Quasi-tree graphs with the largest and the second largest numbers of maximum independent sets

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Abstract

In a graph $G = (V, E)$, an *independent set* is a subset I of $V(G)$ such that no two vertices in I are adjacent. A *maximum independent set* is an independent set of maximum size. A connected graph (respectively, graph) G with vertex set $V(G)$ is called a *quasi-tree graph* (respectively, *quasi-forest graph*), if there exists a vertex $x \in V(G)$ such that $G - x$ is a tree (respectively, forest). In this paper, we study the problem of determining the large and the second largest numbers of maximum independent sets among all quasi-tree graphs and quasi-forest graphs. Extremal graphs achieving these values are also given.

1 Introduction

In a graph $G = (V, E)$, an *independent set* is a subset I of $V(G)$ such that no two vertices in I are adjacent. A *maximum independent set* is an independent set of maximum size. The set of all maximum independent sets of G is denoted by $\mathrm{XI}(G)$ and its cardinality by $xi(G)$.

The problem of determining the largest number of maximum independent sets of a graph was studied for various classes of graphs, including general graphs, trees, forests, (connected) graphs with at most one cycle, connected graphs and triangle-free graphs, see [2]. A connected graph (respectively, graph) G with vertex set $V(G)$ is called a *quasi-tree graph* (respectively, *quasi-forest graph*), if there exists a vertex $x \in V(G)$ such that $G - x$ is a tree (respectively, forest). The concept of quasi-tree graphs was mentioned by H. Liu and M. Lu in [3].

The purpose of this paper is to determine the large and the second largest numbers of maximum independent sets among all quasi-tree graphs and quasi-forest graphs. Extremal graphs achieving these values are also given.

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2 Preliminary

In this section, we describe some notations and preliminary results. For a graph $G = (V, E)$ and a vertex $x \in V(G)$, let $\mathrm{XI}_{-x}(G) = \{I \in \mathrm{XI}(G) : x \notin I\}$ and $\mathrm{XI}_{+x}(G) = \{I \in \mathrm{XI}(G) : x \in I\}$. Note that $xi(G) = |\mathrm{XI}_{-x}(G)| + |\mathrm{XI}_{+x}(G)|$. The cardinality of $V(G)$ is called the *order*, and it is denoted by $|G|$. The *neighborhood* $N_G(x)$ of a vertex $x \in V(G)$ is the set of vertices adjacent to x in G and the *closed neighborhood* $N_G[x]$ is $\{x\} \cup N_G(x)$. Two distinct vertices u and v are called *duplicated vertices* if $N_G(u) = N_G(v)$. The *degree* of x is the cardinality of $N_G(x)$, denoted by $\deg_G(x)$. A vertex x is a *leaf* if $\deg_G(x) = 1$. A vertex is called a *support vertex* if it is adjacent to a leaf. For a set $A \subseteq V(G)$, the *deletion* of A from G is the graph $G - A$ obtained from G by removing all vertices in A and their incident edges. Two graphs G_1 and G_2 are *disjoint* if $V(G_1) \cap V(G_2) = \emptyset$. The *union* of two disjoint graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. nG is the short notation for the union of n copies of disjoint graphs isomorphic to G. A component of odd (respectively, even) order is called an *odd* (respectively, *even*) *component*. Denote by P_n a *path* with n vertices and C_n a *cycle* with n vertices. Throughout this paper, for simplicity, let $r = \sqrt{2}$. We begin with the following useful lemmas.

Lemma 2.1. ([1]) *For any vertex x in a graph G, $xi(G) \le xi(G - x) + xi(G - N_G[x])$.*

Lemma 2.2. ([1]) *If x is a leaf adjacent to y in a graph G, then $xi(G) \le xi(G - N_G[x]) + xi(G - N_G[y])$.*

Lemma 2.3. ([1], [2]) *If $x_1, x_2, \ldots, x_k$ are $k \ge 2$ leaves adjacent to the same vertex y in a graph G, then $xi(G) = xi(G - \{x_1, x_2, \ldots, x_k, y\})$.*

Lemma 2.4. ([1], [2]) *If G is the union of two disjoint graphs G_1 and G_2, then $xi(G) = xi(G_1)xi(G_2)$.*

Lemma 2.5. ([1]) *For an odd integer $n \ge 3$, $xi(C_n) = n$.*

The results of the largest numbers of maximum independent sets among all trees and forests are described in Theorems 2.6 and 2.7, respectively.

Theorem 2.6. ([1], [2]) *If T is a tree with $n \ge 2$ vertices, then*

$$xi(T) \le t(n) = \begin{cases} r^{n-2} + 1, & \text{if } n \text{ is even,} \\ r^{n-3}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $xi(T) = t(n)$ if and only if $T = T(n)$, where $T(n)$ is shown in Figure 1.

Figure 1: The graph $T(n)$

Theorem 2.7. ([1], [2]) *If F is a forest with $n \geq 1$ vertices, then*

$$xi(F) \leq f(n) = \begin{cases} r^n, & \text{if } n \text{ is even,} \\ r^{n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $xi(F) = f(n)$ if and only if $F = F(n)$, where

$$F(n) = \begin{cases} \frac{n}{2}P_2, & \text{if } n \text{ is even,} \\ P_1 \cup \frac{n-1}{2}P_2, & \text{if } n \text{ is odd.} \end{cases}$$

3 The largest number of maximum independent sets

Theorem 3.1. *If Q is a quasi-tree graph with $n \geq 2$ vertices, then*

$$xi(Q) \leq qt(n) = \begin{cases} r^{n-2}+1, & \text{if } n \text{ is even,} \\ r^{n-1}+1, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $xi(Q) = qt(n)$ if and only if $Q = QT(n)$, where

$$QT(n) = \begin{cases} T_e(n), & \text{if } n \text{ is even,} \\ QT_o(n), \text{ or } C_5, & \text{if } n \text{ is odd,} \end{cases}$$

where $QT_o(n)$ is the graph obtained from a cycle C_3 by attaching $\frac{n-3}{2}$ paths of length two to a vertex of the cycle C_3, see Figure 2. The vertex z in $QT_o(n)$ is called the central vertex of $QT_o(n)$.

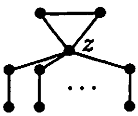


Figure 2: The graph $QT_o(n)$

Proof. It is straightforward to check that $xi(QT(n)) = qt(n)$. Let Q be a quasi-tree graph of order n such that $xi(Q)$ is as large as possible, then $xi(Q) \geq xi(QT(n)) = qt(n)$. Let x be the vertex of Q such that $Q - x$ is a tree. We consider two following cases.

Case 1. n is an even integer. Suppose that Q contains at least one cycle. Then x is on some cycle of Q, it follows that $\deg_Q x \geq 2$. By Theorem 2.6, $xi(Q-x) \leq t(n-1)$. On the other hand, $Q - N_Q[x]$ is a forest with at most $n-3$ vertices, by Theorem 2.7, $xi(Q-N_Q[x]) \leq f(n-3) = f(n-4)$. Thus, by Lemma 2.1, we have $r^{n-2}+1 \leq xi(Q) \leq xi(Q-x) + xi(Q-N_Q[x]) \leq t(n-1)+f(n-3) = t(n-1)+f(n-4) = r^{n-4}+r^{n-4} = r^{n-2}$, which is a contradiction. Hence, by Theorem 2.6, we obtain that $Q = T_e(n)$.

Case 2. n is an odd integer. Since $t(n) < qt(n)$ for $n \geq 3$, Q contains at least one cycle. Then x is on some cycle of Q, it follows that $\deg_Q x \geq 2$. Similar to the arguments in Case 1, we have $r^{n-1}+1 \leq xi(Q) \leq xi(Q - x) + xi(Q-N_Q[x]) \leq t(n-1)+f(n-3) = r^{n-3}+1+r^{n-3} = r^{n-1}+1$. Furthermore, the equalities holding imply that $|XI_{-x}(Q)| = xi(Q-x) = t(n-1)$ and $|XI_{+x}(Q)| = xi(Q-N_Q[x]) = f(n-3)$. By Theorems 2.6 and 2.7, $Q - x = T_e(n-1)$ and $Q - N_Q[x] = \frac{n-3}{2}P_2$. Hence we obtain that $Q \cong QT_o(n)$, or C_5. □

Theorem 3.2. *If Q is a quasi-forest graph with $n \geq 2$ vertices, then*

$$xi(Q) \leq qf(n) = \begin{cases} r^n, & \text{if } n \text{ is even,} \\ 3r^{n-3}, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, $xi(Q) = qf(n)$ if and only if $Q = QF(n)$, where

$$QF(n) = \begin{cases} \frac{n}{2}P_2, & \text{if } n \text{ is even,} \\ C_3 \cup \frac{n-3}{2}P_2, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. It is straightforward to check that $xi(QF(n)) = qf(n)$. Let Q be a quasi-forest graph of order n such that $xi(Q)$ is as large as possible, then $xi(Q) \geq xi(QF(n)) = qf(n)$. Let x be the vertex of Q such that $Q - x$ is a forest. For the case when n is even, suppose that Q contains at least one cycle, then x is on some cycle of Q. It follows that $\deg_Q x \geq 2$. Thus, by Lemma 2.1 and Theorem 2.7, we have that $r^n \leq xi(Q) \leq xi(Q-x)+xi(Q-N_Q[x]) \leq f(n-1)+f(n-3) = f(n-1)+f(n-4) = r^{n-2}+r^{n-4} = 3r^{n-4}$, which is a contradiction. Hence, by Theorem 2.7 again, we obtain that $Q = \frac{n}{2}P_2$.

For the case when n is odd, since $f(n) < qf(n)$ for $n \geq 3$, Q contains at least one cycle. Then x is on some cycle of Q, it follows that $\deg_Q x \geq 2$. Similar to the arguments in the above case, we have that $3r^{n-3} \leq xi(Q) \leq xi(Q-x) + xi(Q-N_Q[x]) \leq f(n-1)+f(n-3) = r^{n-1}+r^{n-3} = 3r^{n-3}$. Furthermore, the equalities holding imply that $|XI_{-x}(Q)| = xi(Q-x) =$

$f(n-1)$ and $|\mathrm{XI}_{+x}(Q)| = xi(Q - N_Q[x]) = f(n-3)$. By Theorem 2.7, $Q - x = \frac{n-1}{2}P_2$ and $Q - N_Q[x] = \frac{n-3}{2}P_2$. Hence we obtain that $Q = C_3 \cup \frac{n-3}{2}P_2$. □

4 The second largest number of maximum independent sets

For even $n \geq 6$, $QT'_e(n)$ is the graph obtained from $QT_o(n-1)$ by adding a vertex and a new edge joining the vertex and the central vertex of $QT_o(n-1)$; T_8 is the graph obtained from two copies of P_4 by adding a new edge joining the support vertices of these two P_4's. For odd $n \geq 7$, $QT'_{o1}(n)$ and $QT'_{o2}(n)$ are the graphs obtained from $QT_o(5)$ by attaching $\frac{n-5}{2}$ paths of length two to a vertex of degree two of $QT_o(5)$; $QT'_{o3}(n)$ is the graph obtained from C_5 by attaching $\frac{n-5}{2}$ paths of length two to a vertex of C_5; $QT'_{o4}(n)$ is the graph obtained from $QT_o(n-4)$ by adding a P_4 and a new edge joining a support vertex of P_4 and the central vertex of $QT_o(n-4)$, see Figure 3.

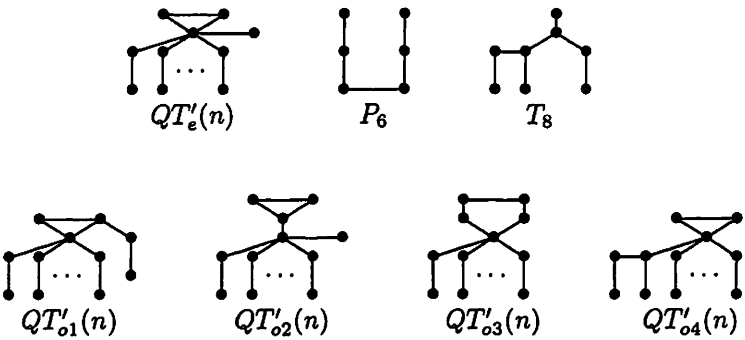


Figure 3: The graph $QT'(n)$

Define the graphs $QT'(n)$ and $QF'(n)$ as follows.

$$QT'(n) = \begin{cases} QT'_e(n), \text{ or } P_6, \text{ or } T_8 & \text{if } n \geq 6 \text{ is even,} \\ QT'_{o1}(n), \text{ or } QT'_{o2}(n), & \\ \qquad \text{or } QT'_{o3}(n), \text{ or } QT'_{o4}(n), & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

and

$$QF'(n) = \begin{cases} P_4 \cup \frac{n-4}{2}P_2, \text{ or } C_3 \cup P_1 \cup \frac{n-4}{2}P_2 & \text{if } n \geq 4 \text{ is even,} \\ QT_o(5) \cup \frac{n-5}{2}P_2, \text{ or } C_5 \cup \frac{n-5}{2}P_2, & \text{if } n \geq 5 \text{ is odd.} \end{cases}$$

Let $qt'(n) = xi(QT'(n))$ and $qf'(n) = xi(QF'(n))$. By simple calculation, we have

$$qt'(n) = \begin{cases} r^{n-2}, & \text{if } n \geq 6 \text{ is even,} \\ 6r^{n-7} + 2, & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

and

$$qf'(n) = \begin{cases} 3r^{n-4}, & \text{if } n \geq 4 \text{ is even,} \\ 5r^{n-5}, & \text{if } n \geq 5 \text{ is odd.} \end{cases}$$

We will prove the following two results.

Theorem 4.1. *If Q is a quasi-tree graph with $n \geq 6$ vertices having $Q \neq QT(n)$, then $xi(Q) \leq qt'(n)$ with the equality holding if and only if $Q = QT'(n)$.*

Theorem 4.2. *If Q is a quasi-forest with $n \geq 4$ vertices having $Q \neq QF(n)$, then $xi(Q) \leq qf'(n)$ with the equality holding if and only if $Q = QF'(n)$.*

We prove Theorems 4.1 and 4.2 by verifying the following four lemmas.

Lemma 4.3. *If Q is a quasi-forest graph of even order $n \geq 4$ having $Q \neq QF(n)$, then $xi(Q) \leq 3r^{n-4}$ with the equality holding if and only if $Q = P_4 \cup \frac{n-4}{2}P_2$, or $C_3 \cup P_1 \cup \frac{n-4}{2}P_2$.*

Proof. It is straightforward to check that $xi(P_4 \cup \frac{n-4}{2}P_2) = xi(C_3 \cup P_1 \cup \frac{n-4}{2}P_2) = 3r^{n-4}$. Let Q be a quasi-forest graph of even order $n \geq 4$ having $Q \neq QF(n)$ such that $xi(Q)$ is as large as possible. Then $xi(Q) \geq 3r^{n-4}$. We consider the following two cases.

Case 1. Q contains no cycle. Suppose that there exist two odd components H_1 and H_2 of Q, where $|H_i| = n_i$ for $i = 1, 2$. By Lemma 2.4, Theorems 2.6 and 2.7, we have that $3r^{n-4} \leq xi(Q) = xi(H_1) \cdot xi(H_2) \cdot xi(Q - (V(H_1) \cup V(H_2)) \leq r^{n_1-3} \cdot r^{n_2-3} \cdot r^{n-n_1-n_2} = r^{n-6} < 3r^{n-4}$. This is a contradiction. Hence Q has no odd component. Since $Q \neq QF(n)$, there exists an even component H of order $m \geq 4$. By Theorem 2.6, $xi(H) \leq t(m) = r^{m-2} + 1$. On the other hand, by Lemma 2.4 and Theorem 2.7, $3r^{n-4} \leq xi(Q) = xi(H) \cdot xi(Q - V(H)) \leq (r^{m-2} + 1) \cdot r^{n-m} = r^{n-2} + r^{n-m} \leq 3r^{n-4}$, where $m \geq 4$. Thus the equality holds, and we can see that $H = P_4$ and $Q - V(H) = \frac{n-4}{2}P_2$. In conclusion, $Q = P_4 \cup \frac{n-4}{2}P_2$.

Case 2. Q contains at least one cycle. Let x be the vertex of Q such that $Q - x$ is a forest of odd order $n - 1$. Then x is on some cycle of Q, it follows that $\deg_Q x \geq 2$. By Theorem 2.7, $xi(Q - x) \leq f(n - 1)$.

On the other hand, $Q - N_Q[x]$ is a forest with at most $n-3$ vertices, by Theorem 2.7, $xi(Q - N_Q[x]) \le f(n-3) = f(n-4)$. Thus, by Lemma 2.1, we have that $3r^{n-4} \le xi(Q) \le xi(Q-x) + xi(Q - N_Q[x]) \le f(n-1) + f(n-3) = f(n-1) + f(n-4) = r^{n-2} + r^{n-4} = 3r^{n-4}$. Furthermore, the equalities holding imply that $|XI_{-x}(Q)| = xi(Q-x) = f(n-1)$ and $|XI_{+x}(Q)| = xi(Q - N_Q[x]) = f(n-3) = f(n-4)$. By Theorem 2.7, $Q - x = P_1 \cup \frac{n-2}{2}P_2$ and $Q - N_Q[x] = P_1 \cup \frac{n-4}{2}P_2$ or $\frac{n-4}{2}P_2$. Hence we obtain that $Q = C_3 \cup P_1 \cup \frac{n-4}{2}P_2$. □

Since every forest is a quasi-forest graph, by Lemma 4.3, we have the following immediately.

Corollary 4.4. *The graph $P_4 \cup \frac{n-4}{2}P_2$ is a forest of order n with the second largest number of maximum independent sets.*

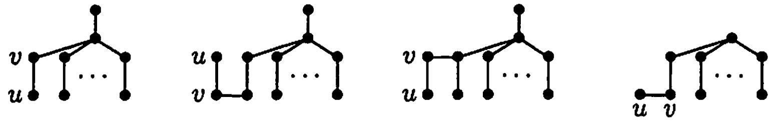
Lemma 4.5. *If Q is a quasi-forest of odd order $n \ge 5$ vertices having $Q \ne QF(n)$, then $xi(Q) \le 5r^{n-5}$ with the equality holding if and only if $Q = QT_o(5) \cup \frac{n-5}{2}P_2$, or $C_5 \cup \frac{n-5}{2}P_2$.*

Proof. It is straightforward to check that $xi(QT_o(5) \cup \frac{n-5}{2}P_2) = xi(C_5 \cup \frac{n-5}{2}P_2) = 5r^{n-5}$. Let Q be a quasi-forest graph of odd order $n \ge 5$ having $Q \ne QF(n)$ such that $xi(Q)$ is as large as possible. Then $xi(Q) \ge 5r^{n-5}$. Since $f(n) = r^{n-1} < 5r^{n-5} \le xi(Q)$ for $n \ge 5$, Q contains at least one cycle. Let H be the component of Q which is not a tree, then $|H| = m \ge 3$. Suppose that m is even, by Theorems 2.7 and 3.1, we have that $5r^{n-5} \le xi(Q) = xi(H) \cdot xi(Q - V(H)) < (r^{m-2}+1) \cdot r^{n-m-1} = r^{n-3} + r^{n-m-1} \le 3r^{n-5}$. This is a contradiction, thus we obtain that m is odd. For the case of $m = 3$, that is, $H = C_3$. It follows from $Q \ne QF(n)$ that $Q - V(H) \ne \frac{n-3}{2}P_2$. By Theorem 4.3 and Corollary 4.4, we have that $5r^{n-5} \le xi(Q) = xi(H) \cdot xi(Q - V(H)) \le 3 \cdot (3r^{n-7}) = 9r^{n-7}$, which is a contradiction. For the case of $m \ge 5$, by Theorems 2.7 and 3.1, we have $5r^{n-5} \le xi(Q) = xi(H) \cdot xi(Q - V(H)) \le (r^{m-1}+1) \cdot r^{n-m} = r^{n-1} + r^{n-m} \le 5r^{n-5}$. Furthermore, the equalities holding imply that $m = 5$. By Theorems 2.7 and 3.1 again, $H = QT_o(5)$ or C_5 and $Q - V(H) = \frac{n-5}{2}P_2$. In conclusion, $Q = QT_o(5) \cup \frac{n-5}{2}P_2$, or $C_5 \cup \frac{n-5}{2}P_2$. □

Lemma 4.6. *If Q is a quasi-tree of even order $n \ge 6$ having $Q \ne QT(n)$, then $xi(Q) \le r^{n-2}$ with the equality holding if and only if $Q = QT'_e(n)$, or P_6, or T_8.*

Proof. It is straightforward to check that $xi(QT'_e(n)) = r^{n-2}$, $xi(P_6) = 4 = r^{6-2}$ and $xi(T_8) = 8 = r^{8-2}$. Let Q be a quasi-tree graph of even order $n \ge 6$ having $Q \ne QT(n)$ such that $xi(Q)$ is as large as possible. By Theorem 3.1, $r^{n-2} \le xi(Q) \le qt(n) - 1 = (r^{n-2}+1) - 1 = r^{n-2}$, hence $xi(Q) = r^{n-2}$. Suppose that Q has duplicated leaves u_1 and u_2

which are adjacent to the same vertex v, by Lemma 2.3 and Theorem 2.7, $r^{n-2} = xi(Q) = xi(Q - \{u_1, u_2, v\}) \leq qf(n-3) = 3r^{n-6}$. This is a contradiction, thus Q has no duplicated leaf. We claim that Q contains at least one cycle except P_6 and T_8. Suppose that Q is a tree and u is a leaf on a longest path of Q, say $P = u, v, \ldots$. The possible graphs Q with the property of $Q - u = T(n-1)$ or $Q - N[u] = T(n-2)$ are shown in Figure 4. The number inside the brackets in Figure 4 indicates the number of maximum independent sets of the corresponding graph. Note that $T^{(1)}(n) = T_e(n)$, $T^{(2)}(6) = T^{(4)}(6) = P_6$ and $T^{(3)}(8) = T_8$. By simple calculation, we have $xi(T^{(i)}(n)) < r^{n-2}$ for $i = 2, 3, 4$ when $n \geq 10$.



$T^{(1)}(n)$ $[r^{n-2}+1]$ $T^{(2)}(n)$ $[3r^{n-6}+1]$ $T^{(3)}(n)$ $[3r^{n-6}+2]$ $T^{(4)}(n)$ $[r^{n-4}+2]$

Figure 4: The possible graphs Q

Thus, by Lemma 2.1, we have that $r^{n-2} = xi(Q) \leq xi(Q-u) + xi(Q - N_Q[u]) \leq (r^{n-4}-1) + (r^{n-4}+1-1) = r^{n-2}-1$, which is a contradiction. It follows that Q contains at least one cycle. Let x be a vertex such that $Q - x$ is a tree of odd order $n-1$. Then x is on some cycle of Q, it follows that $\deg_Q x \geq 2$. By Theorem 2.6, $xi(Q-x) \leq t(n-1)$. On the other hand, $Q - N_Q[x]$ is a forest with at most $n-3$ vertices, by Theorem 2.7, $xi(Q - N_Q[x]) \leq f(n-3) = f(n-4)$. Thus, by Lemma 2.1, we have $r^{n-2} = xi(Q) \leq xi(Q-x) + xi(Q - N_Q[x]) \leq t(n-1) + f(n-3) = t(n-1) + f(n-4) = r^{n-4} + r^{n-4} = r^{n-2}$. Furthermore, the equalities holding imply that $|\mathrm{XI}_{-x}(Q)| = xi(Q-x) = t(n-1)$ and $|\mathrm{XI}_{+x}(Q)| = xi(Q - N_Q[x]) = f(n-3) = f(n-4)$. By Theorems 2.6 and 2.7, $Q - x \cong T_o(n-1)$ and $Q - N_Q[x] \cong P_1 \cup \frac{n-4}{2} P_2$ or $\frac{n-4}{2} P_2$. Hence we obtain that $Q \cong QT'_e(n)$. □

Lemma 4.7. *If Q is a quasi-tree graph of odd order $n \geq 7$ having $Q \neq QT(n)$, then $xi(Q) \leq 6r^{n-7}+2$ with the equality holding if and only if $Q = QT'_{o1}(n)$, or $QT'_{o2}(n)$, or $QT'_{o3}(n)$, or $QT'_{o4}(n)$.*

Proof. It is straightforward to check that $xi(QT'_{o1}(n)) = xi(QT'_{o2}(n)) = xi(QT'_{o3}(n)) = xi(QT'_{o4}(n)) = 6r^{n-7}+2$. Let Q be a quasi-tree graph of odd order $n \geq 7$ having $Q \neq QT(n)$ such that $xi(Q)$ is as large as possible, then $xi(Q) \geq 6r^{n-7}+2$. Suppose that Q has duplicated leaves u_1 and u_2 which are adjacent to the same vertex v, by Lemma 2.3 and Theorem 3.2, $6r^{n-7}+2 \leq xi(Q) = xi(Q - \{u_1, u_2, v\}) \leq qf(n-3) = r^{n-3} < 6r^{n-7}+2$. This is a contradiction, thus Q has no duplicated leaf. We claim that there exists exactly one cycle in Q. Since $t(n) = r^{n-3} < 6r^{n-7} + 2 \leq xi(Q)$ for $n \geq 7$, Q contains at least one cycle. Let x be a vertex such that $Q - x$ is a tree of even order $n - 1$. Suppose that Q contains at least two cycles, then $\deg_Q(x) \geq 3$. By Lemma 2.1 and Theorems 2.6 and 2.7, we have $6r^{n-7} + 2 \leq xi(Q) \leq xi(Q - x) + xi(Q - N_Q[x]) \leq r^{n-3} + 1 + r^{n-5} = 6r^{n-7} + 1 < 6r^{n-7} + 2$, which is a contradiction. In addition, suppose that Q is the cycle C_n, by Lemma 2.5, $xi(C_n) = n < 6r^{n-7} + 2$, hence $Q \neq C_n$.

Let u be a leaf lying on a longest path P joining u and the unique cycle C of Q, say $P = u, v, w, \ldots$ and $\ell(u, C)$ the length from u to C. We claim that $\ell(u, C) \geq 2$. Suppose that $\ell(u, C) = 1$ and u is adjacent to $v \in V(C)$, then $Q - N_Q[u]$ is a tree with $n - 2$ vertices and $Q - N_Q[v]$ is a forest with $n - 4$ vertices. By Lemma 2.2, we have $6r^{n-7} + 2 \leq xi(Q) \leq xi(Q - N_Q[u]) + xi(Q - N_Q[v]) \leq r^{n-5} + r^{n-5} = 4r^{n-7} < 6r^{n-7} + 2$, which is a contradiction. Hence we obtain that $\ell(u, C) \geq 2$.

Now, we certify that the result is true for $n = 7$. Note that $Q \neq QT(7)$ and Q contains an unique cycle C. Since Q has no duplicated leaf and $\ell(u, C) \geq 2$, there are 12 possibilities for Q. See Figure 5. The

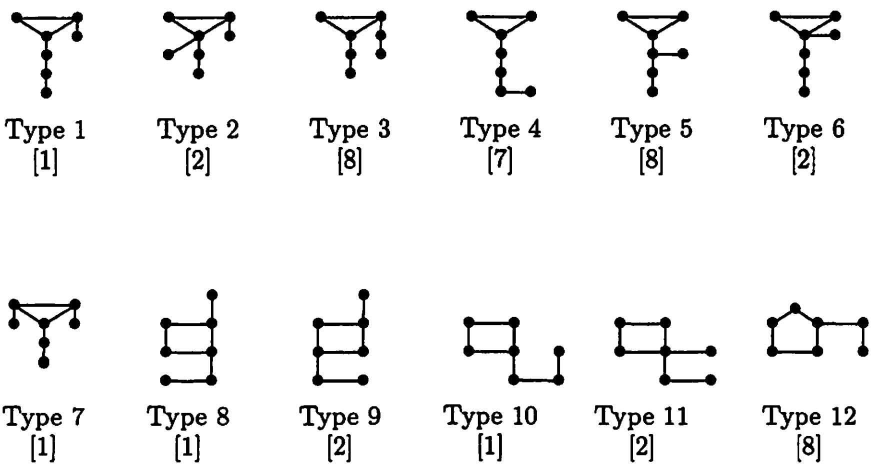


Figure 5: The 12 possibilities for Q

number inside the brackets in Figure 5 indicates the number of maximum independent sets of the corresponding graph of each type. Note that the graph of Type 3 is QT'_{o1}, the graph of Type 5 is QT'_{o2} (or QT'_{o4}) and the graph of Type 12 is QT'_{o3}.

Next, let $n = 2k + 1$, we will prove the result by induction on $k \geq 3$. The result is true for $k = 3$. Assume that it is true for all $k' < k$. We consider the following two cases.

Case 1. $\ell(u, C) = 2$. Let H be the component of $Q - N_Q[v]$ containing some vertices of C. Since P is a longest path joining u and C, it follows that every component of $Q - (N_Q[v] \cup V(H))$ is P_1 or P_2, see Figure 6.

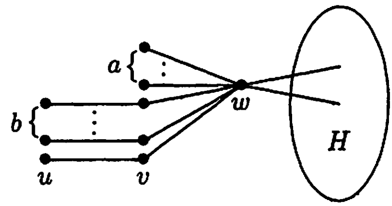


Figure 6: The quasi-tree graph Q in Case 1

So we have that $Q - N_Q[v] = aP_1 \cup bP_2 \cup H$. Since Q has no duplicated leaves, it follows that $a = 0$ or 1. Suppose that $a = 1$, then H is a tree of odd order $n - 4 - 2b$. By Theorem 2.6, $xi(H) \leq r^{n-7-2b}$. By Lemma 2.4, $xi(Q - N_Q[v]) \leq r^{2b} \cdot r^{n-7-2b} = r^{n-7}$. Hence, by Lemma 2.2 and Theorem 3.1, we have $r^{n-3} + 1 = qt(n-2) \geq xi(Q - N_Q[u]) \geq xi(Q) - xi(Q - N_Q[v]) \geq (6r^{n-7} + 2) - r^{n-7} = 5r^{n-7} + 2$, which is a contradiction. Hence we obtain that $a = 0$. It follows that $|H| = n - 3 - 2b \geq 4$ is even and $b \leq \frac{n-7}{2}$. There are two cases depending on the structure of $Q - N_Q[u]$. For the case of $Q - N_Q[u] = QT(n-2)$, it is easy to see that $Q = QT'_{o1}(n)$. For the case of $Q - N_Q[u] \neq QT(n-2)$, by induction hypothesis, $xi(Q - N_Q[u]) \leq 6r^{n-9} + 2$. By Lemmas 2.2, 2.4 and Theorem 2.6, we have that $6r^{n-9} \geq r^{2b} \cdot (r^{n-5-2b} + 1) \geq r^{2b} \cdot xi(H) = xi(Q - N_Q[v]) \geq xi(Q) - xi(Q - N_Q[u]) \geq (6r^{n-7} + 2) - (6r^{n-9} + 2) = 6r^{n-9}$. Hence the equalities holding imply that $b = \frac{n-7}{2}$, $Q - N_Q[u] = QT'(n-2)$ and $H = P_4$. This means that $Q = QT'_{o1}(n)$, or $QT'_{o3}(n)$.

Case 2. $\ell(u, C) \geq 3$. Let H' be the component of $Q - N_Q[v]$ containing some vertices of P. Since P is a longest path joining u and C, it follows that every component of $Q - (N_Q[v] \cup V(H'))$ is P_1 or P_2, see Figure 7.

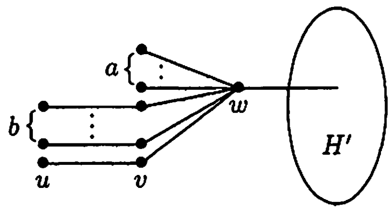


Figure 7: The quasi-tree graph Q in Case 2

So we have that $Q - N_Q[v] = aP_1 \cup bP_2 \cup H'$. Since Q has no duplicated leaves, it follows that $a = 0$ or 1. Suppose that $a = 0$, then H' is a quasi-tree graph containing the unique cycle C of even order $n - 3 - 2b$. Since H' contains a cycle, by Theorem 3.1 and Lemma 4.6, $xi(H') \leq r^{n-5-2b}$. By

Lemma 2.4, $xi(Q - N_Q[v]) \leq r^{2b} \cdot r^{n-5-2b} = r^{n-5}$. Hence, by Lemma 2.2 and Theorem 3.1, we have $r^{n-3}+1 = qt(n-2) \geq xi(Q-N_Q[u]) \geq xi(Q) - xi(Q-N_Q[v]) \geq (6r^{n-7}+2) - r^{n-5} = 4r^{n-7}+2$, which is a contradiction. Hence we obtain that $a = 1$. It follows that $|H'| = n-4-2b \geq 3$ is odd and $b \leq \frac{n-7}{2}$. There are two cases depending on the structure of $Q-N_Q[u]$. For the case of $Q-N_Q[u] = QT(n-2)$, it is easy to see that $Q = QT'_{o4}(n)$. For the case of $Q-N_Q[u] \neq QT(n-2)$, by induction hypothesis, $xi(Q-N_Q[u]) \leq 6r^{n-9}+2$. By Lemmas 2.2, 2.4 and Theorem 3.1, we have that $3r^{n-7} \geq r^{2b} \cdot (r^{n-5-2b}+1) \geq r^{2b} \cdot xi(H') = xi(Q-N_Q[v]) \geq xi(Q) - xi(Q-N_Q[u]) \geq (6r^{n-7}+2) - (6r^{n-9}+2) = 6r^{n-9}$. Hence the equalities holding imply that $b = \frac{n-7}{2}$, $Q - N_Q[u] = QT'(n-2)$ and $H' = C_3$. This means that $Q = QT'_{o2}(n)$. □

Theorems 4.1 and 4.2 now follow from Lemmas 4.3, 4.5, 4.6, and 4.7.

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