On endotype of bipartite graphs[1]

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Abstract: A characterization of E-H-unretractive bipartite graphs is given. Base on this, it is proved that there is no bipartite graph with endotype 1 mod 4.

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1. Introduction and preliminaries

The monoid of endomorphisms of a graph has been the object of researches in the theory of semigroups for quite some time (cf.[1,2]). The graphs for which different endomorphism classes coincide (i.e. various unretractivities) is one of the main themes in this line, and as justification for the investigation one takes the rich algebra structure which is put on a graph by its endomorphism classes and the numerous questions connected with them (cf.[3]). Among open questions raised in [3] are: Do there exist graphs of endotype 9 and 25? What are conditions on a graph G for various unretractivities of G? A general answer to these questions seems to be difficult (c.f. [4]). Undoubtedly bipartite graphs constitute one of the most important families of graphs. In this paper bipartite graphs with E-H- unretractivity are explicitly presented, and furthermore it is proved that there is no bipartite graph with endotype 1 mod 4.

We consider only finite undirected graphs without loops and multiple edges. If G is a graph, we denote by $V(G)$ (or simply G) and $E(G)$ its vertex set and edge set respectively. By K_n we denote a complete graph with n vertices and by C_n a cycle with n vertices. It is well known that a graph is bipartite if and only if it does not contain any C_n where n is an odd number, and therefore trees constitute a special class of bipartite graphs. An empty graph with n vertices is denoted by $\overline{K}_n$. The distance of the vertices a and b in G is denoted by $d_G(a,b)$. The diameter of a connected graph $G(\neq K_1)$ is denoted by $diam(G)$ and define $diam(K_1) = 0$. A subgraph H of G is called *isometric* if for any $x, y \in H$, $d_H(x,y) = d_G(x,y)$. The length of the shortest cycle (if it exists) of a graph G is called the *girth* of G, denoted by $gir(G)$. We use nG to represent a graph composed of n graphs each of

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which is isomorphic to a connected graph G. A complete bipartite graph $G(V_1 \cup V_2, E)$ with $|V_1| = m \geq 1$ and $|V_2| = n \geq 1$ is denoted by $K_{m,n}$. A graph $K_{1,n}$ is also called a *star*. Let G_1 and G_2 be graphs with disjoint vertex sets. The *union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is a graph such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. A component K_1 is also called an *isolated* vertex of G. For a vertex $a \in G$, let $N(a) = \{x \in G | \{x, a\} \in E(G)\}$ (the *neighborhood* of a in G), and $|N(a)|$ is called the *degree* of a in G, denoted by $deg_G(a)$ or simply $deg(a)$ if it is clear which graph G is referred to. The usual concepts such as connected graph, complete bipartite graph, complete graph, empty graph, path, cycle etc., which are not defined in this paper, can be found in [5].

The following definitions of various types of endomorphisms are mainly based on [3]. If G and H are graphs, then a mapping $f : V(G) \to V(H)$ is called a *homomorphism* (or *morphism*) from G to H if $\{a, b\} \in E(G)$ implies that $\{f(a), f(b)\} \in E(H)$ for any $a, b \in G$. Moreover, if f is bijective and its inverse mapping is also a homomorphism (from H to G), then f is called an *isomorphism* from G to H. An *endomorphism* of G is a homomorphism from G to itself. An endomorphism is called a *strong endomorphism* if $\{f(a), f(b)\} \in E(G)$ implies that $\{a, b\} \in E(G)$ for any $a, b \in G$. A bijective endomorphism of a graph G is called an *automorphism* of G. Evidently, an automorphism of a graph G is an isomorphism from G to itself. Let f be an endomorphism of graph G and let $a \in G$. Denote $f^{-1}(a) := \{x \in G | f(x) = a\}$. An endomorphism f is called a *half-strong endomorphism* if $\{f(a), f(b)\} \in E(G)$ implies that there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(G)$. An endomorphism f is called a *locally strong endomorphism* if $\{f(a), f(b)\} \in E(G)$ implies that for any $c \in f^{-1}(f(a))$, there exists $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(G)$. An endomorphism f is called a *quasi-strong endomorphism* if $\{f(a), f(b)\} \in E(G)$ implies that there exists $c \in f^{-1}(f(a))$ such that for any $d \in f^{-1}(f(b))$, $\{c, d\} \in E(G)$, where $a, b, c, d \in G$.

By $End(G)$, $hEnd(G)$, $lEnd(G)$, $qEnd(G)$, $sEnd(G)$ and $Aut(G)$ we denote the set of endomorphisms, half strong endomorphisms, locally strong endomorphisms, quasi-strong endomorphisms, strong endomorphisms and automorphisms of the graph G, respectively. Obviously, $End(G) \supseteq hEnd(G) \supseteq lEnd(G) \supseteq qEnd(G) \supseteq sEnd(G) \supseteq Aut(G)$. It is well-known that $End(G)$ and $sEnd(G)$ are monoids (a *monoid* is a semigroup with an identity element) and that $Aut(G)$ is a group with respect to the composition of mappings, while $hEnd(G)$, $lEnd(G)$ and $qEnd(G)$ are not monoids in general. The coincidence of these endomorphism classes gives rise to various unretractivities of a graph. In particular, a graph G is called *E-H-unretractive* (respectively, *E-S-unretractive* and *E-A-unretractive* etc.) if $End(G) = hEnd(G)$ (respectively, $End(G) = sEnd(G)$ and $End(G) =$

$Aut(G)$ etc.) If graph G is E-A-unretractive, we also call it simply *unretractive*. In [1] E-S-unretractivity, E-A-unretractivity and S-A-unretractivity of a graph are studied. In [6] E-A-unretractivity and S-A-unretractivity of joins and lexicographic products of graphs are characterized. Relationships among endomorphism classes of trees are explored in [7].

Let $f \in End(G)$. A subgraph of G is called the *endomorphic image* of G under f, denoted by I_f, if $V(I_f) = f(G)$, and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(G)$, where $a, b, c, d \in V(G)$ (cf.[8] for the reasonableness of this definition). An element a of a semigroup S is called an *idempotent* if $a^2 = a$ (cf. [9]). The set of idempotents of $End(G)$ is denoted by $Idpt(G)$. Each $f \in Idpt(G)$ is also called a *retraction* of G. If f is a retraction of graph G, the subgraph induced by $f(G)(= \{f(x)|x \in G\})$ (i.e. the induced subgraph with vertex set $f(G)$) is called a *retract* of G (cf.[6,10,11]).

Proposition 1.1.

(1) [1, Example 1.2] *The cycles with odd lengths are unretractive.*

(2) [7, Propositions 2.1 and 3.1] *Any tree is E-H-unretractive.*

Proposition 1.2. [12, Remark 1.3] *Let $f \in End(G)$ for a graph G and let $a, b \in G$.*

(1) *If G is connected , then I_f is connected;*

(2) $d_{I_f}(f(a), f(b)) \leq d_G(a, b)$.

Proposition 1.3.

(1) [11, Theorem 5] *Every isometric tree $T(\neq K_1)$ in a bipartite graph G is a retract of G, i.e. there exists $f \in Idpt(G)$ such that T is a subgraph of G induced by $f(G)$.*

(2) [3, Proposition 2.2] *Idempotent endomorphisms of graph G are elements of $hEnd(G)$.*

(3) [13, Lemma 2.1(1)] *Let G be a graph and let $f \in End(G)$. Then $f \in hEnd(G)$ if and only if I_f is an induced subgraph of G.*

Proposition 1.4. *Let G be a bipartite graph and let P be a path in G. If P is a geodesic, there exists $f \in Idpt(G)$ such that $I_f = P$.*

Proof. Obviously, P is an isometric tree in G, and so by Proposition 1.3(1), there exists $f \in Idpt(G)$ such that P is the subgraph induced by $f(G)$. By Proposition 1.3(2) $f \in hEnd(G)$, and so by Proposition 1.3(3) I_f is an induced subgraph of G. Hence $I_f = P$. □

2. E-H-unretractive bipartite graphs

In this section, we will explicitly characterize bipartite graphs with E-H-unretractivity (Theorem 2.10). First, we consider connected bipartite graphs.

Lemma 2.1. *Let G be a connected bipartite graph with cycles. If $diam(G) \leq gir(G) - 2$, G is E-H-unretractive.*

Proof. Assume G is not E-H-unretractive. Then there exists $f \in End(G) \setminus hEnd(G)$. Thus there exist $a, b \in G$ such that $\{f(a), f(b)\} \in E(G)$. whereas $\{x, y\} \notin E(G)$ for any $x \in f^{-1}(f(a))$ and any $y \in f^{-1}(f(b))$, and so by the definition of the image of an endomorphism, $\{f(a), f(b)\} \notin E(I_f)$. Since G is connected, by Proposition 1.2(1) I_f is connected, and so in I_f there is a geodesic P connecting $f(a)$ and $f(b)$ with length $d_{I_f}(f(a), f(b))$. Therefore $P \cup \{f(a), f(b)\}$ is a cycle in G with length $d_{I_f}(f(a), f(b)) + 1$. Then $gir(G) \leq d_{I_f}(f(a), f(b)) + 1$. Furthermore, $gir(G) \leq d_G(a, b) + 1 \leq diam(G) + 1$ by Proposition 1.2(2), which contradicts $diam(G) \leq gir(G) - 2$. □

Lemma 2.2. *Let G be a graph, and let $f \in Idpt(G)$. Then for any $a \in I_f$, $f(a) = a$.*

Proof. Since $a \in I_f$ and $f^2 = f$, there exists $x \in G$ such that $f(x) = a$ and so $f(a) = f(f(x)) = f(x) = a$. □

The next theorem characterizes connected bipartite graphs with E-H-unretractivity.

Theorem 2.3. *Let G be a connected bipartite graph. Then G is E-H-unretractive if and only if G is a tree or $diam(G) \leq gir(G) - 2$.*

Proof. Sufficiency is by Proposition 1.1(2) and Lemma 2.1.

Necessity. Now suppose G is not a tree and $diam(G) \geq gir(G) - 1$. Let $gir(G) = n (= 4, 6, 8, \ldots)$ and let $diam(G) = d$. So $d \geq n - 1$ and there exists a geodesic P in G with length d, denoted by $P = a_1 a_2 \ldots a_{d+1}$. By Proposition 1.4 there exists $f \in Idpt(G)$ such that $I_f = P$. Let $C_n = b_1 b_2 \ldots b_n$ be a cycle in G with length $n (= gir(G))$. Now we define a mapping g as the composition of two morphisms, one is f and the other maps P to C_n by the rules $a_i \mapsto b_i$ (for $1 \leq i \leq n$), $a_i \mapsto b_{n-1}$ (for i odd and $i \geq n + 1$), and $a_i \mapsto b_n$ otherwise. Clearly, $g \in End(G)$. If $g \in hEnd(G)$, the edge $\{b_1, b_n\}$ is the image of g, say $g(\{x,y\}) = \{b_1, b_n\}$, with $g(x) = b_1$ and $g(y) = b_n$. Then $f(\{x,y\}) = \{a_1, a_j\}$, with $j \in \{n+2, n+4, ...\}$, contradicting the fact that P is a geodesic. □

In particular, we see immediately the following:

Corollary 2.4. *All cycles are E-H-unretractive.*

Proof. Notice all cycles with odd lengths are unretractive (Proposition 1.1(1)) and all cycles $C_{2m} (m \geq 2)$ satisfy the condition in Theorem 2.3, i.e. $diam(C_{2m}) = m \leq m + (m-2) = 2m - 2 = gir(C_{2m}) - 2$. The result follows immediately. □

Now, we consider E-H-unretractivity of non-connected bipartite graphs. First, we list several lemmas as follows:

Lemma 2.5. *Let G be a bipartite graph with $n(\geq 2)$ components.*

(1) *If each component is K_1, i.e. $G = \overline{K}_n$, then G is E-H-unretractive;*

(2) *If each component of G is K_2 or $K_{1,m}(m \geq 2)$ (i.e. a star), then G is E-H-unretractive.*

Proof. (1) By the definition of a half-strong endomorphism, G is trivially E-H-unretractive.

(2) Suppose $f \in End(G)$. Let $a, b \in G$ such that $\{f(a), f(b)\} \in E(G)$. As G has no isolated vertices, there exist $x, y \in G$ with $\{a, x\} \in E(G)$ and $\{b, y\} \in E(G)$. So $\{f(a), f(x)\} \in E(G)$ and $\{f(b), f(y)\} \in E(G)$. If the edge $\{f(a), f(b)\}$ is exactly a component K_2 of G, then $f(x) = f(b)$, i.e. there exist $x \in f^{-1}(f(b))$ and $a \in f^{-1}(f(a))$ such that $\{a, x\} \in E(G)$. If $\{f(a), f(b)\}$ belongs to a component $K_{1,m}$ where $m \geq 2$, then clearly either $deg(f(a)) = 1$ and $deg(f(b)) = m$ or $deg(f(b)) = 1$ and $deg(f(a)) = m$. In the former situation, we have $f(x) = f(b)$, i.e. there exist $x \in f^{-1}(f(b))$ and $a \in f^{-1}(f(a))$ such that $\{a, x\} \in E(G)$. In the latter situation, we have $f(y) = f(a)$, i.e. there exist $y \in f^{-1}(f(a))$ and $b \in f^{-1}(f(b))$ such that $\{b, y\} \in E(G)$. Hence $f \in hEnd(G)$. □

Lemma 2.6. *Let G be a bipartite graph with $n(\geq 2)$ components. If exactly one component is K_1 while any of the other components is K_2, then G is E-H-unretractive.*

Proof. Suppose $f \in End(G)$. Let $a, b \in G$ such that $\{f(a), f(b)\} \in E(G)$. Clearly at least one vertex of a and b, say a, is not an isolated vertex of G. Then there exists $c \in G$ such that $\{a, c\}$ is a component K_2 of G, and so $\{f(a), f(c)\}$ is also a component K_2 of G. Thus $f(b) = f(c)$, i.e. there exist $a \in f^{-1}(f(a))$ and $c \in f^{-1}(f(b))$ such that $\{a, c\} \in E(G)$, which implies $f \in hEnd(G)$. □

Lemma 2.7. *Let G be a bipartite graph with $n(\geq 2)$ components. If exactly one component is K_2 while any of the other components is K_1, then G is E-H-unretractive.*

Proof. Suppose $f \in End(G)$. Let $a, b \in G$ such that $\{f(a), f(b)\} \in E(G)$. Let $\{u, v\}$ be the unique component K_2 of G. Thus $\{f(a), f(b)\} = \{u, v\}$, say, $f(a) = u$ and $f(b) = v$. So $u \in f^{-1}(f(a))$ and $v \in f^{-1}(f(b))$ such that $\{u, v\} \in E(G)$, which implies $f \in hEnd(G)$. □

Lemma 2.8. *Suppose G is a non-connected bipartite graph with isolated vertices. If there exists a component G_i in G such that $diam(G_i) \geq 2$ or $G = mK_2 \cup nK_1$ where $m \geq 2, n \geq 2$, then $End(G) \neq hEnd(G)$.*

Proof. Firstly, suppose there exists a component being $K_1 = \{a\}$ and a component G_1 with $diam(G_1) \geq 2$. Then there exist $u, u_1, u_2 \in G$ such that the path $P = uu_1u_2$ is a geodesic of G. By Proposition 1.4, there exists $f \in Idpt(G \setminus \{a\})$ such that $I_f = u_1u_2$. Now define a mapping g by the rules $g = f$ on $G \setminus \{a\}$ and $g(a) = u$. It is routine to check that $g \in End(G) \setminus hEnd(G)$.

Secondly, suppose G is a union of $n(\geq 2)$ isolated vertices, say, $a_1, a_2, \ldots, a_n$, and $m(\geq 2)$ components K_2s, say, $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_m, y_m\}$. Define a mapping g from $V(G)$ to itself by the following rule:

$g(x_i) = x_1$ and $g(y_i) = y_1$ for any $i \in \{1, 2, \ldots, m\}$;

$g(a_1) = x_2$ and $g(a_j) = y_2$ for any $j \in \{2, 3, \ldots, n\}$.

It is easy to check $g \in End(G)$. Notice $\{g(a_1), g(a_2)\} = \{x_2, y_2\} \in E(G)$. However, since $g^{-1}(g(a_1)) = g^{-1}(x_2) = \{a_1\}$ and $g^{-1}(g(a_2)) = g^{-1}(y_2) = \{a_2, a_3, \ldots, a_n\}$, so for any $s \in g^{-1}(g(a_1))$ and any $t \in g^{-1}(g(a_2))$, $\{s, t\} \notin E(G)$. Thus $g \notin hEnd(G)$. □

Lemma 2.9. *Suppose G is a non-connected bipartite graph without isolated vertices. Then $End(G) \neq hEnd(G)$ if some component of G has a path of length 3.*

Proof. Let $u_1u_2u_3u_4$ be a path of length 3 in G and let G be the union of two disjoint graphs G_1 and G_2. Define f as follows: it maps G_1 to u_1u_2 (with one bipartition class of G_1 mapped to u_1 and the other to u_2), and it maps G_2 to u_3u_4 (with one bipartition class of G_2 mapped to u_3 and the other to u_4). Clearly, $f \in End(G)$, and since no edge is mapped onto $\{u_2, u_3\}$, $f \notin hEnd(G)$. □

Note that for a connected bipartite graph G, $diam(G) = 0$ if and only if $G = K_1$; $diam(G) = 1$ if and only if $G = K_2$; $diam(G) = 2$ if and only if $G = K_{m,n}$ with $max\{m, n\} \geq 2$. Now we characterize E-H-unretractive bipartite graphs in the following theorem, and the proof should be clear by all the foregoing.

Theorem 2.10. *Let G be a bipartite graph with $n(\geq 1)$ components Then G is E-H-unretractive if and only if G belongs to one of the following cases:*

(1) *G is a tree;*

(2) *$n = 1$ and $diam(G) \leq gir(G) - 2$;*

(3) *$n \geq 2$ and each component is K_1, i.e. $G = \overline{K}_n$;*

(4) *$n \geq 2$, and each component is K_2 or $K_{1,m}(m \geq 2)$ (i.e. a star);*

(5) *$n \geq 2$, and exactly one component is K_1 while any of the other components is K_2;*

(6) *$n \geq 2$, and exactly one component is K_2 while any of the other components is K_1.*

3. No bipartite graphs with endotype 1 mod 4.

Lemma 3.1. *Let G be a bipartite graph. If there exists a component G_1 of G such that $diam(G_1) \geq 3$, then $hEnd(G) \neq lEnd(G)$.*

Proof. Since $diam(G_1) \geq 3$, there exists a geodesic $P = a_1a_2a_3a_4$ in G_1. By Proposition 1.4, there exists $f \in Idpt(G)$ such that $I_f = P$. Define a mapping g as the composition of two morphisms, one is f and the other maps P to itself by the rule $a_i \mapsto a_i (i = 1, 2, 3)$ and $a_4 \mapsto a_2$. Note f is half strong (Proposition 1.3(2)). Evidently, $g \in hEnd(G)$.

We now show $g \notin lEnd(G)$. Note $\{g(a_1), g(a_4)\} = \{a_1, a_2\} \in E(G)$. Suppose there exists $x \in g^{-1}(g(a_1))$ such that $\{x, a_4\} \in E(G)$. Then by Lemma 2.2 $\{a_1, a_4\} = \{f(a_1), f(a_4)\} = \{f(x), f(a_4)\} \in E(G)$, contradicting the fact $P = a_1a_2a_3a_4$ is a geodesic. □

Lemma 3.2. *Let G be a non-connected graph but $G \neq \overline{K_n}(n \geq 2)$. If G contains isolated vertices, $hEnd(G) \neq lEnd(G)$.*

Proof. Without loss of generality, let $G_1, G_2, \cdots, G_n (n \geq 2)$ be n components of G such that $G_1 = K_1=\{c\}$ and there exist $a, b \in G_2$ with $\{a, b\} \in E(G_2)(\subseteq E(G))$. Let f be a mapping from $V(G)$ to itself such that $f(c) = a$ and $f(x) = x$ for any $x \in G \setminus \{c\}$. Clearly $f \in hEnd(G)$ with $\{f(c), f(b)\} = \{a, b\} \in E(G)$. Since $c \in f^{-1}(f(c))$ and $f^{-1}(f(b)) = \{b\}$ with $\{c, b\} \notin E(G)$, $f \notin lEnd(G)$. □

Lemma 3.3. *Let G be a non-connected bipartite graph. If each component of G is K_2 or $K_{m,n}(max\{m,n\} \geq 2)$ and $G \neq sK_2(s \geq 2)$, then $hEnd(G) \neq lEnd(G)$.*

Proof. Let $G = G_1 \cup H$ such that $G_1 = K_{m,n}$ with $max\{m,n\} \geq 2$. Then $diam(G_1) = 2$ and so we may let $P = a_1a_2a_3$ be a geodesic of G_1. Then by Proposition 1.4, there exists $f \in Idpt(G_1)$ such that $I_f = P$. Let $e = \{u, v\} \in E(H)$. Then there exists $g \in Idpt(H)$ such that $I_g = e$. Define a mapping h from $V(G)$ to itself by the following rule: $h(x) = f(x)$ if $x \in G_1$; $h(x) = a_2$ if $x \in g^{-1}(u)$; $h(x) = a_3$ if $x \in g^{-1}(v)$. It is easy to see that $h \in hEnd(G)$. We now show that $h \notin lEnd(G)$. Since $g \in Idpt(H)$ and $u \in I_g$, by Lemma 2.2, $g(u) = u$ and so $h(u) = a_2$. Then $\{h(a_1), h(u)\} = \{a_1, a_2\} \in E(G)$. Note $u \in h^{-1}(h(u)) \cap V(H)$ and $h^{-1}(h(a_1)) = h^{-1}(a_1) \subseteq V(G_1)$. So $\{u, x\} \notin E(G)$ for any $x \in h^{-1}(h(a_1))$, which implies $h \notin lEnd(G)$. □

Theorem 3.4. *Let G be a bipartite graph. Then*
(1) $hEnd(G) = lEnd(G)$ if and only if $End(G) = lEnd(G)$;
(2) $hEnd(G) = qEnd(G)$ if and only if $End(G) = qEnd(G)$;
(3) $hEnd(G) = sEnd(G)$ if and only if $End(G) = sEnd(G)$;
(4) $hEnd(G) = Aut(G)$ if and only if $End(G) = Aut(G)$.

Proof. Sufficiency is obvious for all four cases, and we only need to prove necessity for each case.

(1) Let $hEnd(G) = lEnd(G)$. By Lemmas 3.1, 3.2 and 3.3, $G = K_{m,n}$ $(max\{m,n\} \geq 2)$ or $G = nK_2(n \geq 2)$ or $G = \overline{K_n}(n \geq 1)$. Then by Theorem 2.10, $hEnd(G) = End(G)$ and so $End(G) = lEnd(G)$.

(2) Since $hEnd(G) = qEnd(G)$, $hEnd(G) = lEnd(G)$, and so by (1) $End(G) = lEnd(G)$. Hence $End(G) = hEnd(G)$ and furthermore $End(G) = qEnd(G)$.

(3) Since $hEnd(G) = sEnd(G)$, $hEnd(G) = lEnd(G)$, and so by (1) $End(G) = lEnd(G)$. Hence $End(G) = hEnd(G)$ and furthermore $End(G) = sEnd(G)$.

(4) Since $hEnd(G) = Aut(G)$, $hEnd(G) = lEnd(G)$, and so by (1)

End(G) = *lEnd*(G). Hence *End*(G) = *hEnd*(G) and furthermore *End*(G) = *Aut*(G). □

The next corollary follows directly from the above theorem.

Corollary 3.5. *Let G be a bipartite graph. The equality hEnd(G) = End(G) follows from any of the following equalities:* (1) *hEnd*(G) = *lEnd*(G); (2) *hEnd*(G) = *qEnd*(G); (3) *hEnd*(G) = *sEnd*(G); (4) *hEnd*(G) = *Aut*(G).

We now quote the definitions of endospetrum and endotype of a graph G in [3] as follows:

Endospec(G) = $(|End(G)|, |hEnd(G)|, |lEnd(G)|, |qEnd(G)|, |sEnd(G)|, |Aut(G)|)$ and call this 6-tuple the *endospetrum* of G, and a 5-tuple $(s_1, s_2, s_3, s_4, s_5)$ is associated with the above sequence where $s_i = 1$ stands for " $\neq$ " while $s_i = 0$ stands for " = ", e.g. $s_1 = 1$ stands for $|End(G)| \neq |hEnd(G)|$. The integer $\sum_{i=1}^{5} s_i 2^{i-1}$ is called *endotype* of G and is denoted as endotype(G). Thus, in principle for any graph there are 32 possible endotypes: from 0 to 31.

In [3, P56] a question was raised: Do there exist graphs of endotypes 9 and 25? Now by Corollary 3.5 we can easily check a bipartite graph with $s_1 = 1$ while $s_2 = 0$ is non-existent. Thus we have the following theorem, which can partly answer this question.

Theorem 3.6. *There is no bipartite graph with endotype 1 mod 4.*

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