$
$x_9 = a_1^{1-r} a_1^{1-2r} a_1^{4r-2} b_1 a_2^{-1} = a_1^r b_1 a_2^{-1} = a_2^2 b_2,$
$x_{10} = a_1^{1-2r} a_1^{4r-2} b_1 a_2^{-1} a_2^2 b_2 = a_1^{2r-1} b_1 a_1^r b_1 = a_1^{r-1}.$

Let $k \equiv 0 \bmod 10$ and let us suppose that the result holds for all values up to $k+5$, in other words,

$$x_{k+1} = a_1^{\frac{k(2r-2)}{5}+1},\ x_{k+2} = a_1^{-\frac{k(2r-2)}{5}} b_1,\ x_{k+3} = a_2,\ x_{k+4} = b_2,\ x_{k+5} = a_1^{1-r}.$$

Now we have to show that the required result is true for the next ten entries.

$$x_{k+6} = a_1^{-\frac{k(2r-2)}{5}} b_1 a_2 b_2 a_1^{1-r} = a_1^{-\frac{k(2r-2)}{5}} b_1 a_1^r b_1 a_1^{1-r} = a_1^{-\frac{k(2r-2)}{5}-(2r-1)},$$
$$x_{k+7} = a_2 b_2 a_1^{1-r} a_1^{-\frac{k(2r-2)}{5}-(2r-1)} = a_1^r b_1 a_1^{-\frac{k(2r-2)}{5}-3r+2} = a_1^{\frac{k(2r-2)}{5}+4r-2} b_1,$$
$$x_{k+8} = b_2 a_1^{1-r} a_1^{-\frac{k(2r-2)}{5}-(2r-1)} a_1^{\frac{k(2r-2)}{5}+4r-2} b_1 = b_2 a_1^r b_1 = a_2^{-1},$$
$$x_{k+9} = a_1^{1-r} a_1^{-\frac{k(2r-2)}{5}-(2r-1)} a_1^{\frac{k(2r-2)}{5}+4r-2} b_1 a_2^{-1} = a_1^r b_1 a_2^{-1} = a_2^2 b_2,$$
$$x_{k+10} = a_1^{-\frac{k(2r-2)}{5}-(2r-1)} a_1^{\frac{k(2r-2)}{5}+4r-2} b_1 a_2^{-1} a_2^2 b_2 = a_1^{r-1},$$
$$x_{k+11} = a_1^{\frac{k(2r-2)}{5}+4r-2} b_1 a_2^{-1} a_2^2 b_2 a_1^{r-1} = a_1^{\frac{k(2r-2)}{5}+4r-3} = a_1^{\frac{(2r-2)(k+10)}{5}+1},$$
$$x_{k+12} = a_2^{-1} a_2^2 b_2 a_1^{r-1} a_1^{\frac{(2r-2)(k+10)}{5}+1} = a_1^{-\frac{(2r-2)(k+10)}{5}} b_1,$$
$$x_{k+13} = a_2^2 b_2 a_1^{r-1} a_1^{\frac{(2r-2)(k+10)}{5}+1} a_1^{-\frac{(2r-2)(k+10)}{5}} b_1 = a_2^2 b_2 a_2 b_2 = a_2,$$
$$x_{k+14} = a_1^{r-1} a_1^{\frac{(2r-2)(k+10)}{5}+1} a_1^{-\frac{(2r-2)(k+10)}{5}} b_1 a_2 = a_2 b_2 a_2 = b_2,$$
$$x_{k+15} = a_1^{\frac{(2r-2)(k+10)}{5}+1} a_1^{-\frac{(2r-2)(k+10)}{5}} b_1 a_2 b_2 = a_1 b_1 a_2 b_2 = a_1^{1-r}.$$

For simplicity, let us denote $LEN_{(a_1,b_1,a_2,b_2)}(G)$ for short by LEN. Here we require LEN to be the smallest non-trivial integer such that

$$\begin{aligned} a_1^{\frac{(2r-2)LEN}{5}+1} &= a_1, \\ a_1^{-\frac{(2r-2)LEN}{5}} b_1 &= b_1. \end{aligned}$$

After the above progress, for $k \in \mathbb{N}$, we have

$$(2r-2)LEN = 5nk.$$

Let $n \equiv l \mod(2r-2)$. In this step, we have to consider $\frac{n}{(l,2r-2)}$ in two cases. If $\frac{n}{(l,2r-2)}$ is even, then $LEN = \frac{5n}{(l,2r-2)}$ and if $\frac{n}{(l,2r-2)}$ is odd, then $LEN = \frac{10n}{(l,2r-2)}$. Therefore we obtain

$$LEN = \frac{10n}{(l,4r-4)}.$$

On the other hand, it is easy to see that if $n \equiv l \mod 4$, then

$$LEN_{(b_1,a_2,b_2,a_1)}(G) = \frac{10n}{(l,4)}$$

and if $n \equiv l \mod(2r-2)$, then

$$LEN_{(a_2,b_2,a_1,b_1)}(G) = \frac{10n}{(l,2r-2)}.$$

ii) Since the members of the Fibonacci orbit are

$$a_1,\ b_1,\ a_2,\ b_2,\ a_1^{1-\frac{n}{2}} b_1,\ a_1^{-1},\ a_1^2 b_1,\ a_2^{-1},\ a_2^2 b_2,\ a_1^{1-\frac{n}{2}} b_1,\ a_1,\ b_1,\ a_2,\ b_2, \cdots$$

the required length is clearly 10.

Also, $LEN_{(b_1,a_2,b_2,a_1)}(G) = LEN_{(a_2,b_2,a_1,b_1)}(G) = 10$.

Hence the result. ■

Remark 4 *Let $G = D_{2n} *_H D_{2m}$. Let us consider the subgroup $K = < a_2^{r'} b_2 >$ of D_{2m} in the case $r' \neq 1$. We should note that, for any generating set $A = \{a_1, b_1, a_2, b_2\}$, if $H = < a_1^r b_1 >$ $(r \neq 1)$, then $F_A(G)$ is not periodic. We also note that if $H = 1$, then $F_A(G)$ is not a periodic sequence.*

The following result is an easy consequence of Theorem 3.

Corollary 5 *Let $K = < a_2^{d'}, a_2 b_2 >$ and $H = < a_1^d, a_1^r b_1 >$ be two subgroups of D_{2m} and D_{2n}, respectively. If $d = \frac{d'n}{m}$ and $n \equiv l \mod(4r-4)$, then*

$$LEN_{(a_1,b_1,a_2,b_2)}(D_{2n} *_H D_{2m}) = \frac{10n}{(l,4r-4)}.$$

By considering the 3-generator case, we have the following result.

Theorem 6 *Let* $G = D_{2n} *_{<a_1^d,\ a_1b_1>} D_{2m}$ *and* K *be a subgroup of* D_{2m}.

i) Let K *be as in* (1), *then* $LEN_{(b_1,a_2,b_2)}(G) = 8$.

ii) If $K =< a_2b_2 >$, *then* $LEN_{(a_1,b_1,a_2)}(G) = 8$ *and* $LEN_{(a_2,b_2,b_1)}(G) = 4n$.

iii) For an even positive integer m *and* $d = n$ *if* $K =< a_2^{m/2} >$, *then*

$$LEN_{(b_1,a_2,b_2)}(G) = 8.$$

Proof. The proof is similar to the proof of Theorem 1. ∎

We further obtain the following result as a consequence of Theorem 6.

Corollary 7 *Let* $K =< a_2^{d'}, a_2b_2 >$ *and* $G = D_{2n} *_H D_{2m}$.

i) For $d = \frac{d'n}{m}$, if $H =< a_1^d, a_1^r b_1 >$,then $LEN_{(b_2,a_1,b_1)}(G) = 8$.

ii) Let n be even positive integer and $d' = m$. If $H < a_1^{n/2} >$, then $LEN_{(b_2,a_1,b_1)}(G) = 8$.

Example 8 *By* [9], *it is known that the extended Hecke group* $\overline{H}(\lambda_q)$ *is defined by the presentation*

$$< T,\ S,\ R \mid T^2 = S^q = R^2 = 1,\ RT = TR,\ RS = S^{-1}R > .$$

Since $\overline{H}(\lambda_q) \cong D_2 *_{\mathbb{Z}_2} D_q$ *and the members of the Fibonacci orbit are*

$$T,\ S,\ R,\ TSR,\ STS^{-1},\ S^{-1},\ RS^{-2},\ STR,\ T,\ S,\ R, \cdots,$$

the Fibonacci length of $\overline{H}(\lambda_q)$ *is* 8. *Furthermore, as a special case, since the extended Hecke group is called the extended modular group* $\overline{\Gamma}$ *for* $q = 3$, *we also have the Fibonacci length of* $\overline{\Gamma}$ *is* 8.

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