# $\Delta$–Optimum Forbidden Subgraphs and Exclusive Sum Labellings of Graphs *

Jianxin Wei[a,b], Baoqiang Fan[b]
a. School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, P.R. China
b. School of Mathematics and Information, Ludong University, Yantai 264025, P.R. China
E-mails: weijx11@lzu.edu.cn, baoqiangfan@163.com

**Abstract**

The notions of *sum labelling* and *sum number* of graphs were introduced by F. Harary [1] in 1990. A mapping $f$ is called a sum labelling of a graph $G(V, E)$ if it is an injection from $V$ to a set of positive integers such that $uv \in E$ if and only if there exists a vertex $w \in V$ such that $f(w) = f(x) + f(y)$. In this case, $w$ is called *a working vertex*. If $f$ is a sum labelling of $G \cup rK_1$ for some nonnegative integer $r$ and $G$ contains no working vertex, $f$ is defined as *an exclusive sum labelling* of the graph $G$ by M. Miller et al. in paper [2]. The least possible number $r$ of such isolated vertices is called *the exclusive sum number* of $G$, denoted by $\epsilon(G)$. If $\epsilon(G) = \Delta(G)$, the labelling is called *$\Delta$–optimum exclusive sum labelling* and the graph is said to be *$\Delta$–optimum summable*, where $\Delta = \Delta(G)$ denotes the maximum degree of vertices in $G$. By using the notion of *$\Delta$–optimum forbidden subgraph* of a graph the exclusive sum numbers of *crown* $C_n \odot K_1$ and $\overline{(C_n \odot K_1)}$ are given in this paper. Some $\Delta$–optimum forbidden subgraphs of trees are studied and we prove that for any integer $\Delta \geq 3$ there exist trees not $\Delta$–optimum summable, and a nontrivial upper bound of the exclusive sum numbers of trees is also given in this paper.

*This research was supported in part by the National Natural Science Funds of China under grant numbers 11001117.

# 1 Introduction

We follow in general the graph-theoretic notations and terminologies of [3]. All graphs we consider in this paper are finite, simple and undirected graphs.

A mapping $f$ is called *a sum labelling* of a graph $G(V, E)$ if it is an injection from $V$ to a set of positive integers, such that $uv \in E$ if and only if there exists a vertex $w \in V$ such that $f(w) = f(x) + f(y)$. In this case, $w$ is called *a working vertex* and the edge $uv$ is said to be labelled by $f(w)$. A graph $G$ is said to be *a sum graph* if it has a sum labelling $f$, and $f$ is said to give a sum labelling for $G$. In general, a graph $G$ will require some isolated vertices to be a sum graph. The sum number $\sigma(G)$ is the smallest nonnegative integer $m$ such that $G \cup mK_1$, the union of $G$ and $m$ isolated vertices, is a sum graph. These notions were introduced by F. Harary [1] in 1990. For more information about sum graphs see [7]–[12].

M. Miller, D. Patel, J. Ryan, K. Sugeng, Slamin and M. Tuga [2] defined $f$ as *an exclusive sum labelling* of a graph $G$ if it is a sum labelling of $G \cup rK_1$ for some nonnegative integer $r$, and $G$ contains no working vertex. The least possible number $r$ of such isolated vertices is called *the exclusive sum number* of $G$, denoted by $\epsilon(G)$. Obviously, $\epsilon(G) \geq \Delta(G)$ and $\epsilon(G) \geq \sigma(G)$, where $\Delta(G)$ denotes the maximum degree of vertices in $G$. If $\epsilon(G) = \Delta(G)$, the labelling $f$ is called *$\Delta$–optimum exclusive sum labelling*, and the graph is said to be *$\Delta$–optimum summable*. The exclusive sum number of several families of graphs were determined in paper [2]: $\epsilon(K_{p,q}) = p+q-1$, for $p, q \geq 2$; $\epsilon(P_n) = 2$, for $n \geq 2$ and $\epsilon(C_n) = 3$, for $n \geq 3$. In paper [4], Caterpillars and Shrubs were shown to be $\Delta$–optimum summable. Some $\Delta$–optimum exclusive sum labelling of certain graphs with radius one were given in paper [6]. For more information about sum graphs see [13]–[15].

Some useful notions are presented in this paper. Suppose $f$ and $g$ are both exclusive sum labelling of graph $G \cup \epsilon K_1$ and $uv$ is any an edge of $G$, if there exist a one–one mapping $h$ from $f$ to $g$ such that $f(u) + f(v) = i \in f$ if and only if $g(u) + g(v) = h(i) \in g$, then the labelling $f$ and $g$ are said to be the same. If any two exclusive sum labellings of $G \cup \epsilon K_1$ are the same, then the exclusive sum labellings of the graph is said to be unique and $G$ be labelled uniquely. Let $G$ be a graph which has no $\Delta$–optimum exclusive sum labellings, obviously, $G$ must contains a subgraph $H$ not be $\Delta$–optimum summable, then $H$ is called a $\Delta$–optimum forbidden subgraph of $G$, where $\Delta = \Delta(H) = \Delta(G)$. We should emphasize the reason that $H$ can not be $\Delta$–optimum sum labelled always because the structure of $G$. So when we mention $H$ is a $\Delta$–optimum forbidden subgraph, it must be relative to some one supergraph $G$.

Let $G_1$ and $G_2$ are graphs and $G_1$ has $n$ vertices, then $G_1 \odot G_2$ is the graph obtained by taking one copy of $G_1$ and $n$ copies of $G_2$ and joining the $i^{th}$ vertex of $G_1$ with an edge to every vertex in the $i^{th}$ copy of $G_2$. Graph $C_n \odot K_1$ also be called crown [5], where $n \geq 3$. Given a graph $G$, the graph $G'$ is called the subdivision of $G$ if every edge $e \in E(G)$ is substituted by a path $P_3$ [3].

With the help of 3–optimum forbidden subgraph of $C_n \odot K_1$ and $(C_n \odot K_1)'$, the exclusive sum numbers and the exclusive sum labellings of $C_n \odot K_1$ and $(C_n \odot K_1)'$ are given in this paper. Some $\Delta$–optimum forbidden subgraphs of trees are studied and we prove that for any integer $\Delta \geq 3$ there exist trees not $\Delta$–optimum summable, and an upper bound of the exclusive sum numbers of trees is also given in this paper.

## 2 The exclusive sum numbers of $C_n \odot K_1$ and $(C_n \odot K_1)'$

First we give the following lemma about the exclusive sum number of $G \odot K_1$.

**Lemma 2.1.** *Suppose $\epsilon(G) = \epsilon$, then $\epsilon(G \odot K_1) \leq \epsilon + 1$.*

**Proof.** Let $f$ be an exclusive sum labelling of $G \cup \epsilon K_1$, $V(G) = \{u_1, u_2, \cdots, u_n\}$ and $V(\epsilon K_1) = \{p_1, p_2, \cdots, p_\epsilon\}$. In order to give an exclusive sum labelling of $(G \odot K_1) \cup (\epsilon + 1)K_1$, let $V(nK_1) = \{v_1, v_2, \cdots, v_n\}$ be the vertex set of the $n$ copies of $K_1$ and add a vertex $p_{\epsilon+1}$ to $V(\epsilon K_1)$ to form the vertex set of $(\epsilon + 1)K_1$. Let $f'(v) = f(v)$ for $v \in V(G \cup \epsilon K_1)$, $f'(v_i) = k - f(u_i)$ for $i = 1, 2, \cdots, n$ and $f'(p_{\epsilon+1}) = k$, where $k = 3t$ and $t = max\{f(v) : v \in V(G \cup \epsilon K_1)\}$. We claim that $f'$ is an exclusive sum labelling of $(G \odot K_1) \cup (\epsilon + 1)K_1$. Obviously, we require the following (1), (2), (3), (4) to prove the claim.

(1)For any $u, v \in V(G \cup \epsilon K_1)$, $f'(u) + f'(v) \notin \{f'(v) : v \in V(nK_1)\} \cup \{k\}$.

Since $f'(u) + f'(v) < 2t \leq min(\{f'(w) : w \in V(nK_1)\} \cup \{k\})$ for any $u, v \in V(G \cup \epsilon K_1)$, then $f'(u) + f'(v) \notin \{f'(v) : v \in V(nK_1)\} \cup \{k\}$.

(2)For any $u \in V(G \cup \epsilon K_1)$ and any $v_i \in V(nK_1)$, if $u \neq u_i$ then $f'(v_i) + f'(u) \notin \{f'(w) : w \in V((G \odot K_1) \cup (\epsilon + 1)K_1)\}$.

Let $u \neq u_i$, then $f'(v_i) + f'(u) \geq 2t$ for any $u \in V(G \cup \epsilon K_1)$ and any $v_i \in V(nK_1)$. If $f'(v_i) + f'(u) \in \{f'(w) : w \in V((G \odot K_1) \cup (\epsilon + 1)K_1)\}$, there must exist some $v_j \in V(nK_1)$ such that $f'(v_i) + f'(u) = f'(v_j)$ or $f'(v_i) + f'(u) = k$. Suppose $f'(v_i) + f'(u) = f'(v_j)$, then $k - f'(u_i) + f'(u) = k - f'(u_j)$, so $f(u) + f(u_j) = f(u_i)$, which is impossible. Suppose $f'(v_i) + f'(u) = k$, then $f(u) = k - f'(v_i) = f(u_i)$, which is impossible.

(3)For any $v_i, v_j \in V(G \odot K_1)$, $f'(v_i) + f'(v_j) \notin \{f'(w) : w \in V((G \odot K_1) \cup (\epsilon + 1)K_1)\}$.

Since for any $v_i, v_j \in V(G \odot K_1)$, $f'(v_i)+f'(v_j) > 4t > k = max\{f'(w) : w \in V(G \odot K_1) \cup (\epsilon+1)K_1)\}$, $f'(v_i)+f'(v_j) \notin \{f'(w) : w \in V((G \odot K_1) \cup (\epsilon+1)K_1)\}$.

(4)For any $u \in V((G \odot K_1) \cup (\epsilon+1)K_1)$, $f'(u)+k \notin \{f'(w) : w \in V((G \odot K_1) \cup (\epsilon+1)K_1)\}$.

For any $u \in V((G \odot K_1) \cup (\epsilon+1)K_1)$, $f'(u)+k > k = max\{f'(w) : w \in V((G \odot K_1) \cup (\epsilon+1)K_1)\}$, so $f'(u)+k \notin \{f'(w) : w \in V((G \odot K_1) \cup (\epsilon+1)K_1)\}$. □

It is easy to get the following corollary from the proof of *lemma* 2.1.

**Corollary 2.2.** If $G$ is $\Delta$–optimum summable, $G \odot K_1$ is also $\Delta$–optimum summable.

Let $G$ be a subgraph of $H$ and $H$ be a subgraph of $G \odot K_1$, then *lemma* 2.1 holds for $H$, that is, $\epsilon(H) \leq \epsilon(G)+1$. Obviously, *Corollary* 2.2 holds for $H$ if $\Delta(H) = \Delta(G)$, i.e., $H$ is $\Delta$-optimum summable. An example of Graph $C_{10} \odot K_1$ and one of its subgraph $H$ (containing $C_{10}$ as subgraph) are shown in $(i)$ and $(ii)$ of *Figure* 1 respectively.

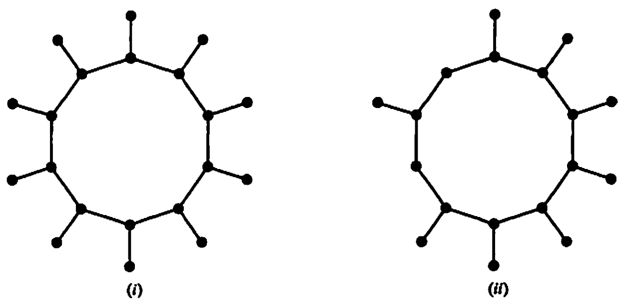

Figure 1: $(i)$ graph $G = C_{10} \odot K_1$ and $(ii)$ a subgraph $H$ of $C_{10} \odot K_1$

Since a caterpillar $H$ is a subgraph of the graph $G$ got from a path by continuous using *lemma* 2.1 for $\Delta - 2$ times and path is 2–optimum summable [2], the following *corollary* 2.3 [4] can be proved to be right, where $\Delta = \Delta(H)$.

**Corollary 2.3.** *Caterpillar is $\Delta$-optimum summable.*

Paper [2] showed that $\epsilon(C_n) = 3$. By this result and *Lemma* 2.1, the following *lemma* 2.4 can be proved to be right easily.

**Lemma 2.4.** $\epsilon(C_n \odot K_1) \leq 4$ *for all positive integers* $n \geq 3$.

In order to get our main results, now we need to prove *lemma* 2.5.

**Lemma 2.5.** *The graph as shown in $(i)$ of Figure 2 is a 3-optimum* *forbidden subgraph of $C_n \odot K_1$ for $n \geq 7$, then $\epsilon(C_n \odot K_1) \geq 4$ for $n \geq 7$.*

**Proof.** Suppose $\epsilon(C_n \odot K_1) = 3$, for all integer $n \geq 7$. Let $V(C_n \odot K_1) \cup 3K_1) = A \cup B \cup I$, where, $A = \{a_1, a_2, \cdots, a_n\}$ is the vertex set of $C_n$, $B = \{b_1, b_2, \cdots, b_n\}$ is the set of $n$ copies of $K_1$, $I = \{c_1, c_2, c_3\}$ is the isolated vertex set of $3K_1$ and $a_t b_t \in E((C_n \odot K_1) \cup 3K_1))$, for $t = 1, 2, \cdots, n$.

Let $f = f(A) \cup f(B) \cup f(I)$ be the exclusive sum labelling of $(C_n \odot K_1) \cup 3K_1$. Since $\epsilon(C_n) = 3$, there must exists a path $P_7$ as the subgraph of $C_n$ such that the edges of $P_7$ are exactly labelled by $f(c_1) = i$, $f(c_2) = j$ and $f(c_3) = k$, i.e., the number of the edges of $P_7$ be labelled by $f(c_t)$ is more than 0, where $t = 1, 2, 3$. Without lost generality, set $P_7 = a_1a_2a_3a_4a_5a_6a_7$. Let $(t_1, t_2, t_3)$ denote the number that the edges of $P_7$ be labelled by $i, j, k$ respectively. Let $a_1p_1a_2p_2a_3p_3a_4p_4a_5p_5a_6p_6a_7$ denote the edges $a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_6a_7$ be labelled by $p_1, p_2, p_3, p_4, p_5, p_6$, respectively, where $p_t \in \{i, j, k\}$, $t = 1, 2, \cdots, 6$. As an example $a_1ia_2ja_3ka_4ia_5ja_6ia_7$ is illustrated in *(ii)* of *Figure* 2. Since $i, j$ and $k$ are only symbols for three numbers, $(t_1, t_2, t_3) = (3, 2, 1)$ or $(2, 2, 2)$. Then the labelling of $P_7$ perhaps are some cases as shown in the following discussions and all these cases are proved to be impossible.

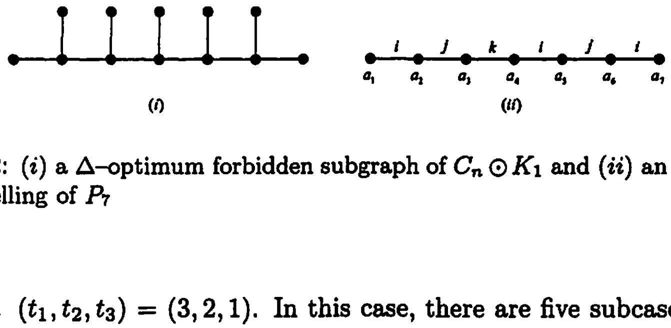


Figure 2: *(i)* a $\Delta$–optimum forbidden subgraph of $C_n \odot K_1$ and *(ii)* an exclusive sum labelling of $P_7$

**Case 1.** $(t_1, t_2, t_3) = (3, 2, 1)$. In this case, there are five subcases of the exclusive sum labelling of $P_7$.

**Subcase 1.1.** $a_1p_1a_2p_2a_3p_3a_4p_4a_5p_5a_6p_6a_7 = a_1ia_2ja_3ka_4ia_5ja_6ia_7$.

We obtain that $f(a_1) = i - f(a_2) = i - (j - f(a_3)) = i - j + (k - f(a_4)) = i - j + k - (i - f(a_5)) = k - j + (j - f(a_6)) = k - f(a_6)$, so $f(a_1) + f(a_6) = k$, a contradiction to $a_1a_6 \notin E((C_n \odot K_1) \cup 3K_1)$.

**Subcase 1.2.** $a_1p_1a_2p_2a_3p_3a_4p_4a_5p_5a_6p_6a_7 = a_1ja_2ia_3ka_4ia_5ja_6ia_7$.

In this subcase, $f(b_3) = j - f(a_3) = j - (k - f(a_4)) = j - k + (i - f(a_5)) = j - k + i - (j - f(a_6)) = i - k + (k - f(b_6)) = i - f(b_6)$, so $f(b_1) + f(b_6) = i$, a contradiction to $b_1b_6 \notin E((C_n \odot K_1) \cup 3K_1)$.

**Subcase 1.3.** $a_1p_1a_2p_2a_3p_3a_4p_4a_5p_5a_6p_6a_7 = a_1ia_2ka_3ja_4ia_5ja_6ia_7$.

Since $a_1 \in V(C_n)$, it must has a neighbor vertex $u \in V(C_n \odot K_1)$ labelled by $j - f(a_1)$. Then $f(u) = j - f(a_1) = j - (i - f(a_2)) = j - i + (k - f(a_3)) = j - i + k - (j - f(a_4)) = k - i + (i - f(a_5)) = k - f(a_5)$, so $f(u) + f(a_5) = k$, a contradiction to $ua_5 \notin E((C_n \odot K_1) \cup 3K_1)$ for $n \geq 7$.

**Subcase 1.4.** $a_1p_1a_2p_2a_3p_3a_4p_4a_5p_5a_6p_6a_7 = a_1ia_2ka_3ia_4ja_5ia_6ja_7$.

We obtain $f(b_2) = j - f(a_2) = j - (k - f(a_3)) = j - k + (i - f(a_4)) = j - k + i - (j - f(a_5)) = i - k + (k - f(b_5)) = i - f(b_5)$, then $f(b_2) + f(b_5) = i$, a contradiction to $b_1b_6 \notin E((C_n \odot K_1) \cup 3K_1)$.

**Subcase 1.5.** $a_1p_1a_2p_2a_3p_3a_4p_4a_5p_5a_6p_6a_7 = a_1ka_2ia_3ja_4ia_5ja_6ia_7$.

Similar to *Subcase* 1.3., $a_1 \in V(C_n)$, so it must has a neighbor vertex $u$ labelled by $j - f(a_1)$. Then $f(u) = j - f(a_1) = j - (k - f(a_2)) = j - k + (i - f(a_3)) = j + i - k - (j - f(a_4)) = i - k + (k - f(b_4)) = i - f(b_4)$, so $f(u) + f(b_4) = i$, a contradiction to $ub_4 \notin E((C_n \odot K_1) \cup 3K_1)$ for $n \geq 7$.

**Case 2.** $(t_1, t_2, t_3) = (2, 2, 2)$.

**Subcase 2.1.** $a_1p_1a_2p_2a_3p_3a_4p_4a_5p_5a_6p_6a_7 = a_1ia_2ja_3ka_4ia_5ja_6ka_7$.

By a similar argument of *subcase* 1.1, a contradiction that $f(a_1) + f(a_6) = k$ can be obtained in this subcase.

**Subcase 2.2.** $a_1p_1a_2p_2a_3p_3a_4p_4a_5p_5a_6p_6a_7 = a_1ia_2ja_3ka_4ja_5ia_6ka_7$.

We can obtain $f(b_3) = i - f(a_3) = i - (k - f(a_4)) = i - k + (j - f(a_5)) = i - k + j - (i - f(a_6)) = j - k + (k - f(a_7)) = j - f(a_7)$, so $f(b_3) + f(a_7) = j$, a contradiction to $b_3a_7 \notin E((C_n \odot K_1) \cup 3K_1)$.

**Subcase 2.3.** $a_1p_1a_2p_2a_3p_3a_4p_4a_5p_5a_6p_6a_7 = a_1ia_2ja_3ia_4ka_5ja_6ka_7$.

It is easy to get $f(b_2) = k - f(a_2) = k - (j - f(a_3)) = k - j + (i - f(a_4)) = k - j + i - (k - f(a_5)) = i - j + (j - f(a_6)) = i - f(a_6)$, so $f(b_2) + f(a_6) = i$, a contradiction to $b_2a_6 \notin E((C_n \odot K_1) \cup 3K_1)$.

**Subcase 2.4.** $a_1p_1a_2p_2a_3p_3a_4p_4a_5p_5a_6p_6a_7 = a_1ia_2ja_3ka_4ia_5ka_6ja_7$.

Since $a_1 \in V(C_n)$, it must has one neighbor vertex $u$ labelled by $k - f(a_1)$. Then $f(u) = k - f(a_1) = k - (i - f(a_2)) = k - i + (j - f(a_3)) = k - i + j - (k - f(a_4)) = j - i + (i - f(a_5)) = j - f(a_5)$, $f(u) + f(a_5) = j$, a contradiction to $ua_5 \notin E((C_n \odot K_1) \cup 3K_1)$ for $n \geq 7$.

**Subcase 2.5.** $a_1p_1a_2p_2a_3p_3a_4p_4a_5p_5a_6p_6a_7 = a_1ja_2ia_3ka_4ia_5ka_6ja_7$.

Since $a_1 \in V(C_n)$, it must has one neighbor vertex $u$ labelled by $k - f(a_1)$. Then we get $f(u) = k - f(a_1) = k - (j - f(a_2)) = k - j + (i - f(a_3)) = k - j + i - (k - f(a_4)) = i - j + (j - f(b_4)) = i - f(b_4)$, $f(u) + f(b_4) = i$, a contradiction to $ub_4 \notin E((C_n \odot K_1) \cup 3K_1)$ for $n \geq 7$.

So the graph as shown in (*i*) *Figure* 2 is a 3–optimum forbidden subgraph of $C_n \odot K_1$ for all positive integer $n \geq 7$ and $\epsilon(C_n \odot K_1) = 3$ is impossible, then $\epsilon(C_n \odot K_1) \geq 4$ for $n \geq 7$.

We should emphasize again that $i, j$, and $k$ only general symbols, in the subcases above, we do not give the same labellings repeatedly. For example, the exclusive sum labellings $a_1ja_2ka_3ja_4ia_5ka_6ia_7$ and $a_1ia_2ja_3ia_4ka_5ja_6k$ $a_7$ are the same if we let $j \leftrightharpoons i, k \rightleftharpoons j$ and $i \rightleftharpoons k$. □

**Theorem 2.6.** $\epsilon(C_n \odot K_1)= \begin{cases} 3, & n=3,6 \\ 4, & n=4,5,7\cdots \end{cases}$.

**Proof.** First we prove that $\epsilon((C_n \odot K_1) \cup 3K_1) = 3$ for $n = 3, 6$. Let $f_1=\{12, 13, 14, 16, 17, 20, 29, 30, 33\}$ and $f_2 = \{10, 17, 21, 24, 26, 28, 33, 35, 37, 42, 44, 53, 54, 61, 70\}$, it is easy to prove that $f_1$ and $f_2$ are the exclusive sum labellings of $(C_3 \odot K_1) \cup 3K_1$ and $(C_6 \odot K_1) \cup 3K_1$, respectively.

By *lemma* 2.4 and *lemma* 2.5, we know that $\epsilon(C_n \odot K_1) = 4$ for $n \geq 7$.

Now we prove that $\epsilon(C_n \odot K_1) = 4$ for $n = 4, 5$. We have known that $\epsilon(C_n \odot K_1) \leq 4$ by *lemma* 2.4 and $\epsilon(C_n \odot K_1) \geq \Delta(C_n \odot K_1) = 3$ for $n = 4, 5$. Suppose that $\epsilon(C_n \odot K_1) = 3$ for $n = 4, 5$ and $f$ is the corresponding exclusive sum labellings. Just like in *lemma* 2.5, let $V((C_n \odot K_1) \cup 3K_1) = A \cup B \cup I$, where $A = \{a_1, a_2, \cdots, a_n\}$ is the vertex set of $C_n$, $B = \{b_1, b_2, \cdots, b_n\}$ is the set of $n$ copies of $K_1$, $I = \{c_1, c_2, c_3\}$ is the isolated vertex set of $3K_1$ and $a_tb_t \in E((C_n \odot K_1) \cup 3K_1))$, for $t = 1, 2, \cdots, n$ and $n = 4, 5$. Let $f(c_1) = i, f(c_2) = j$ and $f(c_3) = k$, then the edges of $(C_n \odot K_1) \cup 3K_1$ can only be labelled as shown in ($i$) and ($ii$) of *Figure* 3 for $n = 4$ and 5 respectively.

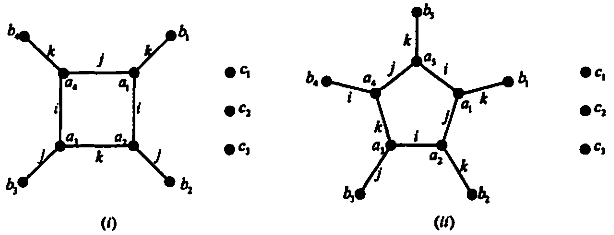

Figure 3: the edges of $(C_n \odot K_1) \cup 3K_1$ be labelled by $i, j, k$ for $n = 4, 5$

From ($i$) of *Figure* 3 we can obtain $f(a_3)+f(a_4) = i, f(a_3)+f(b_3) = k$ and $f(a_4) + f(b_4) = j$, then $k + j = (f(a_3) + f(b_3)) + (f(a_4) + f(b_4)) = (f(a_3) + f(a_4)) + (f(b_3) + f(b_4)) = i + (f(b_3) + f(b_4))$. Since $f(a_1) + f(a_2) = i, f(a_1) + f(a_4) = k$ and $f(a_2) + f(a_3) = j$, then $2i = (f(a_1) + f(a_2)) + (f(a_3) + f(a_4)) = (f(a_1) + f(a_4)) + (f(a_2) + f(a_3)) = k + j$. So $2i = i + (f(b_3) + f(b_4))$, i.e., $f(b_3) + f(b_4) = i$, a contradiction to $b_3b_4 \notin E((C_4 \odot K_1) \cup 3K_1)$. So $\epsilon(C_4 \odot K_1) = 4$.

From $(ii)$ of *Figure* 3 we can obtain $f(a_1)+f(a_2)=j$ and $f(a_1)+f(a_5)=i$, then $i+j=(f(a_1)+f(a_2))+(f(a_1)+f(a_5))=2f(a_1)+(f(a_2)+f(a_5))$. By $f(a_4)+f(a_5)=j$ and $f(a_2)+f(a_3)=i$, then $i+j=(f(a_4)+f(a_5))+(f(a_2)+f(a_3))=(f(a_3)+f(a_4))+(f(a_2)+f(a_5))$. So $f(a_3)+f(a_4)=2f(a_1)=k$ and $k=f(a_1)+f(b_1)$, then $f(a_1)=f(b_1)$, which is impossible. So $\epsilon(C_5\odot K_1)=4$. □

In order to give the exclusive sum number and the exclusive sum labelling of the subdivision graph $(C_n\odot K_1)'$ of $(C_n\odot K_1)$, we need the following *lemma* 2.7.

**Lemma 2.7.** *The graph as shown in* $(i)$ *of figure* 4 *is* 3*-optimum summable and the exclusive sum labelling of this graph is unique.*

**Proof.** For the convenience of clarification, the graph as shown in *Figure*4$(i)$ is denoted by $TG_1$. First we claim that $\epsilon(TG_1)=3$. In fact, it is easy to prove that $f=\{31,38,23,46,40,30,33,28,36,32,37,61,63,69\}$ is an exclusive sum labelling of $TG_1$, then $\epsilon(TG_1)=3$.

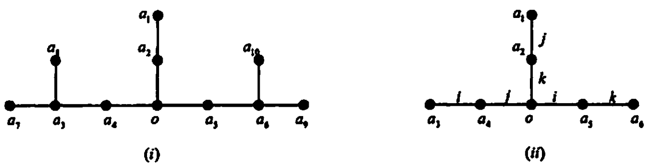

Figure 4: $(i)$ graph $TG_1$ and $(ii)$ a labelling of $TG_1[X]$

Just as shows in $(i)$ of *Figure* 4, Let $V=\{o,a_1,a_2,\cdots,a_{10}\}$ denote the vertex set of $TG_1$. Let $f=\{f(o),f(a_1),f(a_2),\cdots,f(a_{10}),i,j,k\}$ is any one of the exclusive sum labellings of $TG_1\cup 3K_1$, where $i,j,k$ are the labels of the three isolated vertexes of $TG_1\cup 3K_1$. We consider the labelling of the subgraph $TG_1[X]$ of $TG_1$, where $X=\{o,a_1,a_2,a_3,a_4,a_5,a_6\}$. Let $(t_1,t_2,t_3)$ denotes the numbers of the edges of $TG_1[X]$ that be labelled by $i,j$ and $k$, respectively. Let $a_1p_1a_2p_2o\begin{cases}p_3a_4p_4a_3\\ p_5a_5p_6a_6\end{cases}$ denotes the labels of the vertexes of $TG_1[X]$ be labelled meets the following: $f(a_1)+f(a_2)=p_1$, $f(a_2)+f(o)=p_2$, $f(o)+f(a_4)=p_3$, $f(a_4)+f(a_3)=p_4$, $f(o)+f(a_5)=p_5$ and $f(a_5)+f(a_6)=p_6$, where $p_t\in\{i,j,k\}, t=1,2,3$. We illustrate the example of $a_1ja_2ko\begin{cases}ja_4ia_3\\ ia_5ka_6\end{cases}$ in $(ii)$ of *Figure* 4.

Similar to *lemma* 2.5, $i,j$, and $k$ are only three symbols, so $(t_1,t_2,t_3)=(3,2,1)$ or $(2,2,2)$. Obviously, if we prove that the exclusive sum labelling of $TG_1[X]$ is unique, then $TG_1$ is also unique. Now we prove the uniqueness of the exclusive sum labelling of $TG_1[X]$ according to the following cases.

**Case 1.** $(t_1, t_2, t_3) = (3, 2, 1)$.

There are two subcases for the labelling form of $TG_1[X]$ in this case.

**Subcase 1.1.** $a_1p_1a_2p_2o \begin{cases} p_3a_4p_4a_3 \\ p_5a_5p_6a_6 \end{cases} = a_1ia_2ko \begin{cases} ja_4ia_3 \\ ia_5ja_6 \end{cases}$.

In this case, one of the edges $a_3a_7$ and $a_3a_8$ must be labelled by $k$. Suppose $a_3a_8$ be labelled by $k$, then $f(a_8) = k - f(a_3) = k - (i - f(a_4)) = k - i + (j - f(o)) = k - i + j - (k - f(a_2)) = j - i + (i - f(a_1)) = j - f(a_1)$, so $f(a_8) + f(a_1) = j$. Then $a_8a_1 \in E(TG_1)$, which is impossible.

**Subcase 1.2.** $a_1p_1a_2p_2o \begin{cases} p_3a_4p_4a_3 \\ p_5a_5p_6a_6 \end{cases} = a_1ia_2jo \begin{cases} ka_4ia_3 \\ ia_5ja_6 \end{cases}$.

In this case, one of the edges $a_3a_7$ and $a_3a_8$ must be labelled by $j$. Suppose $a_3a_8$ be labelled by $j$, then $f(a_8) = j - f(a_3) = j - (i - f(a_4)) = j - i + (k - f(o)) = j - i + k - (j - f(a_2)) = k - i + (i - f(a_1) = k - f(a_1)$, so $f(a_8) + f(a_1) = k$. Then $a_8a_1 \in E(TG_1)$, which is impossible.

**Case 2.** $(t_1, t_2, t_3) = (2, 2, 2)$.

In this case, there are also two subcases for the labelling form of $TG_1[X]$.

**Subcase 2.1.** $a_1p_1a_2p_2o \begin{cases} p_3a_4p_4a_3 \\ p_5a_5p_6a_6 \end{cases} = a_1ka_2jo \begin{cases} ia_4ka_3 \\ ja_5ia_6 \end{cases}$.

In this case, one of the edges $a_3a_7$ and $a_3a_8$ must be labelled by $j$. Suppose $a_3a_8$ be labelled by $j$, then $f(a_8) = j - f(a_3) = j - (k - f(a_4)) = j - k + (i - f(o)) = j - k + i - (j - f(a_2)) = i - k + (k - f(a_1) = i - f(a_1)$, so $f(a_8) + f(a_1) = i$. Then $a_8a_1 \in E(TG_1)$, which is impossible.

**Subcase 2.2.** $a_1p_1a_2p_2o \begin{cases} p_3a_4p_4a_3 \\ p_5a_5p_6a_6 \end{cases} = a_1ka_2jo \begin{cases} ka_4ia_3 \\ ia_5ja_6 \end{cases}$.

Let $i = 61, j = 63, k = 69$ and $f(o) = 31$, then an exclusive sum labelling of $TG_1[X]$ can be given by the form of this subcase,the labelling $f$ just as given at beginning of the proof of this *lemma*.

So the exclusive sum labelling of $TG_1[X]$ as given in *Subcase*2.2 is the only one. Then the exclusive sum labelling of $TG_1$ is unique. □

**Theorem 2.8.** $\epsilon((C_n \odot K_1)') = 4$, *for* $n \geq 3$.

**Proof.** For all $n \geq 3$, let $V((C_n \odot K_1)') = A \cup B \cup C \cup D$, where $A = \{a_1, a_2, \cdots, a_n\}$, $B = \{b_1, b_2, \cdots, b_n\}$, $C = \{c_1, c_2, \cdots, c_n\}$ and $D = \{d_1, d_2, \cdots, d_n\}$. $A \cup C$ is the vertex set of the subgraph $(C_n)'$ of $(C_n \odot K_1)'$ and $(C_n)' = a_1c_1a_2c_2 \cdots a_nc_na_1$. $A \cup B \cup D$ is the vertex set of the $n$ subgraphs $P_3$(not contained in $(C_n)'$) of $(C_n \odot K_1)'$ and $P_3 = a_id_ib_i$ for $i = 1, 2, \cdots, n$. As an example graph $(C_9 \odot K_1)'$ is shown in *Figure* 5.

First, we will prove $\epsilon((C_n \odot K_1)') \geq 4$ for $n \geq 3$.

Suppose to the contrary that $\epsilon((C_n \odot K_1)') = 3$. Let $I = \{p_1, p_2, p_3\}$ is the three isolated vertexes of $(C_n \odot K_1)' \cup 3K_1$. Let $f = f(A) \cup f(B) \cup f(C) \cup f(D) \cup f(I)$ is an exclusive sum labelling of $(C_n \odot K_1)' \cup 3K_1$ with $f(p_1) = i, f(p_2) = j$ and $f(p_3) = k$. By *lemma* 2.7 we know the exclusive sum labelling of $(C_n \odot K_1)' \cup 3K_1$ can only be the following two cases.

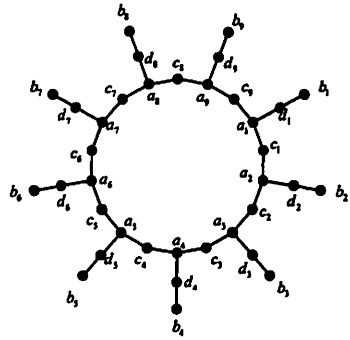

Figure 5: the graph $(C_9 \odot K_1)'$

**Case 1.** $n = 2p, p \geq 2$.

Now we consider the labelling of the subgraph $C_n'$ of $(C_n \odot K_1)' \cup 3K_1$. Without lost generality, let the edges $a_1c_1$ and $c_1a_2$ be labelled by $i$ and $j$ respectively, then by *lemma* 2.7, the other edges $a_2c_2, c_2a_3, a_3c_3, c_3a_4, a_4c_4, c_4a_5, \cdots, a_{2p-1}\,c_{2p-1}, c_{2p-1}a_{2p}, a_{2p}c_{2p}, c_{2p}a_1$ of $C_n'$ must be labelled by $i, k, i, j, i, k, \cdots, i, j, i, k$, respectively. Then $f(a_t) + f(c_t) = i$ for $t = 1, 2, 3, \cdots, 2p$, $f(c_{2t-1}) + f(a_{2t}) = j$ for $t = 1, 2, 3, \cdots, p$, $f(c_{2t}) + f(a_{2t+1}) = k$ for $t = 1, 2, 3, \cdots, p-1$ and $f(c_{2p}) + f(a_1) = k$. So, $\sum_{t=1}^{2p}(f(a_t) + f(c_t)) = 2pi$ and $\sum_{t=1}^{p}(f(c_{2t-1}) + f(a_{2t})) + \sum_{t=1}^{p}(f(c_{2t}) + f(a_{2t+1})) + (f(c_{2p}) + f(a_1)) = \sum_{t=1}^{2p}(f(a_t) + f(c_t)) = p(i+j)$, then $2i = j + k$. Further more, $f(c_2) = i - f(a_2) = i - (j - f(c_1)) = i - j - (i - f(a_1)) = 2i - j - (k - f(c_{2p})) = 2i - j - k + f(c_{2p})$, then $f(c_2) = f(c_{2p})$, which is impossible.

**Case 2.** $n = 2p + 1, p \geq 1$.

Suppose the edges $a_1c_1$ and $c_1a_2$ of $(C_n)'$ are labelled by $i$ and $j$ respectively, then by *lemma*2.7 the other edges $a_2c_2, c_2a_3, a_3c_3, c_3a_4, a_4c_4, c_4a_5, \cdots, a_{2p}c_{2p}, c_{2p}a_{2p+1}, a_{2p+1}c_{2p+1}, c_{2p+1}a_1$ of $(C_n)'$ can only be labelled by $i, k, i, j, i, k, \cdots, i, k, i, j$, respectively. Obviously, $f(a_1) + f(d_1) = k$. By the possibility of the label of the edge $d_1b_1$, there are two subcases in this case.

**Subcase 2.1.** $f(d_1) + f(b_1) = i$.

In this subcase $f(b_1) = i - f(d_1) = i - (k - f(a_1)) = i - k + (j - f(c_{2p+1})) = i - k + j - (i - f(a_{2p+1})) = j - k + (k - f(c_{2p})) = j - f(c_{2p})$, then $f(b_1) + f(c_{2p}) = j$, which is impossible.

**Subcase 2.2.** $f(d_1) + f(b_1) = j$.

In this subcase, $f(b_1) = j - f(d_1) = j-(k-f(a_1)) = j-k+(i-f(c_1)) = j-k+i-(j-f(a_2)) = i-k+(k-f(d_2)) = i-f(d_2)$, then $f(b_1)+f(d_2) = i$, which is impossible.

So we know that $\epsilon((C_n \odot K_1)') \geq 4$ for $n \geq 3$.

Now we prove that $\epsilon((C_n \odot K_1)') \leq 4$ for $n \geq 3$. This just need to give an exclusive sum labelling for $((C_n \odot K_1)') \cup 4K_1$.

Let $f$ be a labelling satisfied the following:

$f(a_t) = 10n + 4t - 3, t = 1, 2, 3, \cdots, n;$

$f(c_t) = 10n - 4t - 7, t = 1, 2, 3, \cdots, n;$

$f(b_1) = 10n + 7$ and $f(b_t) = 8n - 4t + 11, t = 2, 3, \cdots, n;$

$f(d_1) = 30n + 9$ and $f(d_t) = 32n + 4t - 5, t = 2, 3, \cdots, n;$

$f(p_1) = 20n+4$, $f(p_2) = 20n+8$, $f(p_3) = 18n+8$ and $f(p_4) = 40n+16$, where $p_t$ is the isolated vertexes of $((C_n \odot K_1)') \cup 4K_1$, $t = 1, 2, 3, 4$.

It is easy to prove that $f = \{f(a_t), f(b_t), f(c_t), f(d_t), f(p_l) : t = 1, 2, \cdots, n, l = 1, 2, 3, 4\}$ is the exclusive sum number of $((C_n \odot K_1)') \cup 4K_1$. Here we omit to prove the correctness of the labelling to save space.

From the above discussion, $\epsilon((C_n \odot K_1)') = 4$ for $n \geq 3$. □

# 3 About the exclusive sum numbers of Trees

In this section we will show kinds of trees that are not Δ–optimum summable, and give a nontrivial upper bound for the exclusive sum number of trees.

Let $TG_2$ and $TG_3$ denote the graph shown in $(i)$ and $(ii)$ of *Figure* 6 respectively.

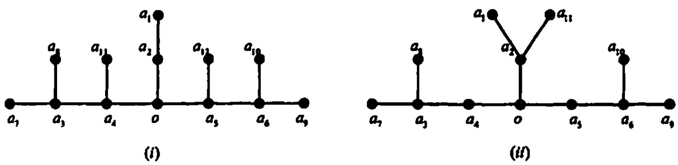


Figure 6: $(i)$ $TG_2$ and $(ii)$ $TG_3$

**Theorem 3.1.** *$TG_2$ and $TG_3$ are 3–optimum forbidden subgraphs of trees $T$ with $\Delta(T) = 3$. Then any tree $T$ with $\Delta(T) = 3$ and containing $TG_2$ or*

*$TG_3$ is not $\Delta$-optimum summable.*

**Proof.** Obviously, $TG_2$ and $TG_3$ both contain $TG_1$ as subgraph and $\Delta(TG_2) = \Delta(TG_3) = 3$.

Suppose $TG_2$ and $TG_3$ are both $\Delta$–optimum summable. Since the sum labelling of graph $TG_1$ is unique by *lemma* 2.7, the exclusive sum labelling of $TG_2$ and $TG_3$ can easily be given.

First we claim that $\epsilon(TG_2) = 3$ is impossible. By *lemma* 2.7, let the edge $a_6a_9$ be labelled by $j$, so $f(a_{11}) = k - f(a_4) = k - (j - f(o)) = k-j+(i-f(a_5)) = k-j+i-(k-f(a_6)) = i-j+(j-f(a_9)) = i-f(a_9)$, then $f(a_{11}) + f(a_9) = i$, which is impossible.

By similar argument, we can also prove that $\epsilon(TG_3) = 3$ is impossible. In fact, suppose the edge $a_7a_3$ and $a_2a_{11}$ be labelled by $k$ and $i$ respectively, then $f(a_7) + f(a_{11}) = j$, a contradiction to $a_7a_{11} \notin E(TG_3 \cup 3K_1)$.

The exclusive sum numbers of $TG_2$ and $TG_3$ both are 4. In fact, it is easy to prove that $S_1 = \{413, 468, 383, 438, 523, 394, 487, 457, 419, 542, 462, 370, 389, 832, 851, 881, 961\}$ and $S_2 = \{364, 407, 333, 440, 376, 356, 415, 384, 368, 403, 329, 337, 732, 740, 771, 847\}$ are exclusive sum labellings of $TG_2 \cup 4K_1$ and $TG_3 \cup 4K_1$ respectively. □

In order to give a generalization of the above theory, we consider the exclusive sum number and the exclusive sum labelling of the tree as shown in $(i)$ of *figure*7 (denoted by $TG_4$), where $D = \Delta = \Delta(TG_4)$ and $\Delta \geq 3$.

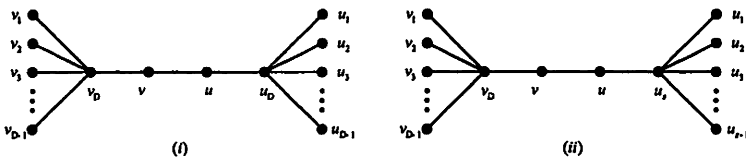


Figure 7: $(i)$ $TG_4$ and $(ii)$ $TG_5$.

**Lemma 3.2.** *$TG_4$ is $\Delta$-optimum summable and the exclusive sum labelling of this graph $TG_4$ is unique.*

**Proof.** Obviously, $TG_4$ is a caterpillar, so it is $\Delta$–optimum summable by *corollary* 2.3. By similar argument of *lemma* 2.7, we can prove that the exclusive sum labelling of $TG_4$ is unique. Let $i_1, i_2, \cdots, i_\Delta$ be the isolated vertexes of $TG_4 \cup \Delta K_1$. Let $f$ is any exclusive sum labelling for $TG_4 \cup \Delta K_1$. Without loss of generality, suppose the edge $v_tv_\Delta$ be labelled by $f(i_t)$ for $t = 1, 2, \cdots, \Delta - 1$, $vv_\Delta$ be labelled by $f(i_\Delta)$ and $vu$ be labelled by some $f(i_l), l \in \{1, 2, \cdots, \Delta - 1\}$. We claim that the edge $uu_\Delta$ must be labelled by $f(i_\Delta)$. Suppose not, $uu_\Delta$ be labelled by some $f(i_k), k \in \{1,2,\cdots,\Delta-1\}$. Then there must exist an edge $u_t u_\Delta$ be labelled by $f(i_\Delta)$, $t \in \{1,2,\cdots,\Delta-1\}$. So $f(v_k) = f(i_k) - f(v_\Delta) = f(i_k) - (f(i_\Delta) - f(v)) = f(i_k) - f(i_\Delta) + (f(i_l) - f(u)) = f(i_k) - f(i_\Delta) + f(i_l) - (f(i_k) - f(u_\Delta)) = f(i_l) - f(i_\Delta) + (f(i_\Delta) - f(u_t)) = f(i_l) - f(u_t)$, then $v_k u_t \in E(TG_4 \cup \Delta K_1)$, which is impossible. So the labelling of $TG_4$ is unique. □

By the proof of *lenmma* 3.2, the following corollary about $TG_5$ (as shown in *(ii)* of *Figure* 7, where $s \le D = \Delta(TG_5)$) can be obtained.

**Corollary 3.3.** *The edges $u_s u_t$ $(t = 1,2,\cdots,s-1)$ and $v_\Delta v \in E(TG_5)$ can not be labelled by the same element of any exclusive sum labelling of $TG_5$.*

**Theorem 3.4.** *For any integer $\Delta \ge 3$, there exist trees $T$ are not $\Delta$–optimum summable.*

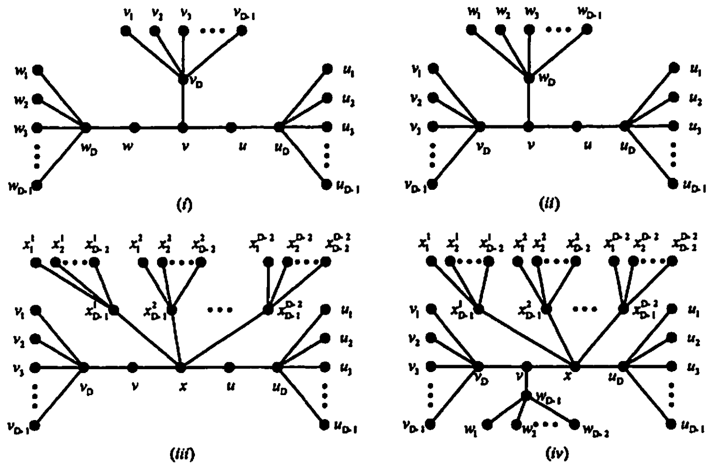

Figure 8: *(i)* $TG_6$, *(ii)* $TG_7$, *(iii)* $TG_8$ and *(iv)* $TG_9$

**Proof.** Obviously,if there exist $\Delta$–optimum forbidden subgraphs of trees for any $\Delta \ge 3$, then this theorem can be proved to be right. $TG_t$ (as shown in Figure 8, where t=6,7,8,9 ) will be proved to be $\Delta$–optimum forbidden subgraph of the trees $T$ with $\Delta = \Delta(T)$. Suppose to the contrary, $TG_6, TG_7, TG_8$ and $TG_9$ all are $\Delta$–optimum summable. Let $f$ is the exclusive sum labelling of $TG_t \cup \Delta K_1$ $(t = 6,7,8,9)$ and the vertex set of $\Delta K_1$ is $\{i_1, i_2, \cdots, i_\Delta\}$. To the convenience of the following arguments, suppose $v_t v_\Delta$ be labelled by one $f(i_t)$ for $t \in \{1,2,\cdots,\Delta-1\}$.

Since $TG_6$ and $TG_7$ both contain $TG_4$, by *lemma* 3.2 we can give an exclusive sum labelling for $TG_t \cup \Delta K_1$, where $t = 6, 7$.

First we consider the exclusive sum labelling of $TG_6 \cup \Delta K_1$. Suppose $TG_7 \cup \Delta K_1 - \{w_1, w_2, \cdots, w_\Delta, w\}$ be labelled just as the same in *lemma* 3.2. We know that the edge $ww_\Delta$ must be labelled by $f(i_\Delta)$, suppose the edge $wv$ be labelled by $f(i_p), p \in \{1, 2, \cdots, \Delta - 1\}$. Then $f(w_\Delta) = f(i_\Delta) - f(w) = f(i_\Delta) - (f(i_p) - f(v)) = f(i_\Delta) - f(i_p) + (f(i_l) - f(u)) = f(i_\Delta) - f(i_p) + f(i_l) - (f(i_\Delta) - f(u_\Delta)) = f(i_l) - f(i_p) + (f(i_p) - f(u_p)) = f(i_l) - f(u_p)$, so $w_\Delta u_p \in E(TG_6 \cup \Delta K_1)$, a contradiction. So $TG_6$ is a $\Delta$–optimum forbidden subgraph of trees. Now we consider the exclusive sum labelling of $TG_7 \cup \Delta K_1$. Suppose $TG_7 \cup \Delta K_1 - \{w_1, w_2, \cdots, w_\Delta\}$ be labelled just as the same in *lemma* 3.2. Then the edge $vw_\Delta$ must be labelled by some $f(i_t), t \in \{1, 2, \cdots, \Delta - 1\}$, and suppose the edge $w_k$ be labelled by $f(i_\Delta)$, then by similar argument in *lemma* 3.2, we have a contradiction to that $w_k$ is adjacent with the vertex $u_k$. Then $TG_6$ and $TG_7$ are both not $\Delta$–optimum summable.

Now we prove that $TG_8$ is a $\Delta$–optimum forbidden subgraph. Since every edge $x_s^t x_{\Delta-1}^t (s, t = 1, 2, \cdots, \Delta - 2)$ and $uu_\Delta$ can not be labelled by $f(i_\Delta)$ by the *corollary* 3.3, suppose the edge $uu_\Delta$ be labelled by $f(i_k)$, where $k < \Delta$, and there must exist some $s, t \in \{1, 2, \cdots, \Delta - 2\}$ such that $x_s^t x_{\Delta-1}^t$ is labelled by $f(i_k)$. Suppose the edges $x_{\Delta-1}^t x$ and $u_{\Delta-1} u_r$ both be labelled by $f(i_l)$ and $xu$ be labelled by $f(i_p)$, then $f(x_s^t) = f(i_k) - f(x_{\Delta-1}^t) = f(i_k) - (f(i_l) - f(x)) = f(i_k) - f(i_l) + (f(i_p) - f(u)) = f(i_k) - f(i_l) + f(i_p) - (f(i_k) - f(u_{\Delta-1})) = f(i_p) - f(i_l) + (f(i_l) - f(u_r)) = f(i_p) - f(u_r)$, i.e., $f(x_s^t) + f(u_r) = f(i_p)$, a contradiction to that $x_s^t u_r \notin E(TG_8 \cup \Delta K_1)$.

Finally we prove that $TG_9$ is a $\Delta$–optimum forbidden subgraph, too. Suppose the edge $vx$ is labelled by $f(i_r)$. By *lemma* 3.2 $xu_\Delta$ must be labelled by $f(i_\Delta)$ and by *corollary* 3.3 no edges of $x_s^t x_{\Delta-1}^t (s, t = 1, 2, \cdots, \Delta - 2)$ be labelled by $f(i_\Delta)$, so there must exist an edge $x_s^t x_{\Delta-1}^t$ be labelled by the same $f(i_k)$ with the edge $vw_{\Delta-1}$ and must exist edges $x_{\Delta-1}^t x$ and $w_p w_{\Delta-1}$ be labelled by the same $f(i_l)$, then $f(w_p) = f(i_l) - f(w_{\Delta-1}) = f(i_l) - (f(i_k) - f(v)) = f(i_l) - f(i_k) + (f(i_r) - f(x)) = f(i_l) - f(i_k) + f(i_r) - (f(i_l) - f(x_{\Delta-1}^t)) = f(i_r) - f(i_k) + (f(i_k) - f(x_s^t)) = f(i_r) - f(x_s^t)$, a contradiction to that $x_s^t w_p \notin E(TG_9 \cup \Delta K_1)$.

Since shrubs are $\Delta$–optimum summable [4], by *lemma* 2.1 and the argument above we know that $\epsilon(TG_t) = \Delta + 1$ for $t = 6, 7, 8, 9$. □

In this final section, we shall consider the exclusive sum number of a special kind of tree. Let $T_r^\Delta$ denote a kind of tree if it meets the following conditions:

*a.* All the vertexes of the tree has the same degree $\Delta$ except the leaves;

*b.* Its center exactly just be one vertex $u$ of this tree;

*c.* $d(u, v) = r$, where $v$ is any leaf vertex of the tree and $r$ is the radius of this tree.

As an example *Figure* 9 shows the tree $T_4^3$. The neighbor vertexes of the center $u$ of $T_r^{\Delta}$ are denoted by $a_i^1$, $i = 1, 2, \cdots, \Delta$, the neighbor vertexes of $a_i^1$ (except the center $u$) are denoted by $a_{ij}^2$, $j = 1, 2, \cdots, \Delta - 1$, and the neighbor vertexes of $a_{ij}^2$ (except the vertex $a_i^1$) are denoted by $a_{ijk}^3$, $k = 1, 2, \cdots, \Delta - 1$, and so on.

Obviously, any tree with maximum degree $\Delta$ and the radius no more than $r$ is a subgraph of $T_r^{\Delta}$. In the next theory we will give a bound as shown in the following *Theorem* 3.5. for the exclusive sum number of the graph $T_r^{\Delta}$ by *theorem* 2.1, so we have a general upper bound for the exclusive sum number for trees.

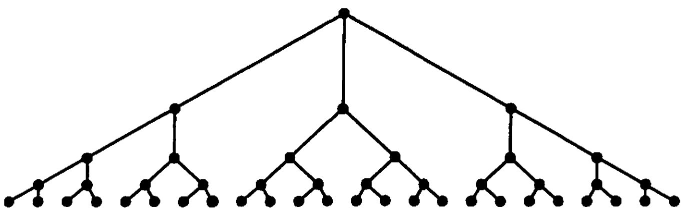

Figure 9: the tree $T_4^3$

**Theorem 3.5.** *For any tree $T$, $\Delta \le \epsilon(T) \le (r-1)\Delta - (r-2)$, where $r$ is the radius of $T$.*

**Proof.** Only need to prove the theory holds for $T_r^{\Delta}$. In fact, we can get $T_r^{\Delta}$ from $T_2^{\Delta}$ by continuous using *theorem*2.1 for $(r-2)(\Delta-1)$ times and every time only add new edges at the leaves of the tree. Since the tree $T_2^{\Delta}$ is a shrub, it is $\Delta$–optimum summable [4]. □

The following theorem shows that the exclusive sum number of a tree $T$ can be very bigger than $\Delta(T) + 1$.

**Theorem 3.6.** *For any $r, \Delta \ge 3$, $\epsilon(T_r^{\Delta}) \ge \lceil \frac{1}{2}(\Delta-1) + \frac{1}{2}\sqrt{(\Delta-1)(5\Delta-1)} \rceil$.*

**Proof.** Assume the exclusive sum number of $T_r^{\Delta}$ is $\epsilon$. Let $f$ is an exclusive sum labelling of $T_r^{\Delta} \bigcup \epsilon K_1$ and $\{i_1, i_2, \cdots, i_\epsilon\}$ is the vertex set of $\epsilon K_1$. Since $\epsilon(T_r^{\Delta}) = \epsilon$, at least $\lceil \frac{\Delta(\Delta-1)}{\epsilon} \rceil$ edges of $\{a_i^1 a_{ij}^2 : i = 1, 2, \cdots, \Delta, j = 1, 2, \cdots, \Delta - 1\}$ be labelled by some one of $f(i_k) \in \{f(i_1), f(i_2), \cdots, f(i_\epsilon)\}$. Suppose the edge $a_i^1 a_{ij}^2$ be labelled by $f(i_k)$, by *lemma* 3.2 the edges $\{a_{ij}^2 a_{ijk}^3 : k = 1, 2, \cdots, \Delta-1\}$ only can be labelled by $(\epsilon-1) - (\lceil \frac{\Delta(\Delta-1)}{\epsilon} \rceil - 1)$ labels of $\{f(i_1), f(i_2), \cdots, f(i_\epsilon)\}$, then $(\epsilon - 1) - (\lceil \frac{\Delta(\Delta-1)}{\epsilon} \rceil - 1) \ge \Delta - 1$, i.e., $\epsilon \ge \lceil \frac{1}{2}(\Delta - 1) + \frac{1}{2}\sqrt{(\Delta-1)(5\Delta-1)} \rceil$. □

## References


[1] F. Harary, Sum graphs and difference graphs, Congressus Numerantium 72 (1990) 101–108.

[2] M. Miller, D. Patel, J. Ryan, K. A. Sugeng, Slamin, M. Tuga, Exclusive sum labeling of graphs. J. Combin. Math. Combin. Comput. 55 (2005) 137–148.

[3] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969.

[4] M. Tuga, M. Miller, J. Ryan, Z. Ryjáček, Exclusive sum labeling of trees. J. Combin. Math. Combin. Comput. 55 (2005) 109–121.

[5] Singh, G. Suresh, G. Santhosh, A note on integral sum crowns. Ars Combin. 66 (2003) 65–77.

[6] M. Tuga, M. Miller, Δ–optimum exclusive sum labeling of certain graphs with radius one. in IJCCGGT 2003, Lecture Notes in Computer Science 3330(J. Akiyama et al.(Eds.)) (2005) 216–225.

[7] M. N. Ellingham, Sum graphs from trees, Ars Combin. 35 (1993) 335–349.

[8] Hartsfield, W. F. Smyth, A family of sparse graphs of large sum number, Discrete Math. 141 (1995) 163–171.

[9] W. Dou, J. Gao, Some results on mod (integral) sum graphs, Ars Combin. 82 (2007) 3–31.

[10] H. Wang, J. Gao, The sum numbers and the integral sum numbers of $\overline{C_n}$ and $\overline{W_n}$, Ars Combin. 96 (2010) 479–488.

[11] M. Miller, J. Ryan, F. Smyth, The sum number of the cocktail party graph, Bull. Inst. Combin. Appl. 22 (1998) 79–90.

[12] H. Nagamochi, M. Miller, Slamin, Bounds on the number of isolates in sum graph labeling, Discrete Math. 240 (2001) 175–185.

[13] D. Bergstrand, F. Harary, K. Hodges, G. Jennings, L. Kuklinski, J. Wiener, The sum numbering of a complete graph, Bull. Malaysian Math. Soc. 12 (1989) 25-28.

[14] M. Sutton, M. Miller, On the sum number of wheels, Discrete Math. 232 (2001) 185-188.

[15] J. Ryan, Exclusive sum labeling of graphs: a survey, AKCE Int. J. Graphs Comb. 6 (2009)no. 1, 113–126.