Minimum Degree, Independence Number and (a, b, k)-Critical Graphs *

Sizhong Zhou †
School of Mathematics and Physics
Jiangsu University of Science and Technology
Mengxi Road 2, Zhenjiang, Jiangsu 212003
People's Republic of China

Abstract

Let G be a graph, and let a, b and k be nonnegative integers with $0 \le a \le b$. A graph G is called an (a, b, k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a, b]$-factor. In this paper, we prove that if $\delta(G) \ge a + k$ and $\alpha(G) \le \frac{4b(\delta(G)-a+1-k)}{(a+1)^2}$, then G is an (a, b, k)-critical graph. Furthermore, it is showed that the result in this paper is best possible in some sense.

Keywords: graph, minimum degree, independence number, $[a, b]$-factor, (a, b, k)-critical graph

AMS(2000) Subject Classification: 05C70

1 Introduction

The graphs considered in this paper will be finite and undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G. Furthermore, we denote the minimum degree and the independence number of G by $\delta(G)$ and $\alpha(G)$,

*This research was supported by Jiangsu Provincial Educational Department (07KJD110048).

†Corresponding author.E-mail address: zsz_cumt@163.com(S. Zhou)

respectively. For a subset $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S and $G-S$ denote the subgraph obtained from G by deleting all the vertices of S together with the edges incident with the vertices of S. We write $N_G(x)$ for the set of vertices adjacent to x, and $N_G[x]$ for $N_G(x)\cup\{x\}$. If S and T be disjoint subsets of $V(G)$, then $e_G(S,T)$ denotes the number of edges that join a vertex in S and a vertex in T.

Let a and b be integers with $0 \leq a \leq b$. An $[a,b]$-factor of a graph G is defined as a spanning subgraph F of G such that $a \leq d_F(x) \leq b$ for each $x \in V(G)$. And if $a = b = r$, then an $[a,b]$-factor of G is called an r-factor of G. A graph G is called an (a,b,k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a,b]$-factor. If G is an (a,b,k)-critical graph, then we also say that G is (a,b,k)-critical. If $a = b = r$, then an (a,b,k)-critical graph is simply called an (r,k)-critical graph. In particular, a $(1,k)$-critical graph is simply called a k-critical graph. Notations and definitions not given here can be found in [1].

Favaron [2] studied the properties of k-critical graphs. Liu and Yu [3] gave the characterization of (r,k)-critical graphs. Liu and Wang [4] gave a necessary and sufficient condition for a graph to be an (a,b,k)-critical graph. Li [5,6] investigated (a,b,k)-critical graphs. Recently, Zhou [7-11] obtained some sufficient conditions for graphs to be (a,b,k)-critical.

In this paper, we obtain a new sufficient condition for a graph to be an (a,b,k)-critical graph. The result is showed in the following.

Theorem 1 *Let G be a graph, and let a, b and k be nonnegative integers with $0 \leq a < b$. If $\delta(G) \geq a + k$ and*

$$\alpha(G) \leq \frac{4b(\delta(G) - a + 1 - k)}{(a+1)^2}, \tag{1}$$

then G is an (a,b,k)-critical graph.

In Theorem 1, if $k = 0$, then we get the following corollary.

Corollary 1 *Let G be a graph, and let a and b be integers such that $0 \leq a < b$. If $\delta(G) \geq a$ and $\alpha(G) \leq \frac{4b(\delta(G)-a+1)}{(a+1)^2}$, then G has an $[a,b]$-factor.*

Condition (1) is best possible in the sense that we cannot replace $\frac{4b(\delta(G)-a+1-k)}{(a+1)^2}$ by $\frac{4b(\delta(G)-a+1-k)}{(a+1)^2} + 1$, which is shown in the following example.

Let $b > a = 1$ and $k \geq 0$ be integers and $G = K_{b+k-1} \bigvee (b(b-a)+1)K_1$. Obviously, we have $\delta(G) = b + k - 1 \geq a + k$ and $\alpha(G) = b(b-a) + 1 =$

$\frac{4b(\delta(G)-a+1-k)}{(a+1)^2}+1$. Let $S=V(K_{b+k-1})\subseteq V(G)$ and $T=V((b(b-a)+1)K_1)\subseteq V(G)$, then $|S|=b+k-1>k$ and $|T|=b(b-a)+1$. Thus, by $a=1$ we have

$$\begin{aligned}\delta_G(S,T) &\geq b|S|+d_{G-S}(T)-a|T| \\ &= b(b+k-1)-a(b(b-a)+1) \\ &= b(b+k-1)-(b(b-1)+1) \\ &= bk-1<bk.\end{aligned}$$

According to the following Lemma 2.1, G is not an (a,b,k)-critical graph.

2 Proof of Theorem 1

The proof of Theorem 1 relies heavily on the following lemma.

Lemma 2.1 [4] *Let a,b and k be nonnegative integers with $a<b$, and let G be a graph of order $n\geq a+k+1$. Then G is (a,b,k)-critical if and only if for any $S\subseteq V(G)$ and $|S|\geq k$*

$$\delta(G)=b|S|+d_{G-S}(T)-a|T|\geq bk,$$

where $T=\{x:x\in V(G)\setminus S, d_{G-S}(x)\leq a-1\}$.

Proof of Theorem 1. Suppose that G satisfies the assumption of the theorem, but it is not an (a,b,k)-critical graph. Then by Lemma 2.1, there exists a subset S of $V(G)$ with $|S|\geq k$ such that

$$\delta_G(S,T)=b|S|+d_{G-S}(T)-a|T|<bk, \tag{2}$$

where $T=\{x:x\in V(G)\setminus S,\ d_{G-S}(x)\leq a-1\}$.

If $T=\varnothing$, then by (2) we obtain $b|S|<bk$, which is a contradiction. Hence, we have $T\neq\varnothing$. Let

$$h=\min\{d_{G-S}(x)|x\in T\}.$$

According to the definition of T, we get

$$0\leq h\leq a-1. \tag{3}$$

Now we consider the subgraph $G[T]$ of G induced by T. Set $T_1=G[T]$. Let x_1 be a vertex with minimum degree in T_1 and $N_1=N_{T_1}[x_1]$. Moreover, for $i \geq 2$, let x_i be a vertex with minimum degree in $T_i = G[T] - \bigcup_{1\leq j<i} N_j$ and $N_i = N_{T_i}[x_i]$. We denote the order of N_i by n_i. We continue these procedures until we reach the situation in which $T_i = \varnothing$ for some i, say for $i = r+1$. Then from the above definition we know that $\{x_1, x_2, \cdots, x_r\}$ is an independent set of G. Since $T \neq \varnothing$, we have $r \geq 1$.

The following properties are easily verified ((4) and (5) are trivial; (6) follows because our choice of x_i implies that all vertices in N_i have degree at least $n_i - 1$ in T_i).

$$\alpha(G[T]) \geq r, \tag{4}$$

$$|T| = \sum_{1\leq i\leq r} n_i, \tag{5}$$

$$\sum_{1\leq i\leq r} \Big(\sum_{x\in N_i} d_{T_i}(x)\Big) \geq \sum_{1\leq i\leq r} (n_i^2 - n_i). \tag{6}$$

From (6), it is easy to see that

$$d_{G-S}(T) \geq \sum_{1\leq i\leq r} (n_i^2 - n_i) + \sum_{1\leq i<j\leq r} e_G(N_i, N_j) \geq \sum_{1\leq i\leq r} (n_i^2 - n_i). \tag{7}$$

In view of (4), the obvious inequality $\alpha(G) \geq \alpha(G[T])$ and the assumption $\alpha(G) \leq \frac{4b(\delta(G)-a+1-k)}{(a+1)^2}$, we have

$$r \leq \frac{4b(\delta(G) - a + 1 - k)}{(a+1)^2}. \tag{8}$$

According to (5), (7), (8) and the obvious inequality $n_i^2 - (a+1)n_i \geq -\frac{(a+1)^2}{4}$, we obtain

$$\begin{aligned}
\delta_G(S,T) &= b|S| + d_{G-S}(T) - a|T| \\
&\geq b|S| + \sum_{1\leq i\leq r} (n_i^2 - n_i) - a \sum_{1\leq i\leq r} n_i \\
&= b|S| + \sum_{1\leq i\leq r} (n_i^2 - (a+1)n_i) \\
&\geq b|S| - \frac{(a+1)^2}{4} r \\
&\geq b|S| - \frac{(a+1)^2}{4} \cdot \frac{4b(\delta(G) - a + 1 - k)}{(a+1)^2} \\
&= b|S| - b(\delta(G) - a + 1 - k),
\end{aligned}$$

that is,

$$\delta_G(S,T) \geq b|S| - b(\delta(G) - a + 1 - k). \tag{9}$$

Clearly, $\delta(G) \leq |S| + h$. Thus we get that

$$|S| \geq \delta(G) - h. \tag{10}$$

In view of (3), (9) and (10), we obtain

$$\begin{aligned}\delta_G(S,T) &\geq b|S| - b(\delta(G) - a + 1 - k)\\ &\geq b(\delta(G) - h) - b(\delta(G) - a + 1 - k)\\ &= b(a - 1 - h) + bk\\ &\geq bk,\end{aligned}$$

which contradicts (2).

From the argument above, we deduce the contradictions. Hence, G is an (a,b,k)-critical graph.
Completing the proof of Theorem 1.

Acknowledgments. The author would like to express his gratitude to the referee for his very helpful and detailed comments in improving this paper.

References

[1] J. A. Bondy, U. S. R. Murty, Graph Theory with Applications, The Macmillan Press, London, 1976.

[2] O. Favaron, On k-factor-critical graph, Discussions Mathematicae Graph Theory 16(1996), 41-51.

[3] G. Liu, Q. Yu, k-Factors and extendability with prescribed components, Congressus Numerantium 139(1999), 77-88.

[4] G. Liu, J. Wang, (a,b,k)-Critical graphs, Advances in Mathematics(in Chinese) 27(6)(1998), 536-540.

[5] J. Li, Sufficient conditions for graphs to be (a,b,n)-critical graphs, Mathematica Applicata (Wuhan) 17(2004), 450-455.

[6] J. Li, A new degree condition for graph to have $[a,b]$-factor, Discrete Mathematics 290(2005), 99-103.

[7] S. Zhou, J. Jiang, Notes on the binding numbers for (a, b, k)-critical graphs, Bulletin of the Australian Mathematical Society 76(2007), 307-314.

[8] S. Zhou, Some sufficient conditions for graphs to have (g, f)-factors, Bulletin of the Australian Mathematical Society 75(2007), 447-452.

[9] S. Zhou, Sufficient conditions for (a, b, k)-critical graphs, Journal of Jilin University (Science Edition)(in Chinese) 43(5)(2005), 607-609.

[10] S. Zhou, M. Zong, Some new sufficient conditions for graphs to be (a, b, k)-critical graphs, Ars Combinatoria, to appear.

[11] S. Zhou, Y. Xu, Neighborhoods of independent sets for (a, b, k)-critical graphs, Bulletin of the Australian Mathematical Society, to appear.