Identities involving harmonic numbers and inverses of binomial coefficients

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Abstract

In this paper, we give some identities involving the harmonic numbers and the inverses of binomial coefficients.

Keywords: Inverses of binomial coefficients; Harmonic numbers; Combinatorial identities

1. Introduction and preliminaries

Binomial coefficients play an important role in many areas of mathematics, including combinatorial analysis, graph theory, number theory, statistics and probability. The inverses of binomial coefficients are also prolific in the mathematical literature, and the readers are referred to the papers [1–4] for many results on the identities involving the inverses of binomial coefficients.

In this paper, we establish some finite sums and some infinite series which involve the harmonic numbers and the inverses of binomial coefficients. The identities of this type might not have been presented before.

Lemma 1 (see[3]). Let n and k be any nonnegative integers, then

$$\binom{n}{k}^{-1} = (n+1)\int_0^1 t^k(1-t)^{n-k}dt.$$

Lemma 2. Let $m \leq n-1$ be a nonnegative integer, then

$$\sum_{k=m+1}^{n}\binom{n}{k}\frac{2^k(-1)^k}{(k-m)} = 2^m\sum_{k=0}^{m}\binom{m}{k}(-1)^k\sum_{j=1}^{n}\binom{k+n-j}{k}\frac{(-1)^j-1}{j}.$$

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Proof. Let $f(z)=\sum_{k=m+1}^{n}\frac{(2z)^k(-1)^k}{k-m}=(-1)^{m+1}(2z)^m\ln(1+2z)$. Then

$$\sum_{k=m+1}^{n}\binom{n}{k}\frac{2^k(-1)^k}{(k-m)}=[z^n]\frac{2^m(-1)^{m+1}}{1-z}\frac{z}{1-z}^m\ln 1+\frac{2z}{1-z}$$

$$=2^m\sum_{k=0}^{m}\binom{m}{k}(-1)^k\sum_{j=1}^{n}\binom{k+n-j}{k}\frac{(-1)^j-1}{j}.$$

This completes the proof. □

In a similar way, we can obtain the following lemmas.

Lemma 3. Let $m\le n-1$ be a nonnegative integer, then

$$\sum_{k=m+1}^{n}\binom{n}{k}\frac{(-1)^k}{k-m}=\sum_{k=0}^{m}\binom{m}{k}(-1)^{m+1-k}\sum_{j=1}^{n}\binom{k+n-j}{k}\frac{1}{j}.$$

Lemma 4. Let j be a nonnegative integer, then

$$\sum_{k=j}^{n}\binom{n}{k}^{-1}=\frac{n+1}{2^{n+2}}\sum_{k=1}^{n+1}\frac{2^k}{k}+\sum_{i=0}^{j}\sum_{k=1}^{n+1-i}\binom{j}{i}(-1)^{j-i}\frac{2^{k+i}}{k},$$

$$\sum_{k=j}^{n}\binom{n}{k}^{-1}(-1)^k=\frac{n+1}{n+2}(-1)^n+(-1)^j\binom{n+1}{j}^{-1}.$$

The first identity of this lemma can be derived from [4, Theorem 2.1], and the second identity can be found in [4, Corollary 2.3].

Lemma 5. Let j be a nonnegative integer, then

$$\sum_{k=j}^{\infty}\binom{n+k}{k}^{-1}=\frac{n}{n-1}\binom{n+j-1}{j}^{-1},$$

$$\sum_{k=j}^{\infty}\binom{n+k}{k}^{-1}(-1)^k=n2^{n-1}\ln 2+n\sum_{k=0}^{n-1}\binom{n-1}{k}\sum_{i=1}^{j+k}\frac{(-1)^i}{i}.$$

The identities presened in this lemma can be found in [4, Corollary 3.7].

2. Summations involving the inverses of binomial coefficients

In this section, we present some summations involving the inverses of binomial coefficients.

Theorem 1. Let j be a positive integer, then

$$\sum_{k=j}^{n} \binom{n}{k}^{-1} \frac{1}{k} = \frac{1}{2^{n+1}} \sum_{k=1}^{n} \frac{2^k}{k} + \sum_{i=0}^{j-1} \sum_{k=1}^{n-i} \binom{j}{i} (-1)^{j-i} \frac{2^{k+i}}{k},$$

$$\sum_{k=j}^{n} \binom{n}{k}^{-1} \frac{(-1)^k}{k} = \frac{1}{n+1} (-1)^n + (-1)^j \binom{n}{j-1}^{-1}.$$

Proof. Note that

$$\sum_{k=j}^{n} \binom{n}{k}^{-1} \frac{1}{k} = \frac{1}{n} \sum_{k=j-1}^{n-1} \binom{n-1}{k}^{-1}$$

and

$$\sum_{k=j}^{n} \binom{n}{k}^{-1} \frac{(-1)^k}{k} = \frac{1}{n} \sum_{k=j-1}^{n-1} \binom{n-1}{k}^{-1} (-1)^{k+1}.$$

Then by Lemma 4, we obtain the desired results. □

Theorem 2. Let j be a nonnegative integer, then

$$\sum_{k=j}^{n} \binom{n+k}{k}^{-1} = \frac{n}{n-1} \binom{n+j-1}{j}^{-1} - \binom{2n}{n+1}^{-1},$$

$$\sum_{k=j}^{n} \binom{n+k}{k}^{-1} (-1)^k = -n(-1)^j \sum_{k=0}^{n-1} \binom{n-1}{k} \sum_{i=j+1}^{n+1} \frac{(-1)^{k+i}}{k+i}.$$

Proof. By Lemma 1, we have

$$\sum_{k=j}^{n} \binom{n+k}{k}^{-1} = (n+1) \int_0^1 (1-t)^n \sum_{k=j}^{n} t^k dt + \int_0^1 (1-t)^n \sum_{k=j}^{n} k t^k dt$$

$$= (n+1+j) \int_0^1 (1-t)^{n-1} t^j dt - (2n+1) \int_0^1 (1-t)^{n-1} t^{n+1} dt$$

$$+ \int_0^1 (1-t)^{n-2} t^{j+1} dt - \int_0^1 (1-t)^{n-2} t^{n+1} dt$$

$$= \frac{n+1+j}{n+j}\binom{n+j-1}{j}^{-1} - \binom{2n}{n+1}^{-1}$$
$$+ \frac{1}{n+j}\binom{n+j-1}{j+1}^{-1} - \frac{1}{2n}\binom{2n-1}{n+1}^{-1}$$
$$= \frac{n}{n-1}\binom{n+j-1}{j}^{-1} - \binom{2n}{n+1}^{-1},$$

which gives the first identity. By Lemmas 1, 2 and 3, we can also establish the second identity. □

Theorem 3. Let j be a positive integer, then

$$\sum_{k=j}^{n}\binom{n+k}{k}^{-1}\frac{1}{k} = \frac{1}{n}\binom{n+j-1}{n}^{-1} - \binom{2n}{n}^{-1},$$
$$\sum_{k=j}^{n}\binom{n+k}{k}^{-1}\frac{(-1)^k}{k} = \frac{(-1)^n}{2(n+1)}\binom{2n+1}{n}^{-1} - (-1)^j\sum_{k=0}^{n}\binom{n}{k}\sum_{i=j}^{n+2}\frac{(-1)^{k+i}}{k+i}.$$

Proof. By Theorem 2, we have

$$\sum_{k=j}^{n}\binom{n+k}{k}^{-1}\frac{1}{k} = \frac{1}{n+1}\sum_{i=j-1}^{n-1}\binom{n+k+1}{k}^{-1}$$
$$= \frac{1}{n}\binom{n+j-1}{n}^{-1} - \binom{2n+2}{n+2}^{-1} - \frac{1}{n+1}\binom{2n+1}{n}^{-1} + \binom{2n+2}{n+1}^{-1}$$
$$= \frac{1}{n}\binom{n+j-1}{n}^{-1} - \binom{2n}{n}^{-1},$$

then the first identity can be obtained. Next, since

$$\sum_{k=j}^{n}\binom{n+k}{k}^{-1}\frac{(-1)^k}{k} = -\frac{1}{n+1}\sum_{i=j-1}^{n-1}\binom{n+k+1}{k}^{-1}(-1)^k$$
$$= (-1)^{j-1}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=j}^{n+2}\frac{(-1)^{k+i}}{k+i} + \frac{(-1)^n}{n+1}\binom{2n+1}{n}^{-1} - \binom{2n+2}{n+1}^{-1},$$

then the second identity can also be obtained. □

Theorem 4. Let j be a positive integer, then

$$\sum_{k=j}^{\infty}\binom{n+k}{k}^{-1}\frac{1}{k}=\frac{1}{n}\binom{n+j-1}{n}^{-1},$$

$$\sum_{k=j}^{\infty}\binom{n+k}{k}^{-1}\frac{(-1)^k}{k}=-2^n\ln 2-\sum_{k=1}^{n}\binom{n}{k}\sum_{i=1}^{j+k-1}\frac{(-1)^i}{i}.$$

Proof. Note that

$$\sum_{k=j}^{\infty}\binom{n+k}{k}^{-1}\frac{(-1)^k}{k}=-\frac{1}{n+1}\sum_{k=j-1}^{\infty}\binom{n+k+1}{k}^{-1}(-1)^k.$$

Thus in view of Lemma 5, we establish the second identity. In a similar way, we can establish the first identity. □

Theorem 5. Let $H_k=\sum_{j=1}^{k}\frac{1}{j}$ be the harmonic numbers, then

$$\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}\frac{1}{k^2}=\frac{\pi^2}{6}+\frac{1}{(n+1)^2}+\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{(-1)^kH_k}{k},$$

$$\begin{aligned}\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}\frac{(-1)^k}{k^2}&=\frac{2^{n+1}}{n+1}\sum_{k=1}^{n+1}\frac{1}{k2^k}-\ln 2-\frac{\pi^2}{12}\\&\quad+\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{1}{k}\sum_{i=1}^{k}\frac{(-1)^i}{i}.\end{aligned}$$

Proof. We have

$$\begin{aligned}\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}\frac{1}{k^2}&=\frac{1}{n+1}\sum_{k=1}^{\infty}\binom{n+k}{k-1}^{-1}\frac{1}{k}\\&=\int_0^1(1-t)^{n+1}\sum_{k=1}^{\infty}\frac{t^{k-1}}{k}dt+\frac{1}{n+1}\int_0^1(1-t)^{n+1}\sum_{k=1}^{\infty}t^{k-1}dt\\&=-\int_0^1\frac{\ln(1-t)}{t}dt+\frac{1}{(n+1)^2}+\sum_{k=1}^{n+1}\binom{n+1}{k}(-1)^k\int_0^1\sum_{i=1}^{\infty}\frac{t^{i+k-1}}{i}dt\\&=\frac{\pi^2}{6}+\frac{1}{(n+1)^2}+\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{(-1)^kH_k}{k},\end{aligned}$$

which is the first identity. Next, let us establish the second identity:

$$\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}\frac{(-1)^k}{k^2}=\frac{1}{n+1}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k+1}(n+1+k)\int_0^1(1-t)^{n+1}t^k dt$$

$$=-\int_0^1\frac{\ln(1+t)}{t}dt-\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{(-1)^k}{k}\ln 2$$

$$+\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{1}{k}\sum_{i=0}^{k}\binom{k}{i}(-1)^i\int_1^2 y^{i-1}dy$$

$$-\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}2^{n+1-k}(-1)^k\int_1^2 y^{k-1}dy$$

$$=\frac{2^{n+1}}{n+1}\sum_{k=1}^{n+1}\frac{1}{k2^k}-\ln 2-\frac{\pi^2}{12}+\sum_{k=1}^{n+1}\binom{n+1}{k}\frac{1}{k}\sum_{i=1}^{k}\frac{(-1)^i}{i}.$$

The proof is complete. □

3. Summations involving harmonic numbers

Let $H_k=\sum_{j=1}^{k}\frac{1}{j}$ be the harmonic numbers. Based on the results given above, we establish in this section several identities involving both the harmonic numbers and the inverses of the binomial coefficients.

Theorem 6. We have

$$\sum_{k=1}^{n}\binom{n}{k}H_k=2^nH_n-\sum_{k=1}^{n}\frac{1}{k2^k}, \tag{3.1}$$

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^kH_k=\frac{1}{n}, \tag{3.2}$$

$$\sum_{k=1}^{n}\binom{n}{k}\frac{H_k}{k}=H_n\sum_{k=1}^{n}\frac{2^k}{k}-H_n-\sum_{j=2}^{n}\frac{1}{j}\sum_{k=1}^{j-1}\binom{n}{k}\frac{1}{k}, \tag{3.3}$$

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^kH_k}{k}=-H_n^2-\sum_{j=2}^{n}\frac{1}{j}\sum_{k=1}^{j-1}\binom{n}{k}\frac{(-1)^k}{k}. \tag{3.4}$$

Proof. Let $f(z) := \sum_{k=1}^{\infty} H_k z^k = \frac{1}{1-z} \ln \frac{1}{1-z}$, then

$$\sum_{k=1}^{n} \binom{n}{k} H_k = [z^n] \frac{1}{1-z} \frac{1}{1-\frac{z}{1-z}} \ln \frac{1}{1-\frac{z}{1-z}}$$
$$= [z^n] \frac{1}{1-2z} \ln \frac{1-z}{1-2z} = 2^n H_n - \sum_{k=1}^{n} \frac{1}{k2^k}.$$

Next, let $g(z) := \sum_{k=j}^{\infty} \frac{z^k}{k} = -\ln(1-z) - \sum_{k=1}^{j-1} \frac{z^k}{k}$. Since

$$\sum_{k=j}^{n} \binom{n}{k} \frac{1}{k} = [z^n] \frac{-1}{1-z} \ln 1 - \frac{z}{1-z} - \sum_{k=1}^{j-1} \frac{z^k}{(1-z)^k k}$$
$$= \sum_{k=1}^{n} \frac{2^k}{k} - H_n - \sum_{k=1}^{j-1} \frac{1}{k} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \binom{i+n}{i}$$
$$= \sum_{k=1}^{n} \frac{2^k}{k} - H_n - \sum_{k=1}^{j-1} \frac{1}{k} \binom{n}{k},$$

then

$$\sum_{k=1}^{n} \binom{n}{k} \frac{H_k}{k} = \sum_{j=1}^{n} \frac{1}{j} \sum_{k=j}^{n} \binom{n}{k} \frac{1}{k} = H_n \sum_{k=1}^{n} \frac{2^k}{k} - H_n - \sum_{j=2}^{n} \frac{1}{j} \sum_{k=1}^{j-1} \binom{n}{k} \frac{1}{k}.$$

Analogously, the identities (3.2) and (3.4) can also be obtained. □

Theorem 7. We have

$$\sum_{k=1}^{n} \binom{n}{k}^{-1} H_k = \frac{n+1}{2^{n+2}} H_n \sum_{k=1}^{n+1} \frac{2^k}{k} - \sum_{j=1}^{n} \sum_{i=1}^{j-1} \binom{j-1}{i} (-1)^{j-i} \sum_{k=1}^{n-i} \frac{2^{i+k+1}}{(i+1)k},$$
$$\sum_{k=1}^{n} \binom{n}{k}^{-1} (-1)^k H_k = \frac{(-1)^n}{n+2} \frac{1}{n+2} + (n+1) H_n - \frac{n+1}{(n+2)^2}.$$

Proof. Note that

$$\sum_{k=1}^{n} \binom{n}{k}^{-1} H_k = \sum_{j=1}^{n} \frac{1}{j} \sum_{k=j}^{n} \binom{n}{k}^{-1}$$

and

$$\sum_{k=1}^{n} \binom{n}{k}^{-1} (-1)^k H_k = \sum_{j=1}^{n} \frac{1}{j} \sum_{k=j}^{n} \binom{n}{k}^{-1} (-1)^k.$$

Then by Lemma 4, we can obtain the desired results. □

Theorem 8. We have

$$\sum_{k=1}^{n}\binom{n}{k}^{-1}\frac{H_k}{k}=\frac{1}{2^{n+1}}\sum_{k=1}^{n}\frac{2^k}{k}H_k+\sum_{j=1}^{n}\frac{1}{j^2}\sum_{k=j}^{n}\binom{k}{j}^{-1}2^k,$$

$$\sum_{k=1}^{n}\binom{n}{k}^{-1}\frac{(-1)^kH_k}{k}=\frac{1}{n+1}(-1)^nH_n-\sum_{k=0}^{n-1}\binom{n}{k}^{-1}\frac{(-1)^k}{k+1}.$$

Proof. To establish this theorem, we should notice that

$$\sum_{k=1}^{n}\binom{n}{k}^{-1}\frac{H_k}{k}=\sum_{j=1}^{n}\frac{1}{j}\sum_{k=j}^{n}\binom{n}{k}^{-1}\frac{1}{k}$$

and

$$\sum_{k=1}^{n}\binom{n}{k}^{-1}\frac{(-1)^kH_k}{k}=\sum_{j=1}^{n}\frac{1}{j}\sum_{k=j}^{n}\binom{n}{k}^{-1}\frac{(-1)^k}{k},$$

and make use of Theorem 1. □

Theorem 9. We have

$$\sum_{k=1}^{n}\binom{n+k}{k}^{-1}H_k=\frac{n}{(n-1)^2}1-\binom{2n-1}{n}^{-1}-\frac{n+1}{2(n-1)}\binom{2n-1}{n}^{-1}H_n,$$

$$n\geq 2,$$

$$\sum_{k=1}^{n}\binom{n+k}{k}^{-1}(-1)^kH_k=-n\sum_{j=1}^{n}\frac{(-1)^j}{j}\sum_{k=0}^{n-1}\binom{n-1}{k}\sum_{i=j+1}^{n+1}\frac{(-1)^{k+i}}{k+i}.$$

Proof. Note that

$$\sum_{k=1}^{n}\binom{n+k}{k}^{-1}H_k=\sum_{j=1}^{n}\frac{1}{j}\sum_{k=j}^{n}\binom{n+k}{k}^{-1},$$

then by Theorems 2 and 3, we obtain the first identity. Note that

$$\sum_{k=1}^{n}\binom{n+k}{k}^{-1}(-1)^kH_k=\sum_{j=1}^{n}\frac{1}{j}\sum_{k=j}^{n}\binom{n+k}{k}^{-1}(-1)^k,$$

then by Theorem 2, the second identity can also be derived. □

Theorem 10. We have

$$\sum_{k=1}^{n}\binom{n+k}{k}^{-1}\frac{H_k}{k}=\frac{1}{n}\sum_{k=0}^{n-1}\binom{n+k}{k}^{-1}\frac{1}{k+1}-\binom{2n}{n}^{-1}H_n,$$

$$\sum_{k=1}^{n}\binom{n+k}{k}^{-1}\frac{(-1)^k H_k}{k}=\frac{(-1)^n}{2(n+1)}\binom{2n+1}{n}^{-1}\sum_{j=1}^{n}\frac{(-1)^j}{j}$$

$$-\sum_{j=1}^{n}\frac{(-1)^j}{j}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=j}^{n+2}\frac{(-1)^{k+i}}{k+i}.$$

Proof. It is sufficient to notice that

$$\sum_{k=1}^{n}\binom{n+k}{k}^{-1}\frac{H_k}{k}=\sum_{j=1}^{n}\frac{1}{j}\sum_{k=j}^{n}\binom{n+k}{k}^{-1}\frac{1}{k}$$

and

$$\sum_{k=1}^{n}\binom{n+k}{k}^{-1}\frac{(-1)^k H_k}{k}=\sum_{j=1}^{n}\frac{1}{j}\sum_{k=j}^{n}\binom{n+k}{k}^{-1}\frac{(-1)^k}{k}$$

and make use of Theorem 3. □

Theorem 11. We have

$$\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}H_k=\frac{n}{(n-1)^2},\quad n\geq 2,$$

$$\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}\frac{H_k}{k}=\frac{\pi^2}{6}+\frac{1}{n^2}-H_n^2-\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{j-1}\binom{n}{k}\frac{(-1)^k}{k}.$$

Proof. First, by Lemma 5 and Theorem 4, we have

$$\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}H_k=\sum_{j=1}^{n}\frac{1}{j}\sum_{k=j}^{\infty}\binom{n+k}{k}^{-1}$$

$$=\frac{n}{n-1}\sum_{j=1}^{n}\frac{1}{j}\binom{n+j-1}{j}^{-1}=\frac{n}{(n-1)^2},$$

which is the first identity. Next, by Theorems 4, 5 and 6, we have

$$\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}\frac{H_k}{k}=\sum_{j=1}^{\infty}\frac{1}{j}\sum_{k=j}^{\infty}\binom{n+k}{k}^{-1}\frac{1}{k}=\frac{1}{n}\sum_{j=1}^{\infty}\frac{1}{j}\binom{n+j-1}{n}^{-1}$$

$$=\sum_{j=1}^{\infty}\frac{1}{j^2}\binom{n+j-1}{j}^{-1}=\frac{\pi^2}{6}+\frac{1}{n^2}-H_n^2-\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{j-1}\binom{n}{k}\frac{(-1)^k}{k}.$$

This gives the second identity. □

Theorem 12. We have

$$\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}\sum_{j=1}^{k}\frac{(-1)^j}{j}=\frac{2^{n-1}n}{n-1}\sum_{k=1}^{n-1}\frac{1}{k2^k}-\ln 2,\quad n\geq 2,$$

$$\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}\frac{1}{k}\sum_{j=1}^{k}\frac{(-1)^j}{j}=\frac{2^n}{n}\sum_{k=1}^{n}\frac{1}{k2^k}-\ln 2-\frac{\pi^2}{12}$$
$$+\sum_{k=1}^{n}\binom{n}{k}\frac{1}{k}\sum_{i=1}^{k}\frac{(-1)^i}{i}+\frac{1}{n}\binom{2n}{n}^{-1}\ln 2.$$

Proof. By Lemma 5, we have

$$\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}\sum_{j=1}^{k}\frac{(-1)^j}{j}=\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\sum_{k=j}^{\infty}\binom{n+k}{k}^{-1}$$
$$=\frac{n}{n-1}\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\binom{n+j-1}{j}^{-1}=\frac{1}{n-1}\sum_{j=1}^{\infty}(-1)^j\binom{n+j-1}{j-1}^{-1}$$
$$=-\frac{1}{n-1}\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}(-1)^k=\frac{2^{n-1}n}{n-1}\sum_{k=1}^{n-1}\frac{1}{k2^k}-\ln 2.$$

This yields the first identity. By Theorems 3 and 5, we have

$$\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}\frac{1}{k}\sum_{j=1}^{k}\frac{(-1)^j}{j}=\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\sum_{k=j}^{\infty}\binom{n+k}{k}^{-1}\frac{1}{k}$$
$$=\frac{1}{n}\sum_{j=1}^{\infty}\binom{n+j-1}{n}^{-1}\frac{(-1)^j}{j}-\binom{2n}{n}^{-1}\sum_{j=1}^{\infty}\frac{(-1)^j}{j}$$
$$=-\frac{1}{n}\sum_{j=0}^{\infty}\binom{n+j}{j}^{-1}\frac{(-1)^j}{j+1}+\frac{1}{n}\binom{2n}{n}^{-1}\ln 2$$
$$=\frac{2^n}{n}\sum_{k=1}^{n}\frac{1}{k2^k}-\ln 2+\sum_{k=1}^{n}\binom{n}{k}\frac{1}{k}\sum_{i=1}^{k}\frac{(-1)^i}{i}-\frac{\pi^2}{12}+\frac{1}{n}\binom{2n}{n}^{-1}\ln 2.$$

This gives the second identity. □

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