A New Algorithm for General Cyclic Heptadiagonal Linear Systems Using Sherman–Morrison–Woodbury formula

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ABSTRACT

In this paper, a new efficient computational algorithm is presented for solving cyclic heptadiagonal linear systems based on using the heptadiagonal linear solver and Sherman–Morrison–Woodbury formula. The implementation of the algorithm using computer algebra systems (CAS) such as MAPLE and MATLAB is straightforward. Two numerical examples are presented for illustration.

Key Words: Cyclic heptadiagonal matrices; LU factorization; Determinants; Inverse matrix; Sherman–Morrison–Woodbury formula; Linear systems; Computer Algebra System(CAS).

1. INTRODUCTION

Cyclic heptadiagonal linear systems occur in several fields such as the numerical solution of ordinary and partial differential equations, interpolation problems, boundary value problems, etc. [1,2]. It is necessary to obtain the solution of cyclic heptadiagonal linear systems.

In this paper, we consider general cyclic heptadiagonal linear systems of the form

$$HX = R, \tag{1.1}$$

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where

$$H = \begin{bmatrix} d_1 & a_1 & A_1 & C_1 & 0 & 0 & \cdots & 0 & B_1 & b_1 \\ b_2 & d_2 & a_2 & A_2 & C_2 & \ddots & \ddots & \cdots & 0 & B_2 \\ B_3 & b_3 & d_3 & a_3 & A_3 & \ddots & \ddots & \ddots & \cdots & 0 \\ D_4 & B_4 & b_4 & d_4 & a_4 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & d_{n-3} & a_{n-3} & A_{n-3} & C_{n-3} \\ 0 & \cdots & \ddots & \ddots & \ddots & \ddots & b_{n-2} & d_{n-2} & a_{n-2} & A_{n-2} \\ A_{n-1} & 0 & \cdots & \ddots & \ddots & \ddots & B_{n-1} & b_{n-1} & d_{n-1} & a_{n-1} \\ a_n & A_n & 0 & \cdots & \cdots & 0 & D_n & B_n & b_n & d_n \end{bmatrix}, \quad (1.2)$$

$X = (x_1, x_2, \ldots, x_n)^T$, $R = (R_1, R_2, \ldots, R_n)^T$, $n \geq 8$.

Karawia [3] gives an efficient symbolic algorithm to obtain the inverse of a heptadiagonal matrix of the form (1.2) and then the solution of the system (1.1) based on LU decomposition. There are many special cases of cyclic heptadiagonal linear systems (See [4-12]).

Recently in [13], a new efficient computational algorithm is presented for solving nearly penta-diagonal linear systems based on the use of any penta-diagonal linear solver. In this paper we compute the solution of a general cyclic heptadiagonal system of the form (1.1) without imposing any restrictive conditions on the elements of the matrix H in (1.2). Our approach is mainly based on getting the natural generalization of the algorithm presented in [13]. The development of a symbolic algorithm is considered in order to remove all cases where the numerical algorithm fails.

The paper is organized as follows. In Section 2, a symbolic computational algorithm for the solution of heptadiagonal linear systems, that will not fail is constructed. In Section 3, the Sherman–Morrison–Woodbury formula is given. Two illustrative examples are given in Section 4. In Section 5, conclusions of the work are presented.

2. Heptadiagonal linear solver

In this section we focus on the construction of new symbolic computational algorithms for computing the solution of general heptadiagonal linear systems of the form:

$$H_h X_h = R_h, \quad (2.1)$$

where

$$
H_h = \begin{bmatrix}
d_1 & a_1 & A_1 & C_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
b_2 & d_2 & a_2 & A_2 & C_2 & \ddots & \ddots & \cdots & 0 & 0 \\
B_3 & b_3 & d_3 & a_3 & A_3 & \ddots & \ddots & \ddots & \cdots & 0 \\
D_4 & B_4 & b_4 & d_4 & a_4 & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & d_{m-3} & a_{m-3} & A_{m-3} & C_{m-3} \\
0 & \cdots & \ddots & \ddots & \ddots & \ddots & b_{m-2} & d_{m-2} & a_{m-2} & A_{m-2} \\
0 & 0 & \cdots & \ddots & \ddots & \ddots & B_{m-1} & b_{m-1} & d_{m-1} & a_{m-1} \\
0 & 0 & 0 & \cdots & \cdots & 0 & D_m & B_m & b_m & d_m
\end{bmatrix} \tag{2.2}
$$

$$X_h = (x_{h_1}, x_{h_2}, \ldots, x_{h_m})^T, \quad R = (R_{h_1}, R_{h_2}, \ldots, R_{h_m})^T, \; m \geq 4.$$

Firstly we begin with computing the LU factorization of the matrix H_h. It is as follows:

$$H_h = LU \tag{2.3}$$

where

$$
L = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
f_2 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
e_3 & f_3 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\frac{D_4}{\alpha_1} & e_4 & f_4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \frac{D_{m-1}}{\alpha_{m-4}} & e_{m-1} & f_{m-1} & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \frac{D_m}{\alpha_{m-3}} & e_m & f_m & 1
\end{bmatrix} \tag{2.4}
$$

and

$$U = \begin{bmatrix} \alpha_1 & g_1 & z_1 & C_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & g_2 & z_2 & \ddots & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & g_3 & \ddots & \ddots & 0 & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & \ddots & \alpha_{m-3} & g_{m-3} & z_{m-3} & C_{m-3} \\ \vdots & & & & & & 0 & \alpha_{m-2} & g_{m-2} & z_{n-2} \\ \vdots & & & & & & 0 & 0 & \alpha_{m-1} & g_{m-1} \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & \alpha_m \end{bmatrix} \tag{2.5}$$

The elements in the matrices L and U in (2.4) and (2.5) satisfy:

$$\alpha_i = \begin{cases} d_1 & \text{if } i = 1 \\ d_2 - f_2 g_1 & \text{if } i = 2 \\ d_3 - e_3 z_1 - f_3 g_2 & \text{if } i = 3 \\ d_i - \dfrac{D_i}{\alpha_{i-3}} C_{i-3} - e_i z_{i-2} - f_i g_{i-1} & \text{if } i = 4,5,\ldots,m, \end{cases} \tag{2.6}$$

$$\alpha_i \neq 0, \quad i = 1,2,\ldots,m,$$

$$f_i = \begin{cases} \dfrac{b_2}{\alpha_1} & \text{if } i = 2 \\ \dfrac{b_3 - e_3 g_1}{\alpha_2} & \text{if } i = 3 \\ \dfrac{1}{\alpha_{i-1}}\left(b_i - \dfrac{D_i}{\alpha_{i-3}} z_{i-3} - e_i g_{i-2}\right) & \text{if } i = 4,5,\ldots,m, \end{cases} \tag{2.7}$$

$$e_i = \begin{cases} \dfrac{B_3}{\alpha_1} & \text{if } i = 3 \\ \dfrac{1}{\alpha_{i-2}}\left(B_i - \dfrac{D_i}{\alpha_{i-3}} g_{i-3}\right) & \text{if } i = 4,5,\ldots,m, \end{cases} \tag{2.8}$$

$$g_i = \begin{cases} a_1 & \text{if } i = 1 \\ a_2 - f_2 z_1 & \text{if } i = 2 \\ a_i - f_i z_{i-1} - e_i C_{i-2} & \text{if } i = 3,4,\ldots,m-1, \end{cases} \tag{2.9}$$

$$z_i = \begin{cases} A_1 & \text{if } i = 1 \\ A_i - f_i C_{i-1} & \text{if } i = 2,3,\ldots,m-2. \end{cases} \quad (2.10)$$

We also have:

$$\det H_h = \prod_{i=1}^{m} \alpha_i . \quad (2.11)$$

Then, the solution is given by

$$x_{h_i} = \begin{cases} \dfrac{Q_{h_m}}{\alpha_m} & \text{if } i = m \\ \dfrac{1}{\alpha_{m-1}}\left(Q_{h_{m-1}} - g_{m-1} x_{h_m}\right) & \text{if } i = m-1 \\ \dfrac{1}{\alpha_{m-2}}\left(Q_{h_{m-2}} - g_{m-2} x_{h_{m-1}} - z_{m-2} x_{h_m}\right) & \text{if } i = m-2 \\ \dfrac{1}{\alpha_i}\left(Q_{h_i} - g_i x_{h_{i+1}} - z_i x_{h_{i+2}} - C_i x_{h_{i+3}}\right) & \text{if } i = m-3, m-4, \ldots, 1, \end{cases} \quad (2.12)$$

where

$$Q_{h_i} = \begin{cases} R_{h_1} & \text{if } i = 1 \\ R_{h_2} - f_2 Q_{h_1} & \text{if } i = 2 \\ R_{h_3} - e_3 Q_{h_1} - f_3 Q_{h_2} & \text{if } i = 3 \\ R_{h_i} - \dfrac{D_i}{\alpha_{i-3}} Q_{h_{i-3}} - e_i Q_{h_{i-2}} - f_i Q_{h_{i-1}} & \text{if } i = 4,5,\ldots,m. \end{cases} \quad (2.13)$$

At this point it is convenient to formulate our first result. It is a symbolic algorithm for computing the solution of a heptadiagonal linear system of the form (2.1).

Algorithm 2.1. To compute the solution of a heptadiagonal linear system of the form (2.1), we may proceed as follows:

Step 1: Set $\alpha_1=d_1$. If $\alpha_1=0$ then $\alpha_1= t$ (just symbol) end if. Set $g_1=a_1$, $z_1=A_1$, $k_1=A_{n-1}/ \alpha_1$, $f_2=b_2/ \alpha_1$, $e_3=B_3/ \alpha_1$, $\alpha_2=d_2-f_2 *g_1$. If $\alpha_2=0$ then $\alpha_2= t$ end if. Set $g_2=a_2-f_2*z_1$, $f_3=(b_3-e_3*g_1)/\alpha_2$, $\alpha_3=d_3-e_3*z_1-f_3*g_2$. If $\alpha_3=0$ then $\alpha_3= t$ end if.

Step 2: Compute and simplify:

For i from 4 to m do

$e_i=(B_i-D_i* g_{i-3}/ \alpha_{i-3})/ \alpha_{i-2}$

$f_i=(b_i- D_i* z_{i-3}/ \alpha_{i-3}-e_i*g_{i-2})/ \alpha_{i-1}$

$z_{i-2}=A_{i-2}-f_{i-2}*C_{i-3}$

$\alpha_i=(d_i- D_i* C_{i-3}/ \alpha_{i-3}-e_i z_{i-2}-f_i*g_{i-1})$

If $\alpha_i=0$ then $\alpha_i= t$ end if

End do

Step 3: Compute det $H_h = \left(\prod_{i=1}^{m} \alpha_i\right)_{t=0}$.

Step 4: If det $H_h \neq 0$ then do

Step 5: Set $Q_{h1}=R_{h1}$, $Q_{h2}=R_{h2}-f_2*Q_{h1}$, $Q_{h3}=R_{h3}-e_3*Q_{h1}-f_3*Q_{h2}$,

Compute and simplify:

For i from 4 to m do

$Q_{hi}=R_{hi}-D_i*Q_{hi-3}/ \alpha_{i-3}- e_i*Q_{hi-2}-f_i*Q_{hi-1}$

End do

Step 6: Set $x_{hm}=Q_{hm}/ \alpha_m$, $x_{hm-1}=(Q_{hm-1}-g_{m-1}*x_{hm})/ \alpha_{m-1}$, $x_{hm-2}=(Q_{hm-2}-g_{m-2} *x_{hm-1}-z_{m-2}*x_{hm})/ \alpha_{m-2}$,

Compute and simplify:

For i from m-3 by -1 to 1 do

$x_{hi}=(Q_{hi}-g_i *x_{hi+1}-z_i*x_{hi+2}-C_i*x_{hi+3})/ \alpha_i$.

End do

Step 8: Compute and simplify the solution:

For i from 1 to m do

$X_{hi}=(x_{hi})_{t=0}$.

End do

Else

OUTPUT("The matrix H_h is singular"); Stop.

End If

The new algorithm 2.1 is very useful to check the nonsingularity of the matrix H_h.

3. Sherman–Morrison–Woodbury formula[14]

In this section, we formulate a new computational algorithm for solving cyclic heptadiagonal linear systems of the form (1.1) based on the previous heptadiagonal linear solver. The heptadiagonal linear system of the form (1.1) can be written in the form:

$$\begin{pmatrix} M_1 & V \\ U^T & M_2 \end{pmatrix}\begin{pmatrix} x' \\ x'' \end{pmatrix}=\begin{pmatrix} R' \\ R'' \end{pmatrix}, \tag{3.1}$$

where

$$M_1 = \begin{bmatrix} d_1 & a_1 & A_1 & C_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_2 & d_2 & a_2 & A_2 & C_2 & \ddots & \ddots & \cdots & 0 & 0 \\ B_3 & b_3 & d_3 & a_3 & A_3 & \ddots & \ddots & \ddots & \cdots & 0 \\ D_4 & B_4 & b_4 & d_4 & a_4 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & d_{n-5} & a_{n-5} & A_{n-5} & C_{n-5} \\ 0 & \cdots & \ddots & \ddots & \ddots & \ddots & b_{n-4} & d_{n-4} & a_{n-4} & A_{n-4} \\ 0 & 0 & \cdots & \ddots & \ddots & \ddots & B_{n-3} & b_{n-3} & d_{n-3} & a_{n-3} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & D_{n-2} & B_{n-2} & b_{n-2} & d_{n-2} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} d_{n-1} & a_{n-1} \\ b_n & d_n \end{bmatrix},$$

$$V = \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}^T = \begin{bmatrix} B_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & C_{n-4} & A_{n-3} & a_{n-2} \\ b_1 & B_2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & C_{n-3} & A_{n-2} \end{bmatrix}^T,$$

$$U^T = \begin{bmatrix} A_{n-1} & 0 & 0 & \cdots & \cdots & \cdots & 0 & D_{n-1} & B_{n-1} & b_{n-1} \\ a_n & A_n & 0 & \cdots & \cdots & \cdots & 0 & 0 & D_n & B_n \end{bmatrix},$$

$x' = (x_1, x_2, \ldots, x_{n-2})^T$, $x'' = (x_{n-1}, x_n)^T$, $R' = (R_1, R_2, \ldots, R_{n-2})^T$, and $R'' = (R_{n-1}, R_n)^T$.

Thus (3.1) is equivalent to

$$\begin{aligned} M_1 x' + V\ x'' &= R' \\ U^T x' + M_2 x'' &= R''. \end{aligned} \tag{3.2}$$

Assume that M_2 is nonsingular. After elimination of x'' from (3.2), we obtain the linear system

$$M\ x' = \hat{R} \tag{3.3}$$

where $M = M_1 - VM_2^{-1}U^T$, $\hat{R} = R' - VM_2^{-1}R''$.

Applying the Sherman–Morrison–Woodbury formula to M, we obtain

$$M^{-1} = M_1^{-1} + M_1^{-1}V\,(M_2 - U^T M_1^{-1}V\,)^{-1} U^T M_1^{-1} \text{ and}$$
$$x' = M^{-1}\hat{R} = y + M_1^{-1}V\,(M_2 - U^T M_1^{-1}V\,)^{-1} U^T y,$$

where y is the solution of $M_1 y = \hat{R}$. It is clear that the solution x'' can be found from the above formula by successive calculation of the expressions

$$y = M_1^{-1}\hat{R},\ M_1^{-1}V,\ U^T M_1^{-1}V,\ (M_2 - U^T M_1^{-1}V\,)^{-1}, \text{and } (M_2 - U^T M_1^{-1}V\,)^{-1} U^T y.$$

The main part of the above calculations is finding the first two expressions, which is equivalent to solving three (n - 2)-by-(n - 2) heptadiagonal linear systems with the same coefficient matrix M_1 and different right-hand sides. After finding x', we can obtain x'' from the second equation of (3.2) by the formula

$$x'' = M_2^{-1}(R'' - U^T x').$$

At this point it is convenient to formulate our second result. It is a symbolic algorithm for computing the solution of a cyclic heptadiagonal linear system of the form (1.1) and can be considered as a natural generalization of the symbolic algorithm 1 in [13].

Algorithm 3.1. To compute the solution of a cyclic heptadiagonal linear system of the form (1.1), we may proceed as follows:

Step 1: Find M_1, M_2, U^T, V, R', R'', and $\hat{R} = R' - VM_2^{-1}R''$.

Step 2: Solve $M_1 y = \hat{R}, M_1 q_1 = v_1$, and $M_1 q_2 = v_2$ by algorithm 2.1, then obtain y

and $M_1^{-1} v = (q_1, q_2)$.

Step 3: Compute $x' = y + (q_1, q_2)(M_2 - U^T(q_1, q_2))^{-1} U^T y, x'' = M_2^{-1}(R'' - U^T x')$.

Step 4: Compute the solution $x = \begin{pmatrix} x' \\ x'' \end{pmatrix}_{t=0}$.

Three systems $M_1 y = \hat{R}, M_1 q_1 = v_1$, and $M_1 q_2 = v_2$ in algorithm 3.1 can be solved in parallel.

4. Illustrative examples

We give two simple examples to illustrate the effectiveness of our algorithm.

Example 4.1(case I: $\alpha_i \neq 0$ for all i)

Consider the 10-by-10 cyclic heptadiagonal systems coming from [3]

$$\begin{bmatrix} 1 & -1 & 1 & -2 & 0 & 0 & 0 & 0 & 2 & -1 \\ 1 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & -1 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 2 & -2 & 3 & 1 & 5 & -6 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 & 3 & 1 & -3 \\ 0 & 0 & 0 & 0 & -2 & -2 & 1 & 1 & 3 & 5 \\ 3 & 0 & 0 & 0 & 0 & 3 & 1 & 3 & 4 & -1 \\ 2 & 4 & 0 & 0 & 0 & 0 & 2 & 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \\ 33 \\ 0 \\ 43 \\ -24 \\ 47 \\ 70 \\ 78 \\ 94 \end{bmatrix} \quad (4.1)$$

Solution: The application of Algorithm 3.1 gives

Step1: $M_1 = \begin{bmatrix} 1 & -1 & 1 & -2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 2 & 1 & -1 & 1 & 2 & 3 & 0 & 0 \\ 2 & -2 & 3 & 1 & 5 & -6 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & -2 & -2 & 1 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 4 & -1 \\ 4 & 1 \end{bmatrix},$

$$U^T = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 3 & 1 & 3 \\ 2 & 4 & 0 & 0 & 0 & 0 & 2 & 3 \end{bmatrix},$$

$$V = \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ -1 & 1 & 0 & 0 & 0 & 0 & -3 & 5 \end{bmatrix}^T,$$

$$R' = [\,2 \quad 15 \quad 33 \quad 0 \quad 43 \quad -24 \quad 47 \quad 70\,]^T, R'' = [78 \quad 94]^T,$$

$$\hat{R} = R' - VM_2^{-1}R'' = (-33, 7, 33, 0, 43, \frac{-91}{2}, \frac{99}{2}, \frac{-69}{2})^T.$$

Step2: $y = \left(\frac{-2814}{199}, \frac{3345}{199}, \frac{2208}{199}, \frac{1308}{199}, \frac{2654}{199}, \frac{4442}{597}, \frac{15739}{597}, \frac{-7685}{398} \right)^T,$

$$M_1^{-1}V = (q_1, q_2)$$

$$= \begin{pmatrix} \frac{242}{199} & \frac{-98}{199} & \frac{-150}{199} & \frac{-104}{199} & \frac{-110}{199} & \frac{-212}{597} & \frac{-184}{597} & \frac{297}{199} \\ \frac{6}{199} & \frac{861}{199} & \frac{-132}{199} & \frac{-394}{199} & \frac{142}{199} & \frac{-895}{597} & \frac{4630}{597} & \frac{-861}{199} \end{pmatrix}^T.$$

Step3: $x' = y + (q_1, q_2)(M_2 - U^T(q_1, q_2))^{-1} U^T y = (1, 2, 3, 4, 5, 6, 7, 8)^{\mathrm{T}}$,

$$x'' = M_2^{-1}(R'' - U^T x') = (9, 10)^{\mathrm{T}}.$$

Step 4: The solution of (4.1) is $x = \begin{pmatrix} x' \\ x'' \end{pmatrix} = (1,2,3,4,5,6,7,8,9,10)^T$.

Example 4.2. (case I: $\alpha_i = 0$ for some i)

Consider the following 10-by-10 cyclic heptadiagonal systems:

$$\begin{bmatrix} 2 & 2 & -5 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\ -1 & -1 & 1 & 3 & -2 & 0 & 0 & 0 & 0 & 3 \\ 7 & 3 & -5 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 8 & 1 & 4 & -2 & 1 & 5 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & -10 & 6 & 1 & 7 & 8 & 0 & 0 \\ 0 & 0 & -4 & -3 & 2 & 9 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 5 & -6 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 4 & 2 & 5 & 3 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & -7 & 2 & 5 & 1 & 4 \\ -4 & 5 & 0 & 0 & 0 & 0 & -2 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} = \begin{bmatrix} 24 \\ 32 \\ 18 \\ 56 \\ 122 \\ 72 \\ 30 \\ 119 \\ 62 \\ 85 \end{bmatrix} \quad (4.2)$$

Solution: The application of Algorithm 3.1 gives

Step1: $$M_1 = \begin{bmatrix} 2 & 2 & -5 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 3 & -2 & 0 & 0 & 0 \\ 7 & 3 & -5 & 1 & 2 & 1 & 0 & 0 \\ 8 & 1 & 4 & -2 & 1 & 5 & 1 & 0 \\ 0 & 2 & 3 & -10 & 6 & 1 & 7 & 8 \\ 0 & 0 & -4 & -3 & 2 & 9 & 1 & 2 \\ 0 & 0 & 0 & 5 & -6 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 4 & 2 & 5 & 3 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 4 \\ 1 & 6 \end{bmatrix},$$

$$U^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -7 & 2 & 5 \\ -4 & 5 & 0 & 0 & 0 & 0 & -2 & 3 \end{bmatrix},$$

$$V = \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 2 & 3 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^T,$$

$$R' = \begin{bmatrix} 24 & 32 & 18 & 56 & 122 & 72 & 30 & 119 \end{bmatrix}^T, R'' = \begin{bmatrix} 62 & 85 \end{bmatrix}^T,$$

$$\hat{R} = R' - VM_2^{-1}R'' = (-15, -\frac{5}{2}, 18, 56, 122, 56, \frac{5}{2}\frac{151}{2})^T.$$

Step2: The application of Algorithm 2.1 gives ($\alpha_2 = 0$)

$$y = \left(\frac{189}{5}\left(\frac{7t+9}{115t+7}\right), \frac{9639}{10}\left(\frac{1}{115t+7}\right), \frac{1}{30}\left(\frac{17950t-8267}{115t+7}\right), \frac{1}{30}\left(\frac{22124t-7063}{115t+7}\right),\right.$$

$$\left.\frac{1}{15}\left(\frac{14023t-2555}{115t+7}\right), \frac{1}{30}\left(\frac{18068t-4963}{115t+7}\right), \frac{1}{5}\left(\frac{2707t-1316}{115t+7}\right), \frac{1}{10}\left(\frac{21485t+749}{115t+7}\right)\right)^T\Bigg|_{t=0}$$

$$= \left[\frac{243}{5}, -\frac{1377}{10}, -\frac{1181}{30}, -\frac{1009}{30}, -\frac{73}{3}, -\frac{709}{30}, \frac{188}{5}, \frac{107}{10}\right],$$

$$M_1^{-1}V = (q_1, q_2)$$

$$= \begin{pmatrix} \frac{-1077}{665} & \frac{3034}{665} & \frac{794}{665} & \frac{103}{95} & \frac{100}{133} & \frac{98}{95} & \frac{-82}{665} & \frac{-544}{665} \\ \frac{-1378}{57} & \frac{1365}{199} & \frac{3877}{171} & \frac{3425}{171} & \frac{2744}{171} & \frac{2555}{171} & \frac{-1130}{57} & \frac{115}{57} \end{pmatrix}^T.$$ **S**

tep3: $x' = y + (q_1, q_2)(M_2 - U^T(q_1, q_2))^{-1}U^T y = (1, 2, 3, 4, 5, 6, 7, 8)^T$,

$$x'' = M_2^{-1}(R'' - U^T x') = (9, 10)^T.$$

Step 4: The solution of (4.2) is $$x = \begin{pmatrix} x' \\ x'' \end{pmatrix} = (1,2,3,4,5,6,7,8,9,10)^T.$$

5. Conclusions

In this paper, we derived a computational algorithm for solving cyclic heptadiagonal linear systems. Since the algorithm uses the symbolic algorithm 2.1 for the LU factorization of the heptadiagonal matrix, the factorization never suffers from breakdown, and this leads to fast and reliable solution of cyclic heptadiagonal linear systems. The realization of the method needs O(n) operations. The algorithm is a natural generalization of some algorithms in current use.

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