Longest path starting at a vertex*

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Abstract

Let G be a connected graph of order n. Denote $p_u(G)$ the order of a longest path starting at vertex u in G. In this paper, we prove that if G has more than $t\binom{k}{2} + \binom{p+1}{2}$ edges, where $k \geq 2$, $n = t(k-1)+p+1$, $t \geq 0$ and $0 \leq p \leq k-1$, then $p_u(G) > k$ for each vertex u in G. By this result, we give an alternative proof of a result obtained by P. Wang et al. that if G is a 2-connected graph on n vertices and with more than $t\binom{k-2}{2} + \binom{p}{2} + (2n-3)$ edges, where $k \geq 3$, $n-2 = t(k-2)+p$, $t \geq 0$ and $0 \leq p < k-2$, then each edge of G lies on a cycle of order more than k.

1 Introduction

All graphs considered in this paper are finite, undirected and without loops or multiple edges. We denote the sets of vertices and edges of a graph G by $V(G)$ and $E(G)$, respectively. The order of a graph G is the number of its vertices. $e(G)$ denotes the number of edges of G. Let H be a subgraph of G, $N_H(x)$ is the set of the neighbors of x which are in H, and $d_H(x) = |N_H(x)|$ is the degree of x in H. When there is no confusion,

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we shall write $N(x)$ and $d(x)$, instead of $N_G(x)$ and $d_G(x)$. $G \setminus H$ denotes the graph obtained from G by deleting all the vertices of H and all the edges with at least one end in H. An edge uv of G is said to be contracted if it is deleted and the end vertices u and v are identified, the resulting graph is denoted by G/uv. Let $S \subseteq V(G)$, a subgraph H is induced by S if $V(H) = S$ and $xy \in E(H)$ if and only if $xy \in E(G)$, denote $H = G[S]$. And S is a vertex cut of a connected graph G if $G \setminus S$ is disconnected. The union of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The union of l disjoint copies of the same graph G is denoted by lG. The join of two disjoint graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is obtained from their union by joining each vertex of G_1 to each vertex of G_2. Denote $p(G)$ the order of a longest path in G, and $p_u(G)$ the order of a longest path starting at vertex u in G.

In [3], Woodall proved that if G is a graph on n vertices with more than $t\binom{k}{2} + \binom{p+1}{2}$ edges, where $k \geq 2$, $n = t(k-1) + p + 1$, $t \geq 0$ and $0 \leq p \leq k-1$, then G contains a cycle of order more than k, and this result is best possible. In this paper, it's interesting to show that if the number of edges of a connected graph G is more than this value, then for any vertex u in G, there is a path of order more than k which starts at u. That is the following theorem.

Theorem 1.1. *Let G be a connected graph of order n, if*

$$e(G) > f(n,k),$$

where $f(n,k) = t\binom{k}{2} + \binom{p+1}{2}$, $k \geq 2$, $n = t(k-1) + p + 1$, $t \geq 0$ and $0 \leq p \leq k-1$. Then for any vertex u in G, we have that $p_u(G) > k$.

Consider the graph $G = (tK_{k-1} \cup K_p) \vee \{u\}$, $e(G) = t\binom{k}{2} + \binom{p+1}{2}$, but $p_u(G) = k$. In this sense, Theorem 1.1 is best possible.

Let $c_e(G)$ be the order of a longest cycle which contains e in G. In [4],

P. Wang and X. Lv gave an estimation of the number of edges of a 2-connected graph G such that there is an edge e in G with $c_e(G) \leq k$ in their main theorem (Theorem 1.2 [3, p114]). They also gave the extremal graphs in their result. In section 3, as an application of Theorem 1.1, we shall give an alternative proof of the extremal number in their result.

Theorem 1.2 [4]. *For integers $n \geq 3$, and $k \geq 3$, let G be a 2-connected graph on n vertices. If there exists an edge uv such that $c_{uv}(G) \leq k$, then*

$$e(G) \leq g(n, k),$$

where $g(n,k) = t\binom{k-2}{2} + \binom{p}{2} + (2n-3)$, $n-2 = t(k-2)+p$, $t \geq 0$ and $0 \leq p < k-2$.

2 Proof of Theorem 1.1

The method of edge-switching was defined by G. Fan in [1]. Let uv be an edge in a graph G. An edge-switching from v to u is to delete edges $\{vw|w \in W\}$ and add edges $\{uw|w \in W\}$, where $W = N(v) \setminus (N(u) \cup \{u\})$. The resulting graph, denoted by $G[v \to u]$, is called an edge-switching graph of G (from v to u). Note that if G is connected, then an edge-switching graph of G is also connected. Denote the edges $\{uw|w \in W\}$ in $G[v \to u]$ by F. Then we have the following lemma.

Lemma 2.1. *G is a connected graph and uv is an edge of G. Let $G' = G[v \to u]$. Then we have that $p_u(G') \leq p_u(G)$.*

Proof. Suppose, to the contrary, that $p_u(G') > p_u(G)$. That is, there is a path P' in G', which starts at u and with $|V(P')| > p_u(G)$. In the following, we shall always find a path P in G, which starts at u and with $|V(P)| \geq |V(P')| > p_u(G)$. It's contrary to the definition of $p_u(G)$, which completes the proof.

If $E(P') \cap F = \emptyset$, then we choose $P = P'$.

If $|E(P') \cap F| = 1$, say $uw \in E(P') \cap F$, where $w \in N_G(v) \setminus (N_G(u) \cup \{u\})$. If $v \notin P'$, let $P = (P' \setminus \{uw\}) \cup \{uv, vw\}$. If v is an end vertex of P', say $P' = uw \cdots yv$, where $y \in N_G(u) \cap N_G(v)$. Then let $P = (P' \setminus \{uw\}) \cup \{uv, vw\}$. If v is an inner vertex of P', say $P' = uw \cdots xvy \cdots z$, where $x, y \in N_G(u) \cap N_G(v)$. Then we can choose $P = (P' \setminus \{uw, vx\}) \cup \{ux, vw\}$.

Since $|E(P') \cap F| \leq 1$, we complete the proof. □

In [2], Faudree and Schelp gave an estimation of the number of edges of a graph G such that $p(G) \leq k$, which is the following lemma. We shall use it in the proof of Theorem 1.1.

Lemma 2.2 [2]. *Let G be a graph on n vertices. If*

$$e(G) > h(n, k),$$

where $h(n,k) = t\binom{k}{2} + \binom{p}{2}$, $n = kt + p$, $t \geq 0$ and $0 \leq p < k$, then there is a path of order larger than k in G.

The graph $G = tK_k \cup K_p$ with $e(G) = t\binom{k}{2} + \binom{p}{2}$ and $p(G) = k$ shows that the result is best possible.

Proof of Theorem 1.1. Denote $\mathcal{G} = \{G |\ G$ is a connected graph on n vertices and $p_u(G) \leq k$ for some vertex u in $G\}$. For a graph $G \in \mathcal{G}$, let

$$l(G) = \max\{d_G(u) | u \in V(G) \text{ and } p_u(G) \leq k\}.$$

Choose $G_0 \in \mathcal{G}$ with maximum number of edges, and subject to this, let $l(G_0)$ be as large as possible. We only need to show that $e(G_0) \leq f(n,k)$. Let u be a vertex in G_0 such that $p_u(G_0) \leq k$ and $d_{G_0}(u) = l(G_0)$. Then we have that

$$d_{G_0}(u) = l(G_0) = n - 1.$$

If not, there exists a vertex v in G_0 such that $uv \notin E(G_0)$. Since G_0 is connected, we can choose a shortest path $P = uu_1u_2 \cdots v$ from u to v in G_0. Clearly, $uu_2 \notin E(G_0)$. We do edge-switching from u_1 to u in G_0. Let

$G_1 = G_0[u_1 \rightarrow u]$. By Lemma 2.1, $p_u(G_1) \leq p_u(G_0) \leq k$. And, clearly we have that $d_{G_1}(u) > d_{G_0}(u)$ and $e(G_1) = e(G_0)$. It's contrary to our choice of G_0 that $l(G_0)$ is as large as possible.

Since $d_{G_0}(u) = n-1$ and $p_u(G_0) \leq k$, we have that $p(G_0 \setminus \{u\}) \leq k-1$. By Lemma 2.2,

$$e(G_0 \setminus \{u\}) \leq h(n-1, k-1),$$

where $n-1 = t(k-1)+p$, $t \geq 0$ and $0 \leq p < k-1$.

Thus,

$$\begin{aligned} e(G_0) &= d_{G_0}(u) + e(G_0 \setminus \{u\}) \\ &\leq n-1+h(n-1,k-1) \\ &\leq f(n,k). \end{aligned}$$

□

3 Application

Denote $c_e(G)$ the order of a longest cycle which contains e in G. If there is no cycle contains e, then let $c_e(G) = 0$.

Lemma 3.1. *G is a connected graph and uv is an edge of G. Let $G' = G[v \rightarrow u]$. Then for any edge $e = ux, x \in N_G(u)$, we have that $c_e(G') \leq c_e(G)$.*

Proof. Suppose, to the contrary, that there is an edge $e = ux, x \in N_G(u)$, such that $c_e(G') > c_e(G)$. That is, there is a cycle C' in G', which contains e and with $|V(C')| > c_e(G)$. In the following, we shall always find a cycle C in G, such that $e \in C$ and $|V(C)| \geq |V(C')| > c_e(G)$. That's a contradiction which completes the proof.

If $E(C') \cap F = \emptyset$, then we can choose $C = C'$. Thus, we can assume that $E(C') \cap F \neq \emptyset$. Since $|E(C') \cap F| \leq 1$, we can assume that $E(C') \cap F =$

$\{uy\}$, where $y \in N_G(v) \setminus (N_G(u) \cup \{u\})$. There are two cases to discuss.

(i) $e = uv$. Without loss of generality, let $C' = uvz \cdots yu$, where $z \in N_G(u) \cap N_G(v)$. Then we choose $C = (C' \setminus \{uy, vz\}) \cup \{uz, vy\}$.

(ii) $e = ux$, $x \in N_G(u) \setminus \{v\}$. If $v \notin C'$, then let $C = (C' \setminus \{uy\}) \cup \{uv, vy\}$. We assume that $v \in C'$. Let $C' = ux \cdots z_1vz_2 \cdots yu$, where $\{z_1, z_2\} \subseteq N_G(u) \cap N_G(v)$. Then we choose $C = (C' \setminus \{uy, vz_2\}) \cup \{uz_2, vy\}$. □

The following lemma is easy to be proved, we omit the details here.

Lemma 3.2. *G is a 2-connected graph and uv is an edge in G.*

(i) If G/uv isn't 2-connected, then $\{u, v\}$ is a vertex cut of G.

(ii) If $N(u) \cap N(v) \neq \emptyset$, and the edge-switching graph $G[v \to u]$ isn't 2-connected, then $\{u, v\}$ is a vertex cut of G.

Alternative proof of Theorem 1.2.

We apply induction on n ($n \geq 3$).

If $n = 3$, $k \geq 3$, $e(G) \leq 3 = g(3, k)$.

Now we assume that the result is true for all graphs with fewer than n ($n \geq 4$) vertices. Denote $\mathcal{G} = \{G|\ G$ is a 2-connected graph on n vertices, and $c_e(G) \leq k$ for some edge e in $G\}$. For a graph $G \in \mathcal{G}$, let $l(G)$=max$\{d_G(u)|\ u$ is an end vertex of some edge e in G such that $c_e(G) \leq k\}$. Choose $G_0 \in \mathcal{G}$ with maximum number of edges, and subject to this, let $l(G_0)$ be as large as possible. We only need to show that $e(G_0) \leq g(n, k)$. Let u be a vertex in G_0 such that $d_{G_0}(u) = l(G_0)$, and uv is an edge with $c_{uv}(G_0) \leq k$.

Claim 1. If G_0 has a vertex cut $\{x, y\}$ with $xy \in E(G_0)$, then $e(G_0) \leq g(n, k)$.

If $\{x, y\}$ is vertex cut of G_0 and $xy \in E(G_0)$. Let H_1 be one component of $G_0 \setminus \{x, y\}$ and $H_2 = G_0 \setminus (\{x, y\} \cup H_1)$. Denote $G_1 = G_0[V(H_1) \cup \{x, y\}]$

and $G_2 = G_0[V(H_2) \cup \{x,y\}]$. Let $n_1 = |V(G_1)|$ and $n_2 = |V(G_2)|$. Clearly, G_i is 2-connected, $i = 1,2$. If $e(G_i) > g(n_i,k)$, for some i, say $e(G_1) > g(n_1,k)$, then by induction hypothesis, $c_e(G_1) > k$, for any edge $e \in G_1$. For an edge $e' \in G_2$ and $e' \neq xy$, since G_2 is 2-connected, e' and xy must lie on a common cycle (it can be easily proved by Menger's theorem). That is, there is a path P in G_2 from x to y which contains e' with $|V(P)| \geq 3$. Since $c_{xy}(G_1) > k$, $c_{e'}(G_0) \geq |V(P)| + c_{xy}(G_1) - 2 > k$. Thus, $c_e(G_0) > k$, for any edge $e \in G_0$. It's a contradiction. Hence,

$$e(G_i) \leq g(n_i,k), i = 1,2.$$

Thus,

$$\begin{aligned} e(G) &= e(G_1) + e(G_2) - 1 \\ &\leq g(n_1,k) + g(n_2,k) - 1 \\ &\leq g(n,k). \end{aligned}$$

By Claim 1, we can assume that G_0 has no vertex cut $\{x,y\}$ with $xy \in E(G_0)$.

In the following, we shall prove that $d_{G_0}(u) = l(G_0) = n-1$.

Suppose, to the contrary, that there exists a vertex z in G_0 such that $uz \notin E(G_0)$. Since G_0 is 2-connected, there is a path from u to z, which doesn't contain v in G_0. Choose $P = uu_1u_2\cdots z$ as the shortest path from u to z such that $v \notin V(P)$ in G_0. Clearly, $uu_2 \notin E(G_0)$. If $N_{G_0}(u) \cap N_{G_0}(u_1) = \emptyset$, then we contract the edge uu_1. Let $G_3 = G_0/uu_1$. By Lemma 3.2 (i) and Claim 1, we can assume that G_3 is 2-connected. Since $uu_1 \neq uv$, $c_{uv}(G_3) \leq c_{uv}(G_0) \leq k$. By induction hypothesis, $e(G_3) \leq g(n-1,k)$, then

$$\begin{aligned} e(G_0) &= e(G_3) + 1 \\ &\leq g(n-1,k) + 1 \\ &\leq g(n,k). \end{aligned}$$

Thus, we can assume that $N_{G_0}(u) \cap N_{G_0}(u_1) \neq \emptyset$. Then we do edge-switching from u_1 to u in G_0. Let $G_4 = G_0[u_1 \to u]$. By Lemma 3.2 (ii) and Claim 1, we can assume that G_4 is 2-connected. Clearly, $d_{G_4}(u) > d_{G_0}(u)$ since $uu_2 \notin E(G_0)$. And by Lemma 3.1, $c_{uv}(G_4) \leq c_{uv}(G_0) \leq k$. It's contrary to our choice of G_0 that $l(G_0)$ is as large as possible.

Since $d_{G_0}(u) = n-1$ and $c_{uv}(G_0) \leq k$, we have that $p_v(G_0 \setminus \{u\}) \leq k-1$. Since G_0 is 2-connected, $G_0 \setminus \{u\}$ is connected. By Theorem 1.1,

$$e(G_0 \setminus \{u\}) \leq f(n-1, k-1),$$

where $n-1 = t(k-2)+p+1$, $t \geq 0$ and $0 \leq p < k-2$. Thus,

$$\begin{aligned} e(G_0) &= d_{G_0}(u) + e(G_0 \setminus \{u\}) \\ &\leq n-1+f(n-1,k-1) \\ &\leq g(n,k). \end{aligned}$$

This completes the proof of Theorem 1.2. □

References

[1] Genghua Fan, Subgraph coverings and edge switchings, J. Combin. Theory B **84** (2002) 54 – 83.

[2] R. J. Faudree, R. H. Schelp, Path Ramsey numbers in multicolorings, J. Combin. Theory B **19** (1975) 150 – 160.

[3] D. R. Woodall, Maximal circuits of graphs, I, Acta Math. Acad. Sci. Hungar. **28** (1976) 77 – 80.

[4] P. Wang, X. Lv, The codiameter of a 2-connected graph, Discrete Math. **308** (2008) 113 – 122.